

DEVET RAZNIH NAČINA IZRAČUNAVANJA  $tg \frac{5\pi}{24}$   
(Nine different manners of calculation of  $tg \frac{5\pi}{24}$ )  
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**Sažetak:** U radu je dato devet raznih načina izračunavanja vrijednosti tangensa ugla od  $\frac{5\pi}{24}$  (ili  $\frac{1}{2} \cdot 75^{\circ} = 37^{\circ} 30'$ ).

**Ključne riječi:** ugao, tangens ugla, tangens zbira, tangens razlike, Pitagorina teorema, jednakostranični trougao, jednakokraki trougao, sinusna teorema, simetrala ugla.

**Abstract:** In this paper we give nine different manners of calculation of  $tg \frac{5\pi}{24}$  (or  $\frac{1}{2} \cdot 75^{\circ} = 37^{\circ} 30'$ ).

**Key words:** angle, tangent of angle, tangent of sum, tangent of difference, Pythagoras theorem, equilateral triangle, isosceles triangle, law of sines, angle-bisector.

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ZDM Subject Classification (2010): **G40**

U ovom radu ćemo dati devet raznih načina izračunavanja tangensa ugla od  $\frac{5\pi}{24}$  (ili  $37^{\circ} 30'$ ), tj.  $tg \frac{5\pi}{24}$ . Pri tome ćemo koristiti razne činjenice iz algebre, geometrije i trigonometrije. Mišljenja smo da je izuzetno važno riješiti jedan matematički zadatak na dva ili više načina. Svakako, to ne mogu svi učenici (studenti) već oni bolji koji pokazuju veći interes za matematiku i za koje možemo reći da su nadareni za matematiku. Evo tih devet rješenja.

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**Rješenje 1:** Korištenjem formule za tangens zbira  $tg(\alpha + \beta) = \frac{tg\alpha + tg\beta}{1 - tg\alpha tg\beta}$  i činjenice

da je  $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$  dobijamo:

$$tg \frac{5\pi}{12} = \frac{tg \frac{\pi}{4} + tg \frac{\pi}{6}}{1 - tg \frac{\pi}{4} tg \frac{\pi}{6}} = \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{(3 + \sqrt{3})^2}{9 - 3} = \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}.$$

Dalje, koristeći formulu  $cos \alpha = \frac{1}{\sqrt{1 + tg^2 \alpha}}$ , imamo:

$$cos \frac{5\pi}{12} = \frac{1}{\sqrt{1 + (2 + \sqrt{3})^2}} = \frac{1}{\sqrt{8 + 4\sqrt{3}}} = \frac{1}{2\sqrt{2 + \sqrt{3}}}.$$

Sada iz formule  $tg \frac{\alpha}{2} = \sqrt{\frac{1 - cos \alpha}{1 + cos \alpha}}$  slijedi:

$$\begin{aligned} tg \frac{5\pi}{24} &= \sqrt{\frac{1 - \frac{1}{2\sqrt{2 + \sqrt{3}}}}{1 + \frac{1}{2\sqrt{2 + \sqrt{3}}}}} = \sqrt{\frac{2\sqrt{2 + \sqrt{3}} - 1}{2\sqrt{2 + \sqrt{3}} + 1}} = \\ &= \sqrt{\frac{(2\sqrt{2 + \sqrt{3}} - 1)^2}{4(2 + \sqrt{3}) - 1}} = \frac{2\sqrt{2 + \sqrt{3}} - 1}{\sqrt{7 + 4\sqrt{3}}} = \frac{2\sqrt{\frac{4 + 2\sqrt{3}}{2}} - 1}{\sqrt{(2 + \sqrt{3})^2}} = \\ &= \frac{\sqrt{2}\sqrt{(\sqrt{3} + 1)^2} - 1}{2 + \sqrt{3}} = \frac{\sqrt{2}(\sqrt{3} + 1) - 1}{2 + \sqrt{3}} = \frac{\sqrt{6} + \sqrt{2} - 1}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \\ &= \frac{2\sqrt{6} + 2\sqrt{2} - 2 - \sqrt{18} - \sqrt{6} + \sqrt{3}}{4 - 3} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2. \end{aligned}$$

**Rješenje 2:** Korištenjem formule za tangens razlike  $tg(\alpha - \beta) = \frac{tg\alpha - tg\beta}{1 + tg\alpha tg\beta}$  i

činjenice da je  $\frac{\pi}{4} - \frac{\pi}{24} = \frac{5\pi}{24}$ , dobijamo  $tg \frac{5\pi}{24} = tg\left(\frac{\pi}{4} - \frac{\pi}{24}\right) = \frac{tg \frac{\pi}{4} - tg \frac{\pi}{24}}{1 + tg \frac{\pi}{4} tg \frac{\pi}{24}}$ , tj.

$$\operatorname{tg} \frac{5\pi}{24} = \frac{1 - \operatorname{tg} \frac{\pi}{24}}{1 + \operatorname{tg} \frac{\pi}{24}}. \quad (1)$$

S obzirom da je

$$\operatorname{tg} \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \frac{2 \sin \frac{x}{2}}{2 \sin \frac{x}{2}} = \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}},$$

odnosno

$$\operatorname{tg} \frac{x}{2} = \frac{1 - \cos x}{\sin x},$$

imamo

$$\operatorname{tg} \frac{\pi}{24} = \frac{1 - \cos \frac{\pi}{12}}{\sin \frac{\pi}{12}}. \quad (2)$$

Kako je

$$\cos \frac{\pi}{12} = \cos \left( \frac{\pi}{4} - \frac{\pi}{6} \right) = \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{1}{4} (\sqrt{6} + \sqrt{2})$$

i

$$\sin \frac{\pi}{12} = \sin \left( \frac{\pi}{4} - \frac{\pi}{6} \right) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} = \frac{1}{4} (\sqrt{6} - \sqrt{2}),$$

iz (2) slijedi

$$\operatorname{tg} \frac{\pi}{24} = \frac{1 - \frac{1}{4} (\sqrt{6} + \sqrt{2})}{\frac{1}{4} (\sqrt{6} - \sqrt{2})},$$

a oдавде, poslije proširivanja razlomka sa  $\sqrt{6} + \sqrt{2}$ :

$$\operatorname{tg} \frac{\pi}{24} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2. \quad (3)$$

Iz jednakosti (1) i (3) slijedi

$$\operatorname{tg} \frac{5\pi}{24} = \frac{1 - (\sqrt{6} - \sqrt{3} + \sqrt{2} - 2)}{1 + (\sqrt{6} - \sqrt{3} + \sqrt{2} - 2)} = \frac{(\sqrt{3} - \sqrt{2})(\sqrt{3} + 1)}{(\sqrt{2} - 1)(\sqrt{3} + 1)} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1} \cdot \frac{\sqrt{2} + 1}{\sqrt{2} + 1},$$

a oдавде:

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

**Rješenje 3:** Kako je

$$\operatorname{tg} \frac{5\pi}{24} = \operatorname{tg} \left( \frac{\pi}{3} - \frac{\pi}{8} \right) = \frac{\operatorname{tg} \frac{\pi}{3} - \operatorname{tg} \frac{\pi}{8}}{1 + \operatorname{tg} \frac{\pi}{3} \operatorname{tg} \frac{\pi}{8}} = \frac{\sqrt{3} - \operatorname{tg} \frac{\pi}{8}}{1 + \sqrt{3} \operatorname{tg} \frac{\pi}{8}}$$

i

$$\operatorname{tg} \frac{\pi}{8} = \frac{1 - \cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{1 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2} - 1, \quad (4)$$

imamo

$$\operatorname{tg} \frac{5\pi}{24} = \frac{\sqrt{3} - (\sqrt{2} - 1)}{1 + \sqrt{3}(\sqrt{2} - 1)} = \frac{\sqrt{3} - \sqrt{2} + 1}{\sqrt{6} - \sqrt{3} + 1},$$

odakle je

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

**Rješenje 4:** Korištenjem formule za tangens zbira  $\operatorname{tg}(\alpha + \beta) = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta}$  i činjenice

da  $\frac{\pi}{6} + \frac{\pi}{24} = \frac{5\pi}{24}$ , imamo

$$\operatorname{tg} \frac{5\pi}{24} = \operatorname{tg} \left( \frac{\pi}{6} + \frac{\pi}{24} \right) = \frac{\operatorname{tg} \frac{\pi}{6} + \operatorname{tg} \frac{\pi}{24}}{1 - \operatorname{tg} \frac{\pi}{6} \operatorname{tg} \frac{\pi}{24}} = \frac{\frac{1}{\sqrt{3}} + \operatorname{tg} \frac{\pi}{24}}{1 - \frac{1}{\sqrt{3}} \operatorname{tg} \frac{\pi}{24}},$$

tj.

$$\operatorname{tg} \frac{5\pi}{24} = \frac{1 + \sqrt{3} \operatorname{tg} \frac{\pi}{24}}{\sqrt{3} - \operatorname{tg} \frac{\pi}{24}}. \quad (5)$$

Uvrštavanjem (3) u (5) dobijamo

$$\operatorname{tg} \frac{5\pi}{24} = \frac{1 + \sqrt{3}(\sqrt{6} - \sqrt{3} + \sqrt{2} - 2)}{\sqrt{3} - (\sqrt{6} - \sqrt{3} + \sqrt{2} - 2)} = \frac{(\sqrt{6} - 2)(\sqrt{3} + 1)}{(2 - \sqrt{3})(\sqrt{3} + 1)} = \frac{\sqrt{6} - 2}{2 - \sqrt{2}} \cdot \frac{2 + \sqrt{2}}{2 + \sqrt{2}},$$

tj.

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

**Rješenje 5:** Imamo redom:

$$\operatorname{tg} \frac{5\pi}{24} = \frac{\sin \frac{5\pi}{24}}{\cos \frac{5\pi}{24}} = \frac{\cos \left( \frac{\pi}{2} - \frac{5\pi}{24} \right)}{\cos \frac{5\pi}{24}} = \frac{\cos \frac{7\pi}{24}}{\cos \frac{5\pi}{24}}.$$

Poslije proširivanja posljednjeg razlomka sa  $2 \sin \frac{\pi}{24}$  i primjene formule

$$\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} \text{ dobijamo}$$

$$\operatorname{tg} \frac{5\pi}{24} = \frac{2 \cos \frac{7\pi}{24} \sin \frac{5\pi}{24}}{2 \cos \frac{5\pi}{24} \sin \frac{\pi}{24}} = \frac{\sin \frac{\pi}{3} - \sin \frac{\pi}{4}}{\sin \frac{\pi}{4} - \sin \frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} - \frac{1}{2}} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2} - 1} = (\sqrt{3} - \sqrt{2})(\sqrt{2} + 1),$$

tj.

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

**Rješenje 6:** Budući da je

$$\operatorname{tg} \frac{5\pi}{24} = \operatorname{tg} \left( \frac{\pi}{8} + \frac{\pi}{12} \right) = \frac{\operatorname{tg} \frac{\pi}{8} + \operatorname{tg} \frac{\pi}{12}}{1 - \operatorname{tg} \frac{\pi}{8} \operatorname{tg} \frac{\pi}{12}}, \text{ te } \operatorname{tg} \frac{\pi}{8} = \sqrt{2} - 1$$

i

$$\operatorname{tg} \frac{\pi}{12} = \frac{1 - \cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} = 2 - \sqrt{3},$$

dobijamo

$$\operatorname{tg} \frac{5\pi}{24} = \frac{\sqrt{2} - 1 + 2 - \sqrt{3}}{1 - (\sqrt{2} - 1)(2 - \sqrt{3})} = \frac{-\sqrt{3} + \sqrt{2} + 1}{\sqrt{6} - \sqrt{3} - 2\sqrt{2} + 3},$$

odakle je

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

**Rješenje 7:** Neka je dat pravougli trougao  $\triangle ABC$  ( $\angle BCA = 90^\circ$ ) kod koga je  $\angle BAC = 30^\circ$  i  $\overline{AC} = 1$  (sl.1). Naspram ugla od  $30^\circ$  u pravouglom trouglu nalazi se kateta koja je dva puta kraća od hipotenuze, što znači da je  $\overline{BC} = \frac{1}{2} \overline{AB}$ . Primjenom Pitagorine teoreme na pravougli trougao  $\triangle ABC$  imamo

$$\overline{AB}^2 = \overline{BC}^2 + \overline{CA}^2 = \left( \frac{1}{2} \overline{AB} \right)^2 + 1^2,$$

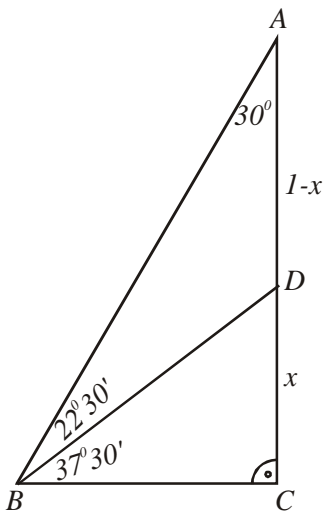
odakle je  $\overline{AB} = \frac{2}{\sqrt{3}}$ . Zbog toga je  $\overline{BC} = \frac{1}{\sqrt{3}}$ .

Na kateti  $AC$  datog trougla odredimo tačku  $D$  tako da  $\angle CBD = 37^\circ 30'$ , pa je  $\angle ABD = 22^\circ 30'$ . Ako sa  $x$  označimo dužinu duži  $CD$ , onda je  $\overline{AD} = 1 - x$ .

Primjenom sinusne teoreme na trougao  $\triangle ABD$  dobijamo  $\frac{1-x}{\sin 22^{\circ}30'} = \frac{\overline{BD}}{\sin 30^{\circ}}$ , a

odavde zbog  $\sin 22^{\circ}30' = \frac{\sqrt{1-\cos 45^{\circ}}}{2} = \frac{1}{2}\sqrt{2-\sqrt{2}}$  imamo

$$\overline{BD} = \frac{1-x}{\sqrt{2-\sqrt{2}}}. \tag{6}$$



Na osnovu Pitagorine teoreme primjenjene na pravougli trougao  $\triangle BCD$  je  $\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2$  ili

$$\overline{BD}^2 = \left(\frac{1}{\sqrt{3}}\right)^2 + x^2, \text{ tj.}$$

$$\overline{BD}^2 = \frac{1}{3} + x^2. \tag{7}$$

Iz jednakosti (6) i (7) slijedi

$$\left(\frac{1-x}{\sqrt{2-\sqrt{2}}}\right)^2 = \frac{1}{3} + x^2, \text{ tj.}$$

$$x^2 - 2(\sqrt{2}+1)x + \frac{1}{3}(\sqrt{2}+1)^2 = 0.$$

sl.1

Rješavanjem ove jednačine dobijamo

$$x = \frac{\sqrt{2}+1}{\sqrt{3}}(\sqrt{3}-\sqrt{2}),$$

pa je

$$\operatorname{tg} \frac{5\pi}{24} = \operatorname{tg} 37^{\circ}30' = \frac{\overline{CD}}{\overline{BC}} = \frac{\frac{1}{\sqrt{3}}(\sqrt{2}+1)(\sqrt{3}-\sqrt{2})}{\frac{1}{\sqrt{3}}},$$

tj.

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

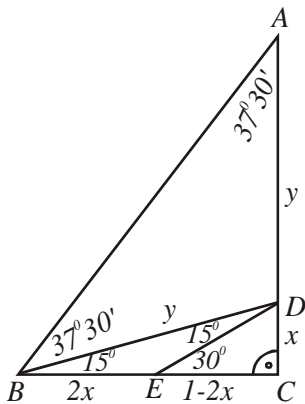
**Rješenje 8:** Nacrtajmo trougao  $\triangle ABC$  kod koga je  $\overline{BC} = 1$ ,  $\angle BAC = 37^{\circ}30'$  i  $\angle BCA = 90^{\circ}$  (sl.2). Odredimo na stranici  $AC$  tačku  $D$  tako da  $\angle DBE = 15^{\circ}$ . Otuda je

$\angle DBA = 37^{\circ}30'$ . Trougao  $\triangle ABD$  je jednakokraki, što znači da je  $\overline{BD} = \overline{AD} = y$ . Na stranici  $BC$  odredimo tačku  $E$  tako da je  $\angle BDE = 15^{\circ}$ . Trougao  $\triangle BED$  je jednakokraki pa je  $\overline{BE} = \overline{ED} = 2x$  i  $\overline{CE} = \overline{BC} - \overline{BE} = 1 - 2x$ . Kako je u trouglu  $\triangle CDE$   $\angle DEC = 30^{\circ}$ , to je  $\frac{1}{2} = \sin 30^{\circ} = \frac{\overline{CD}}{\overline{CE}} = \frac{\overline{CD}}{1 - 2x}$  odakle slijedi da je  $\overline{CD} = x$ . Primjenom Pitagorine teoreme na pravougli trougao  $\triangle CDE$  dobijamo:  $(2x)^2 = x^2 + (1 - 2x)^2$ , odakle je  $x = 2 - \sqrt{3}$ , jer mora biti  $x < 1$ . Trougao  $\triangle BDC$  je pravougli također, pa imamo  $y^2 = 1^2 + (2 - \sqrt{3})^2$ , pa je  $y = 2\sqrt{2 - \sqrt{3}}$ , tj.  $y = \sqrt{6} - \sqrt{2}$  (što nije teško provjeriti). Sada, iz trougla  $\triangle ABC$ , je:

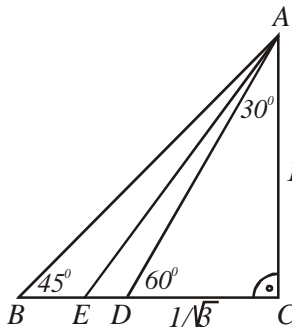
$$\begin{aligned} \operatorname{tg} \frac{5\pi}{24} &= \operatorname{tg} 37^{\circ}30' = \frac{\overline{BC}}{\overline{AC}} = \frac{1}{x + y} = \frac{1}{2 - \sqrt{3} + \sqrt{6} - \sqrt{2}} = \\ &= \frac{1}{(\sqrt{6} - \sqrt{2}) + (2 - \sqrt{3})} \cdot \frac{(\sqrt{6} - \sqrt{2}) - (2 - \sqrt{3})}{(\sqrt{6} - \sqrt{2}) - (2 - \sqrt{3})} = \frac{\sqrt{6} - \sqrt{2} - 2 + \sqrt{3}}{(\sqrt{6} - \sqrt{2})^2 - (2 - \sqrt{3})^2}, \end{aligned}$$

a odavde

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$



sl.2



sl.3

**Rješenje 9:** Konstruišimo ugao  $\angle CAD = 30^{\circ}$  u jednakokrakom pravouglom trouglu  $\triangle ABC$  ( $\overline{AB}$  - dužina hipotenuze), a  $D \in BC$  (sl.3). Neka je  $\overline{AE}$  simetrala ugla  $\angle DAB = 45^{\circ} - 30^{\circ} = 15^{\circ}$ . Tada je  $\angle CAE = 37^{\circ}30' = \frac{5\pi}{24}$ . Ako je  $\overline{AC} = \overline{BC} = 1$ , onda je  $\operatorname{tg} \frac{5\pi}{24} = \frac{\overline{EC}}{\overline{AC}}$ . Kako je  $\overline{AD} = 2 \cdot \overline{CD}$  i  $\overline{AC} = 1$ , imamo:  $\overline{CD}^2 = (2 \cdot \overline{CD})^2 - \overline{AC}^2$ , odnosno  $3 \cdot \overline{CD}^2 = 1$ , odakle je  $\overline{CD} = \frac{1}{\sqrt{3}}$ . Zbog toga je  $\overline{AD} = \frac{2}{\sqrt{3}}$ . Na bazi teoreme o

simetrali unutrašnjeg ugla trougla  $\triangle ABD$  je  $\overline{BE} : \overline{ED} = \overline{AB} : \overline{AD}$ , odakle zbog  $\overline{AB} = \sqrt{2}$  i  $\overline{BE} + \overline{ED} = 1 - \frac{1}{\sqrt{3}}$ , slijedi  $\left(1 - \frac{1}{\sqrt{3}} - \overline{ED}\right) : \overline{ED} = \sqrt{2} : \frac{2}{\sqrt{3}}$ , tj.

$$\overline{ED} = \frac{1}{3}(3\sqrt{6} + 2\sqrt{3} - 3\sqrt{2} - 6).$$

Tada je

$$\overline{EC} = \overline{ED} + \overline{DC} = \frac{1}{3}(3\sqrt{6} + 2\sqrt{3} - 3\sqrt{2} - 6) + \frac{1}{\sqrt{3}} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2,$$

tj.

$$\operatorname{tg} \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

### Zaključak

Vidimo da se za prvih šest rješenja koriste standardne formule za tangens zbira i razlike, kosinus polovičnog ugla kao i formule za pretvaranje razlike sinusa u proizvod. U preostala tri rješenja koriste se znanja iz geometrije i trigonometrije (Pitagorina teorema, sinusna teorema, teorema o simetrali unutrašnjeg ugla trougla kao i značajne osobine jednakostraničnog i jednakokrakog trougla). Možda ovaj članak bude inspiracija budućim čitaocima da daju još koje rješenje ovog zadatka ili pak da neki drugi zadatak riješe na više načina.

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