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Integralni analog nejednakosti Koši-Bunjakovski-Šarca i njena primjena

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Sažetak: U ovom radu ćemo dati i dokazati integralni analogon nejednakosti Koši-Bunjakovski-Švarca (CBS). Dokazaćemo najpoznatiji oblik CBS nejednakosti, zatim preći na integralni oblik, dati dokaz i navesti primjere.

Ključne riječi: CBS nejednakost, integralni analogon, primjene.

Abstract: The aim of this paper is to state and prove Cauchy-Buniakowski-Schwarz inequality. Firstly, the most common from of CBS inequality will be proven, then its integral form will be dealth with and the proofs and, finally, its application will be given.

Key words: CBS inequality, Integrated analogue, Application.

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1 Uvod

1.1 Koši-Bunjakovski-Švarcova nejednakost

Važno je napomenuti da CBS nejednakost proizilazi iz nejednakosti

$$(\sum_{i=1}^{n} p_i \cdot a_i^2) \cdot (\sum_{i=1}^{n} p_i \cdot b_i^2) \geq (\sum_{i=1}^{n} p_i \cdot a_i \cdot b_i)^2 \qquad (*)$$

gdje su a i b dva konana niza realnih brojeva, a $p_i > 0 \ \ (i = 1, ..., n)$.
Nejednakost $(*)$ predstavlja teinsku verziju CBS nejednakosti. Brojevi p_i $(i = 1, ..., n)$ su teine.

Stavljajui u nejednakost $(*)$ $p_i = \dfrac{1}{n}$, dobijamo:

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Cauchy, A. (1789.-1857.) - francuski matematiar
Buniakovsky V.J. (1804.-1889.) - ruski matematiar
Schwarz, H.A. (1834.-1921.) - njemaki matematiar

$$(\frac{1}{n}a_1^2+\frac{1}{n}a_2^2+...+\frac{1}{n}a_n^2)\cdot(\frac{1}{n}b_1^2+\frac{1}{n}b_2^2+...+\frac{1}{n}b_n^2) \geq (\frac{1}{n}a_1b_1+\frac{1}{n}a_2b_2+...+\frac{1}{n}a_nb_n)^2$$

a odavde

$$(a_1^2 + a_2^2 + ... + a_n^2) \cdot (b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1b_1 + a_2b_2 + ... + a_nb_n)^2,$$

a ovo je najjednostavniji oblik nejednakosti CBS.

Teorem 1.1 *Za bilo koja dva konana niza realnih brojeva* $a = (a_1, a_2, ..., a_n)$ *i* $b = (b_1, b_2, ..., b_n)$ *vrijedi*

$$(\sum_{k=1}^{n} a_kb_k)^2 \leq (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) \tag{1}$$

odnosno,

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2).$$

Pri tome jednakost vrijedi u (1) *ako i samo ako su nizovi a i b proporcionalni, tj.*

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}.$$

Dokaz 1:

Posmatrajmo kvadratnu funkciju $f : R \to R$ definisanu formulom:

$$f(x) = \sum_{k=1}^{n}(a_kx - b_k)^2 = (\sum_{k=1}^{n} a_k^2)x^2 - 2(\sum_{k=1}^{n} a_kb_k)x + \sum_{k=1}^{n} b_k^2.$$

Znamo da je $Ax^2+Bx+C \geq 0 \Leftrightarrow A > 0 \ \wedge \ B^2-4AC \leq 0$. Poto je f nenegativna funkcija ($f(x) \geq 0$ za svako $x \in R$) i $\sum_{k=1}^{n} a_k^2 > 0$, njena diskriminanta ne moe biti pozitivna, tj. mora biti

$$D = 4(\sum_{k=1}^{n} a_kb_k)^2 - 4(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) \leq 0,$$

odakle slijedi traena nejednakost

$$(\sum_{k=1}^{n} a_kb_k)^2 \leq (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) \leq 0,$$

odnosno

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2).$$

Znak jednakosti u (1) vrijedi ako i samo ako je $a_kx - b_k = 0, \forall k \in N$, a odavde

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} (= \frac{1}{x}, x \neq 0).$$

Ovim je jednakost (1) dokazana.

Dokaz 2:

Neka je $A_n = a_1^2 + a_2^2 + ... + a_n^2$, $B_n = a_1b_1 + a_2b_2 + ... + a_nb_n$, $C_n = b_1^2 + b_2^2 + ... + b_n^2$.
Na osnovu $A - G$ nejednakosti imamo

$$\frac{A_nC_n}{B_n^2} + 1 = \sum_{i=1}^{n} \frac{a_i^2 C_n}{B_n^2} + \sum_{i=1}^{n} \frac{b_i^2}{C_n} = \sum_{i=1}^{n} (\frac{a_i^2 C_n}{B_n^2} + \frac{b_i^2}{C_n}) \geq 2 \sum_{i=1}^{n} \sqrt{\frac{a_i^2 C_n b_i^2}{B_n^2 C_n}} = 2 \sum_{i=1}^{n} \frac{a_i b_i}{B_n} = 2,$$

jer je $\sum_{i=1}^{n} \frac{a_i b_i}{B_n} = \frac{a_1b_1 + a_2b_2 + ... + a_nb_n}{a_1b_1 + a_2b_2 + ... + a_nb_n} = 1.$
Dakle,

$$\frac{A_nC_n}{B_n^2} + 1 \geq 2 \Rightarrow \frac{A_nC_n}{B_n^2} \geq 1$$

odnosno

$$A_nC_n \geq B_n^2, \text{ tj.}$$

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2).$$

Dokaz 3:

Dokaz CBS nejednakosti ćemo izvesti matematičkom indukcijom. Za $n = 1$ slučaj je trivijalan, tj. $a_1^2 b_1^2 \geq (a_1b_1)^2$. Dalje vrijedi zbog očigledne nejednakosti $(a_1b_2 - a_2b_1)^2 \geq 0$:

$$\begin{aligned}(a_1b_1 + a_2b_2)^2 = a_1^2b_1^2 + 2a_1b_1a_2b_2 + a_2^2b_2^2 &\leq a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2 \\ &= (a_1^2 + a_2^2)(b_1^2 + b_2^2)\end{aligned}$$

Što znači da (1) važi za $n = 2$. Ova se nejednakost može pisati u obliku

$$\text{(2)} \qquad \sqrt{a_1^2 + a_2^2} \cdot \sqrt{b_1^2 + b_2^2} \geq |a_1b_1 + a_2b_2| \geq a_1b_1 + a_2b_2.$$

Pretpostavimo da nejednakost (1) vrijedi za proizvoljan prirodan broj $k \geq 2$, tj.

$$(\sum_{i=1}^{k} a_i b_i)^2 \leq (\sum_{i=1}^{k} a_i^2)(\sum_{i=1}^{k} b_i^2).$$

Koristeći induktivnu pretpostavku i nejednakost (2) imamo:

$$\sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k+1} b_i^2} = \sqrt{\sum_{i=1}^{k} a_i^2 + a_{k+1}^2}\sqrt{\sum_{i=1}^{k} b_i^2 + b_{k+1}^2}$$
$$\geq \sqrt{\sum_{i=1}^{k} a_i^2} \cdot \sqrt{\sum_{i=1}^{k} b_i^2} + |a_{k+1} \cdot b_{k+1}|$$
$$\geq \sum_{i=1}^{k} |a_i b_i| + |a_{k+1} b_{k+1}| = \sum_{i=1}^{k+1} |a_i b_i|.$$

Vidimo da nejednakost (1) vrijedi i za $n = k + 1$, pa tako zaključujemo da (1) vrijede za sve prirodne brojeve n.

2 Glavni rezultat (main results)

2.1 Integralni analogon CBS nejednakosti

Teorem 2.1 *Neka su funkcije $f, g : [a, b] \to R$ Riman integrabilne. Tada vai nejednakost:*

$$\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right) \geq \left(\int_a^b f(x) \cdot g(x)dx\right)^2.$$

Dokaz: Neka $\lambda \in R$. Funkcija $f + \lambda g$ je Riman integrabilna i imamo:

$$(f(x) + \lambda g(x))^2 \geq 0,$$

odnosno

$$\left(\int_a^b (f(x) + \lambda g(x))dx\right)^2 \geq 0.$$

Imamo,

$$\lambda^2 \int_a^b g^2(x)dx + 2\lambda \int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \geq 0.$$

Dvije situacije treba ispitati:

$$(1) \int_a^b g(x)dx = 0$$
$$(2) \int_a^b f(x)dx \geq 0$$

(1) Ako je $\int_a^b g(x)dx = 0$ tada je g skoro svuda nula i nejednakost postaje jednakost.
(2) Ako je $\int_a^b g(x)dx \neq 0$, imamo

$$(\int_a^b f^2(x)dx)(\int_a^b g^2(x)dx) \geq (\int_a^b f(x)g(x)dx)^2.$$

Komentar:

Nejednakost Koi-Bunjakovski-varca postaje jednakost u dva sluaja:
1) Ako je g skoro svuda nula, f je proizvoljna.
2) Ako je $f = k \cdot g$, gdje je $k \in R$.

3 Primjena (Application)

Zadatak 1: Ako je $f : [a, b] \to R$ integrabilna funkcija dokazati da je

$$(\int_a^b f(x) \sin x dx)^2 + (\int_a^b f(x) \cos x dx)^2 \leq (b-a) \int_a^b f^2(x)dx.$$

Rjeenje: Na osnovu CBS nejednakosti imamo

$$(\int_a^b f(x) \sin x)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b \sin^2 x dx$$
$$(\int_a^b f(x) \cos x)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b \cos^2 x dx.$$

Nakon sabiranja dobijamo,

$$(\int_a^b f(x) \sin x dx)^2 + (\int_a^b f(x) \cos x dx)^2 \leq (b-a) \int_a^b f^2(x)dx.$$

Zadatak 2: Neka je funkcija $f : [0, 1] \to R$ diferencijabilna za koju vai $f(1) - f(0) = a$. Tada vai $\int_0^1 (f'(x))^2 dx \geq a^2$.

Dokaz: Na osnovu CBS nejednakosti imamo

$$\int_0^1 (f'(x))^2 dx = \int_0^1 (f'(x))^2 dx \cdot \int_0^1 1^2 dx \geq (\int_0^1 f'(x)dx)^2 = a^2.$$

Zadatak 3: Neka je $f:[0,1]\to R$ diferencijabilna za koju vrijedi $f(1)-f(0)=0$. Tada vai

$$(\int_0^1 f(x)dx)^2 \leq \frac{1}{12}\int_0^1 (f'(x))^2dx.$$

Dokaz: Parcijalnom integracijom imamo

$$\int_0^1 f(x)dx = xf(x)|_0^1 - \int_0^1 xf'(x)dx = -\int_0^1 xf'(x)dx,$$

poto je

$$\int_0^1 f'(x)dx = f(x)|_0^1 = f(1)-f(0) = 0,$$

dobijamo relaciju:

$$2\int_0^1 f(x)dx = \int_0^1 (1-2x)f'(x)dx.$$

Na osnovu CBS nejednakosti imamo:

$$\begin{aligned}
4(\int_0^1 f(x)dx)^2 =& (\int_0^1 (1-2x)f'(x)dx)^2 \leq \int_0^1 (1-2x)^2dx \cdot \int_0^1 (f'(x))^2dx,\\
&(\int_0^1 f(x)dx)^2 \leq \frac{1}{4}[\int_0^1 (1-4x+4x^2)dx \cdot \int_0^1 (f'(x))^2dx],\\
&(\int_0^1 f(x)dx)^2 \leq \frac{1}{4}\int_0^1 (f'(x))^2dx \cdot (x-2x^2+\frac{4}{3}x^3)|_0^1,\\
&(\int_0^1 f(x)dx)^2 \leq \frac{1}{4}\cdot\frac{1}{3}\cdot\int_0^1 (f'(x))^2dx,
\end{aligned}$$

dobijamo

$$(\int_0^1 f(x)dx)^2 \leq \frac{1}{12}\int_0^1 (f'(x))^2dx.$$

Zadatak 4: Ako je $f:[0,1]\to R$ integrabilna funkcija, dokazati da vai

$$(\int_0^1 x^2f(x)dx)^2 \leq \frac{1}{3}\int_0^1 x^2(f(x))^2dx.$$

Dokaz: Na osnovu CBS nejednakosti imamo

$$(\int_0^1 x^2 f(x)dx)^2 = (\int_0^1 x \cdot xf(x)dx)^2 \leq (\int_0^1 x^2 dx)(\int_0^1 x^2 f^2(x)dx) =$$
$$= \frac{x^3}{3}|_0^1 \cdot \int_0^1 x^2 f^2(x)dx = \frac{1}{3}\int_0^1 x^2 f^2(x)dx.$$

Zadatak 5: Dokazati da vrijedi nejednakost

$$\int_0^{\pi/2} \sqrt{\cos x} dx \leq 2\sqrt{\sqrt{2}-1}.$$

Dokaz: Uvedimo smjenu $x = \frac{\pi}{2} - y$; dobijamo

$$I = \int_0^{\pi/2} \sqrt{\cos x} dx = \int_{\pi/2}^0 \sqrt{cos(\frac{\pi}{2} - y)}(-dy) = \int_0^{\pi/2} \sqrt{\sin y} dy.$$

Na osnovu CBS nejednakosti slijedi

$$I^2 = (\int_0^{\pi/2} \sqrt{\sin x} dx)^2 = 2(\int_0^{\frac{\pi}{2}} \sqrt{\sin \frac{x}{2}} \cdot \sqrt{\cos \frac{x}{2}} dx)^2$$
$$\leq 2(\int_0^{\frac{\pi}{2}} \sin \frac{x}{2} dx) \cdot (\int_0^{\frac{\pi}{2}} \cos \frac{x}{2} dx)$$
$$= 2(-2\cos \frac{x}{2}|_0^{\frac{\pi}{2}}) \cdot (2\sin \frac{x}{2}|_0^{\frac{\pi}{2}}) = 2 \cdot [-2(\frac{\sqrt{2}}{2} + 1)] \cdot \sqrt{2} = 4(\sqrt{2} - 1)$$

Dakle imamo

$$I \leq \sqrt{4(\sqrt{2}-1)}, tj.$$
$$I \leq 2\sqrt{\sqrt{2}-1}, \quad q.e.d.$$

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