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Osnove matematike

'*Livada*' – nova algebarska struktura

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Sažetak. U nedavno publikovanim radovima [1]-[8], uvedena je (u radu [6]) i analizirana nova algebraska struktura nazvana '*livada*' [eng. Meadow / medɛu]. To je komutativni prsten sa jedinicom i jednom unarnom totalnom operacijom $^{-1}$: $x \to x^{-1}$ sa osobinama $(x^{-1})^{-1} = x$ (reflection) i $x \cdot (x \cdot x^{-1}) = x$ (restricted inverse law). U ovom tekstu, autor prezentuju jednu od bitnih karakteristika te nove algebarske strukture $0^{-1} = 0$.

Abstract. In this paper author propose one view of a new algebraic structure – meadow. A meadow is a commutative ring with unit equipped with a total unary operation $^{-1}$: $x \to x^{-1}$, named inverse, that satisfies these additional equations: $(x^{-1})^{-1} = x$ and $x \cdot (x \cdot x^{-1}) = x$. Meadows are total algebras in which $0^{-1} = 0$.

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U radu [6], 2007. Bergstra i Tucker uvode novu algebarsku strukturu. Ime '*livada* [*meadow*]' je uvedeno za označavanje komutativnog prstena (*CR*)

$$(x + y) + z = x + (y + z)$$
$$x + y = y + x$$
$$x + 0 = x$$
$$x + (-x) = 0$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$
$$x \cdot y = y \cdot x$$
$$x \cdot 1 = x$$
$$x \cdot (y + z) = x \cdot y + x \cdot z$$

koji ima multiplikativni neutral 1 sa totalnom operacijom $^{-1}$- (inverzijom) koja zadovoljava slijedeća dva uslova:

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(*Ref*) $(x^{-1})^{-1} = x$ (reflection)

(*Ril*) $x \cdot (x \cdot x^{-1}) = x$ (restricted inverse law).

Označimo sa *Md* prethodnih devet aksioma: *Md* = *CR* + *Ref* + *Ril*

U stvari, u tom radu Bergstra and Tucker dodatno zahtijevali ispunjavanje i slijedeća dva uslova:

$$(-x)^{-1} = -x^{-1} \text{ i}$$
$$(x\cdot y)^{-1} = x^{-1}\cdot y^{-1}.$$

Ove jednakosti se mogu dedukovati iz *Md* ([1], Proposition 2.7, Proposition 2.8).

Pravo, pokažimo da ako je $x \cdot y = 1$ i $x \cdot z = 1$, tada je $y = z$. Zaista, budući da iz $x \cdot y = 1$ i $x \cdot z = 1$ slijedi $x \cdot (y - z) = 0$, imamo:

$$y - z = 1 \cdot (y - z) = (x \cdot y) \cdot (y - z) = x \cdot (y - z) \cdot y = 0 \cdot y = 0.$$

Odavde slijedi $y = z$. Dakle, ako inverz elementa x postoji, on je jedinstven. Označavamo sa x^{-1}. Dakle, $CR + Ril \vdash x \cdot y = 1 \Rightarrow x^{-1} = y$. Imamo:

$$\begin{aligned} x^{-1} = 1\cdot x^{-1} &= (x \cdot y) \cdot x^{-1} = (x \cdot x^{-1}) \cdot y = (x \cdot x^{-1} + 0) \cdot y = \\ &(x \cdot x^{-1} + 0 \cdot x^{-1}) \cdot y = (x \cdot x^{-1} + (x - x) \cdot x^{-1}) \cdot y = \\ &(x \cdot x^{-1} + (x \cdot 1 - x \cdot x \cdot x^{-1}) \cdot x^{-1}) \cdot y = \\ &(x \cdot x^{-1} + (x \cdot x \cdot y - x \cdot x \cdot x^{-1}) \cdot x^{-1}) \cdot y = \\ &(x \cdot x^{-1} + x \cdot x \cdot (y - x^{-1}) \cdot x^{-1}) \cdot y = \\ &(x \cdot x^{-1} + x \cdot x \cdot x^{-1} \cdot (y - x^{-1})) \cdot y = (x \cdot x^{-1} + x \cdot (y - x^{-1})) \cdot y = \\ &(x \cdot x^{-1} + x \cdot y - x \cdot x^{-1})) \cdot y = x \cdot y \cdot y = y. \end{aligned}$$

Pokažimo sada

$$Md \vdash (xy)^{-1} = x^{-1}\cdot y^{-1} \quad \text{i} \quad Md \vdash (-x)^{-1} = -(x^{-1}).$$

(i) Vrijedi $(x \cdot y)^{-1} = (x \cdot y)^{-1} \cdot 1 \cdot 1 = (x \cdot y)^{-1} \cdot (x \cdot x^{-1}) \cdot (y \cdot y^{-1})$

$$= (x \cdot y)^{-1} \cdot (x \cdot y) \cdot (x^{-1} \cdot y^{-1}) = 1 \cdot x^{-1} \cdot y^{-1} = x^{-1} \cdot y^{-1}.$$

(ii) Za $x = -1 = y$, iz tvrdnje $x \cdot y = 1 \Rightarrow x^{-1} = y$ slijedi da je $(-1)^{-1} = -1$. Da je $(-1) \cdot (-1) = 1$, slijedi iz tvrdnji (b) i (f) niže. Sada imamo:

$$(-x)^{-1} = (-1 \cdot x)^{-1} = (-1)^{-1} \cdot x^{-1} = -1 \cdot x^{-1} = -x^{-1}.$$

Iz *CR* axioma mogu se dedukovati slijedeći identiteti (pogledati, na primjer, [1], Lemma 2.1):

$$0 \cdot x = 0,\ \ (-1) \cdot x = -x,\ \ (-x) \cdot y = -(x \cdot y),\ \ -0 = 0,$$

$$(-x) + (-y) = -(x + y), \quad -(-x) = x.$$

(a) Računamo:

$0 + 0 = 0$	prema $CR3$
$(0 + 0) \cdot x = 0 \cdot x$	množenje obje strane sa x
$0 \cdot x + 0 \cdot x = 0 \cdot x$	prema $CR8$ i $CR6$
$(0 \cdot x + 0 \cdot x) + (-(0 \cdot x)) = 0 \cdot x + (-(0 \cdot x))$	dodajemo na obje strane po $(-(0 \cdot x))$
$0 \cdot x + (0 \cdot x + (-(0 \cdot x))) = 0$	prema $CR1$ i $CR4$
$0 \cdot x + 0 = 0$	prema $CR4$
$0 \cdot x = 0$	zbog $CR3$.

(b) Imamo:

$(-1) \cdot x = (-1) \cdot x + (x - x)$	prema $CR3$ i $CR4$
$= ((-1) \cdot x + (x \cdot 1)) - x$	prema $CR7$ i $CR1$
$= ((-1) \cdot x + (1 \cdot x)) - x$	prema $CR6$
$= ((-1) + 1) \cdot x - x$	prema $CR8$
$= (1 + (-1)) \cdot x - x$	prema $CR2$
$= 0 \cdot x - x$	prema $CR4$
$= 0 - x$	prema tvrdnji (a)
$= -x$	prema $CR3$.

(c) $(-x) \cdot y = ((-1) \cdot x) \cdot y$	prema (b)
$= (-1) \cdot (x \cdot y)$	prema $CR5$
$= -(x \cdot y)$	opet prema tvrdnji (b).

(d) $-0 = (-1) \cdot 0$	prema (b)
$= 0$	prema (a).

(e) Imamo:

$(-x) + (-y) = 0 + ((-x) + (-y))$	prema $CR3$
$= (-(x + y) + (x + y)) + ((-x) + (-y))$	prema $CR3$
$= -(x + y) + ((x + -x) + (y + -y))$	prema $CR1$ i $CR2$
$= -(x + y) + (0 + 0)$	prema $CR4$
$= -(x + y) + 0$	prema $CR3$
$= -(x + y)$	prema $CR3$.

(f) $-(-x) = 0 + -(-x)$	prema $CR3$
$= (x + (-x)) + -(-x)$	prema $CR4$
$= x + ((-x) + -(-x))$	prema $CR1$
$= x + 0$	prema $CR3$
$= x$	prema $CR3$.

Ovo nam omogućava da pokažemo (pogledati, na primjer, [1], Theorem 2.2) da vrijedi $0^{-1} = 0$.

Primjetimo, prvo, da je

$0 = 0^{-1} + -(0^{-1})$ prema $CR4$
$= 0^{-1} + (-0)^{-1}$ zbog $(-x)^{-1} = -(x^{-1})$
$= 0^{-1} + 0^{-1}$ prema (d).

Odavde, dalje, imamo:

$0^{-1} = (0^{-1} + 0^{-1})^{-1}$
$= (1 \cdot 0^{-1} + 1 \cdot 0^{-1})^{-1}$ prema $CR6$ i $CR7$
$= ((1 + 1) \cdot 0^{-1})^{-1}$ prema $CR8$
$= (1 + 1)^{-1} \cdot (0^{-1})^{-1}$ zbog $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$
$= (1 + 1)^{-1} \cdot 0$ zbog $(x^{-1})^{-1} = x$
$= 0$ prema (a) i $CR2$.

Takođe se može pokazati da livada [meadow] nema nenultih nilpotentnih elemenata (Inge Bethke and Piet Rodenburg (2008)). Zaista, pretpostavimo da postoji neko x takvo da je $x \cdot x = 0$. Tada imamo:

$$x = x \cdot (x \cdot x^{-1}) = (x \cdot x) \cdot x^{-1} = 0 \cdot x^{-1} = x^{-1} \cdot 0 = 0.$$

Polja su livade ako kompletiramo inverznu operaciju u polju sa $0^{-1} = 0$. Tako dobijena struktura se naziva *nula-totalno polje* [*zero-totalized field*].

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