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QUASI-CONJUGATIVE RELATIONS ON SETS

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Abstract. In this paper the concept of quasi-conjugative relation on sets is introduced. A characterizations of quasi-conjugative relations are obtained. In addition, particulary we show when the anti-order relation $\leqslant^C$ is quasi-conjugative.

1. Introduction

The regularity of binary relations was first characterized by Zareckiĭ ([**9**]). Further criteria for regularity were given by Markowsky ([**6**]), Schein ([**8**]) and Xu Xiao-quan and Liu Yingming ([**10**]) (see also [**1**] and [**2**]). The concepts of conjugative relations, dually conjugative relations and dually normal relations were introduced by Guanghao Jiang and Luoshan Xu ([**3**], [**4**]), and a characterization of normal relations was introduced and analyzed by Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen ([**5**]). In this paper, we introduce and analyze quasi-conjugative relations on sets.

The following are some basic concepts needed in the sequel, for other nonexplicitly stated elementary notions please refer to [**10**].

For a set X, we call ρ a binary relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ a semigroup. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

2. Quasi-conjugative relations

The following classes of elements in the semigroup $\mathbf{B}(X)$ are known:

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Definition 2.1. For relation $\alpha \in \mathcal{B}(X)$ we say that it is:
(1) *regular* if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha.$$

(2) ([**5**]) *normal* if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ (\alpha^C)^{-1}.$$

(3) ([**4**]) *dually normal* if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

(4) ([**3**]) *conjugative* if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

(5) ([**3**]) *dually conjugative* if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

(6) ([**7**]) *quasi-regular* if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^C \circ \beta \circ \alpha.$$

In the following definition we introduce a new class of elements in $\mathcal{B}(X)$.

Definition 2.2. For relation $\alpha \in \mathcal{B}(X)$ we say that it is a *quasi-conjugative* relation on X if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^C.$$

Our first lemma is an adaptation of Schein's result exposed in [**8**], Theorem 1. (See, also, [**2**], Lemma 1.)

Lemma 2.1. *For a binary relation $\alpha \in \mathcal{B}(X)$, relation*

$$\alpha^* = ((\alpha \circ \alpha^C \circ (\alpha^C)^{-1})^C$$

is the maximal element in family of all relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha.$$

Proof. First, remember ourself that

$$max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha\} = \cup\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha\}.$$

Let $\beta \in B(X)$ be an arbitrary relation such that $\alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha$. We will prove that $\beta \subseteq \alpha^*$. If not, there is $(x, y) \in \beta$ such that $\neg((x, y) \in \alpha^*)$. The last gives:
$(x, y) \in \alpha \circ \alpha^C \circ (\alpha^C)^{-1} \Longleftrightarrow$
$(\exists u, v \in X)((x, u) \in (\alpha^C)^{-1} \wedge (u, v) \in \alpha^C \wedge (v, y) \in \alpha) \Longleftrightarrow$
$(\exists u, v \in X)((u, x) \in \alpha^C \wedge (u, v) \in \alpha^C \wedge (y, v) \in \alpha^{-1}) \Longrightarrow$
$(\exists u, v \in X)((u, x) \in (\alpha)^C \wedge (x, y) \in \beta \wedge (y, v) \in \alpha^{1-} \wedge (u, v) \in \alpha^C) \Longrightarrow$
$(\exists u, v) \in X)((u, v) \in \alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha \ \wedge \ (u, v) \in \alpha^C)$
We got a contradiction. So, must be $\beta \subseteq \alpha^*$.

On the other hand, we should prove that

$$\alpha^{-1} \circ \alpha^* \circ \alpha^C \subseteq \alpha.$$

Let $(x, y) \in \alpha^{-1} \circ \alpha^* \circ \alpha^C$ be an arbitrary element. Then, there are elements $u, v \in X$ such that $(x, u) \in \alpha^C$, $(u, v) \in \alpha^*$ and $(v, y) \in \alpha$. So, from

$$(x, u) \in \alpha,\ \neg((u, v) \in (\alpha^C)^{-1} \circ \alpha^C \circ \alpha^{-1}),\ (v, y) \in \alpha^C,$$

we have $\neg((x,y) \in \alpha^C)$. Indeed. Suppose that $(x,y) \in \alpha^C$. Then, we have $(u,v) \in (\alpha^C)^{-1} \circ \alpha^C \circ \alpha^{-1}$, which is impossible. Hence, we have to $(x,y) \in \alpha$ and, there fore, $\alpha^C \circ \alpha^* \circ \alpha \subseteq \alpha$.

Finally, we conclude that α^* is the maximal element of the family of all relations $\beta \in \mathcal{B}(X)$ such that $\alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha$. □

The family of all quasi-conjugative relations on set X is not empty. For example, for relation $\triangledown_X$ on X, defined by $(x,y) \in \triangledown_X \Longleftrightarrow x \neq y$, holds

$$\triangledown_X \;=\; \triangledown_X \circ Id_X \circ Id_X = \triangledown_X^{-1} \circ Id_X \circ \triangledown_X^C$$

because the relation $\triangledown_X$ is a symmetric relation on X and $\triangledown_X^C = Id_X$ holds. So, the relation $\triangledown_X$ is a quasi-conjugative relation on X.

In the following proposition we give an intrinsic characterization of quasi-conjugative relations.

Theorem 2.1. *For a binary relation α on a set X, the following conditions are equivalent:*
(1) *α is a quasi-conjugative relation.*
(2) *For all $x,z \in X$, if $(x,y) \in \alpha$, there exist $u,v \in X$ such that:*
  (a) $(x,u) \in \alpha^C \wedge (y,v) \in \alpha$,
  (b) $(\forall s,t \in X)((s,u) \in \alpha^C \wedge (t,v) \in \alpha \Longrightarrow (s,t) \in \alpha)$.
(3) $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^C$.

Proof. (1) $\Longrightarrow$ (2). Let α be a quasi-conjugative relation, i.e. let there exists a relation β such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^C$. Let $(x,y) \in \alpha$. Then there exist elements $u,v \in X$ such that

$$(x,u) \in \alpha^C,\ (u,v) \in \beta,\ (v,y) \in \alpha^{-1}.$$

Follows that there exist elements $u,v \in X$ such that $(x,u) \in \alpha^C$ and $(y,v) \in \alpha$. This proves condition (a). Now, we check the condition (b). Let $s,t \in X$ be arbitrary elements such that $(s,u) \in \alpha^C$ and $(t,v) \in \alpha$. Now, from $(s,u) \in \alpha^C$, $(u,v) \in \beta$ and $(v,t) \in \alpha^{-1}$ follows $(s,t) \in \alpha^{-1} \circ \beta \circ \alpha^C = \alpha$.

(2) $\Longrightarrow$ (1). Define a binary relation

$$\alpha' = \{(u,v) \in X \times X : (\forall s,t \in X)((s,u) \in \alpha^C \wedge ((t,v) \in \alpha \Longrightarrow (s,t) \in \alpha)\}$$

and show that $\alpha^{-1} \circ \alpha' \circ \alpha^C = \alpha$ is valid. Let $(x,y) \in \alpha$. Then there exist elements $u,v \in X$ such that the conditions (a) and (b) are hold. We have $(u,v) \in \alpha'$ by definition of relation α'.

Further, from $(x,u) \in \alpha^C$, $(u,v) \in \alpha'$ and $(v,y) \in \alpha^{-1}$ follows $(x,y) \in \alpha^{-1} \circ \alpha' \circ \alpha^C$. Hence, we have $\alpha \subseteq \alpha^{-1} \circ \alpha' \circ \alpha^C$. Contrary, let $(x,y) \in \alpha^{-1} \circ \alpha' \circ \alpha^C$ be an arbitrary pair. There exist elements $u,v \in X$ such that $(x,u) \in \alpha^C$, $(u,v) \in \alpha'$ and $(v,y) \in \alpha^{-1}$, i.e. such that $(x,u) \in \alpha^C$ and $(y,v) \in \alpha$, Hence, by definition of relation α', follows $(x,y) \in \alpha$ since $(u,v) \in \alpha'$. Therefore, $\alpha^{-1} \circ \alpha' \circ \alpha^C \subseteq \alpha$. So, the relation α is a quasi-conjugative relation on X since there exists a relation α' such that $\alpha^{-1} \circ \alpha' \circ \alpha^C = \alpha$.

(1) $\Longleftrightarrow$ (3). Let α be a quasi-conjugative relation. Then there a relation β such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^C$. Since $\alpha^* = max\{\beta \in \mathcal{B}(X) : \alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha\}$, we have $\beta \subseteq \alpha^*$ and $\alpha = \alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^C$. Contrary, let holds $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^C$,

for a relation α. Then, we have $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^C \subseteq \alpha$. So, the relation α is a quasi-conjugative relation on set X. $\square$

Corollary 2.1. *Let $(X, \leqslant)$ be a poset. Relation $\leqslant^C$ is a quasi-conjugative relation on X if and only if for all $x, y \in X$ such that $x \leqslant^C y$ there exist elements $u, v \in X$ such that :*
(a') $x \leqslant u \wedge y \leqslant^C v$, *and*
(b') $(\forall z \in X)(z \leqslant^C u \vee z \leqslant v)$.

Proof. Let $\leqslant^C$ be a quasi-conjugative relation on set X, and let $x, y \in X$ be elements such that $x \leqslant^C y$. Then, by previous Theorem, there exist elements $u, v \in X$ such that:
(a) $x \leqslant u \wedge y \leqslant^C v$;
(b) $(\forall s, t \in X)((s \leqslant u \wedge t \leqslant^C v) \Longrightarrow s \leqslant^C t)$.
Let z be an arbitrary element and if we put $z = s = t$ in formula (b), we have

$$(z \leqslant u \wedge z \leqslant^C v) \Longrightarrow z \leqslant^C z.$$

It is a contradiction. Hence, $\neg(z \leqslant u \wedge z \leqslant^C v)$. Follows

$$z \leqslant^C u \vee z \leqslant v.$$

Contrary, let $x, y \in X$ be arbitrary elements such that $x \leqslant^C y$. There exist elements $u, v \in X$ such that
(a') $x \leqslant u \wedge y \leqslant^C v$, and
(b') $(\forall z \in X)(z \leqslant^C u \vee z \leqslant v)$.
Let $s, t \in X$ be arbitrary elements such that $s \leqslant u$ and $t \leqslant^C v$. For $s = z$, we have $s \leqslant^C u \vee s \leqslant v$. Since, the option $s \leqslant u \wedge t \leqslant^C v \wedge s \leqslant^C u$ is a contradiction, left to us the possibility $s \leqslant u \wedge t \leqslant^C v \wedge s \leqslant v$. If we suppose that $s \leqslant t$, we will have $s \leqslant^C v$ what is in contradiction with $s \leqslant v$. So, must be $s \leqslant^C t$. This prove that the relation $\leqslant^C$ satisfies condition (b) of previous theorem. Hence, the relation $\leqslant^C$ is a quasi-conjugative relation on X. $\square$

Example 2.1. Let α be a quasi-conjugative relation on set X. Then there exists a relation β on X such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^C$. If θ is an equivalence relation on X, we define relation $\alpha/\theta = \{(a\theta, b\theta) \in X/\theta \times X/\theta : (a, b) \in \alpha\}$ and β/θ by analogy. We have

$$\alpha/\theta = (\alpha/\theta)^{-1} \circ \beta/\theta \circ (\alpha/\theta)^C.$$

So, the relation α/θ is a quasi-conjugative relation on X/θ.

Example 2.2. Let α be a quasi-conjugative element in $\mathcal{B}(X')$. Then there exists a relation $\beta \in \mathcal{B}(X')$ such that $\alpha = \alpha^{-1} \circ \beta \circ \alpha^C$. For a mapping $f : X \longrightarrow X'$ and a relation $\gamma \in \mathcal{B}(X')$ we define $f^{-1}(\gamma)$ by

$$(x, y) \in f^{-1}(\gamma) \Longleftrightarrow (f(x), f(y)) \in \gamma.$$

If f is a surjective mapping, we have:

$$(x, y) \in f^{-1}(\alpha) \Longleftrightarrow (x, y) \in (f^{-1}(\alpha))^{-1} \circ f^{-1}(\beta) \circ (f^{-1}(\alpha))^C.$$

So, the relation $f^{-1}(\alpha)$ is a quasi-conjugative relation in $\mathcal{B}(X)$.

EXAMPLE 2.3. Let θ be an equivalence relation on X. There is a natural surjective mapping $\pi : X \longrightarrow X/\theta$. Then, the relation $\nabla_{X/\theta}$, by comment after the Lemma 2.1, is a quasi-conjugative relation on X/θ. Thus, by Example 2.2, the relation $\pi^{-1}(\nabla_{X/\theta})$ is a quasi-conjugative relation on X.

REMARK 2.1. There is a possibility to introduce the notion of dually quasi-conjugative relation on semigroup.

For relation $\alpha \in \mathcal{B}(X)$ we say that it is a *dually quasi-conjugative relation* on X if and only if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^C \circ \beta \circ \alpha^{-1}.$$

For these relations we can state dual statements of Lemma 2.1, Theorem 2.1 and Corollary 2.1.

On the other hand, since for a quasi-conjugative relation α holds

$$\alpha^{-1} = (\alpha^{-1} \circ \beta \circ \alpha^C)^{-1} = (\alpha^{-1})^C \circ \beta^{-1} \circ (\alpha^{-1})^{-1}$$

we conclude that the relation α is a quasi-conjugative relation on X if and only if the relation α^{-1} is a dually quasi-conjugative relation on X.

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