Bull.Soc.Math. Banja Luka
2(1995), 9-14

A NOTE ON POSITIVE HERMITIAN MATRICES

by
Milan Janjic
Faculty of Sciences, Department of mathematics and Informatics,
78000 Banja Luka, 2 Mladena Stojanovica Street,
Republic of Srpska, Bosnia and Herzegovina

AMS Subject Classification (1991) : 15A21, 15A57

Abstract: In this note we first prove that arbitrary matrix is equivalent to a diagonal matrix such that these matrices have equal all nonzero principal minors of the same order. Applying this method to Hermitian matrices we shall prove the theorem of reducing a Hermitian matrix to its canonical form. From this we easily derive Silvesters' criterion for positive definite matrices. Finally we give a criterion for positive definite matrices connected with a property of diagonal elements of these matrices.

Except in Theorem 1, which holds for matrices over an arbitrary field, for all matrices will be supposed to have complex elements. We shall denote by $E_{ij}, E_i(\alpha)$ and $E_{ij}(\alpha)$ the standard elementary matrices and by E_n the identity matrix of order n. By A^T and $\bar{A}$ will be denoted the transpose and conjugate of the matrix A. As usual $\left(\begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array}\right)$ will be a block matrix and diag(α_1, α_2 ,..., α_r) diagonal square matrix of the order r .We denote n-dimensional unitary space over the field of the complex numbers C by C^n. By $\|\cdot\|$ and $\langle \cdot , \cdot \rangle$ will be denoted the norm and scalar product in this space . We also denote by D_k the principal minor of the order k of the matrix A set in the upper left-hand corner of A .

The first theorem that we shall prove is a slight modification of well-known procedure of reducing a matrix to its Hermitian canonical form . Namely, if we apply elementary transformation in a special order we may preserve all nonzero principal minors. This theorem will easily imply well-known formula of representing a matrix as a product of a left triangular, a diagonal and a right triangular matrices .

Theorem 1. *Let* $A=\left(a_{ij}\right)_{m\times n}$ *be a matrix over arbitrary field. Then this matrix is equivalent to the matrix* $B=\left(\begin{array}{c|c} B_r & 0 \\ \hline 0 & 0 \end{array}\right)$, *where* $B_r = diag(\alpha_1, \alpha_2, \dots, \alpha_r)$, $r = rang\, A$. *Moreover, if* $D_k \neq 0$ $(k \leq r)$ *is the principal minor of the matrix A then* $D_k=\alpha_1 \cdot \alpha_2 \cdots \alpha_k$.

Proof. If all principal minors of the matrix A are equal zero then for the matrix B we may choose Hermitian canonical form of the matrix A . So we may suppose that there exists a nonzero principal minor . Suppose that $a_{11} = D_1 \neq 0$. Then the matrix $A_1 = \prod_{k=n}^{2} E_{1k}\left(-\frac{a_{k1}}{a_{11}}\right) \cdot A \cdot \prod_{k=2}^{n} E_{k1}\left(-\frac{a_{1k}}{a_{11}}\right)$ is equivalent to the matrix A and is of the form

$$(1) \qquad B=\left(\begin{array}{c|c} \alpha_1 & 0 \\ \hline 0 & B_1 \end{array}\right), \alpha_1 = a_{11}.$$

If $a_{11} = 0$ we choose i_0 to be minimal i such that $a_{i1} \neq 0$ and j_0 to be minimal j such that $a_{1j} \neq 0$. Take $k_0 = \max\{i_0, j_0\}$, then $D_1 = D_2 = \cdots = D_{k_0-1} = 0$ and the matrix $A' = E_{i_0 1}(1) \cdot A (\text{or } A \cdot E_{1 j_0})$ has a nonzero element in the upper left-hand corner. In the same time all principal minors $D_{k_0}, D_{k_0+1}, \dots, D_n$ of the matrix A are equal to the corresponding principal minors of the matrix A' . In this case we may apply the preceding procedure and get (1) again. Note that , in both cases, nonzero principal minors of the same order of matrices A and A' are equal . If rang A = 1 then $B_1 = 0$ which proves Theorem 1. If rang A > 1 we only need to repeat the preceding by the matrix B instead of A. This procedure will obviously be finished after r (= rang A) steps which completes the proof of the Theorem 1.

As an immediate consequence of this theorem we have

Corollary 1. (1, formula 93.1) *If all principal minor of a rectangular matrix A except eventually the highest-order minor are different from zero then the matrix A can be represented as a product* $A = L \cdot D \cdot U$ *where L is a left triangular matrix with unit* *diagonal elements, D is a diagonal matrix and U is right triangular matrix with unit diagonal elements .*

Proof. In this case we first have $a_{11} \neq 0$ and we get the matrix A_1 in the preceding theorem multiplying the matrix A by left triangular matrices on the left and by right triangular matrices on the right . By the assumption we have the same situation in every further step in the proof of the Theorem 1 and so the matrix L is a product of left triangular matrices with unit diagonal elements, consisting of the inverse matrices of the matrices by which the matrix A is multiplied on the left while U is a product of right triangular matrices with unit diagonal elements consisting of the inverse matrices of the matrices by which the matrix A is multiplied on the right .This completes the proof of Corollary 1

We shall now apply the method of the preceding theorem to prove well-known result of the reduction of a Hermitian matrix to its canonical form preserving , in the same time , all nonzero principal minor of the matrix A. From this we shall easily conclude that all diagonal elements of a positive definite matrix are positive . As an immediate consequence we may get formulas (93.5) and (93.6) in [1] that are analogues of the representation of the matrix A in Corollary 1 for symmetric and Hermitian matrices

Theorem 2. *Let $A=(a_{ij})_{n \times n}$ be a Hermitian matrix , then there exists a regular matrix Q such that the matrix A is equivalent to the real matrix*

$$B=Q^T \cdot A \cdot \overline{Q} = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 0 \end{array} \right), \qquad (2)$$

where $C = diag\,(\beta_1,\beta_2,\ldots,\beta_r)$, r =rang A. Moreover , if the first r principal minors of A are different from zero , then the principal minors of the same order of matrices A and B are equal. There further exists a regular matrix P such that

$$B_1 = P^T \cdot A \cdot \overline{P} = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array} \right) \qquad (3)$$

$D = diag\,(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n),\varepsilon_i = \pm 1 (i = 1,2,\ldots,r)$.

Proof. Since the matrix A is a Hermitian matrix we have $a_{ij} = \overline{a_{ji}}, E_{ij}^T(a) = \overline{E_{ji}}\,(\overline{a})(i,j = 1,\ldots,n)$ If $a_{11} \neq 0$ then the matrix A_1 in (1) is the matrix $\prod_{k=n}^{2} E_{k1}^T(\frac{a_{k1}}{a_{11}}) \cdot A \cdot \prod \overline{E_{k1}\left(-\frac{a_{k1}}{a_{11}}\right)}$.
If $a_{11} = 0$ then we have $i_0 = j_0 = k_0$ so that we may take the matrix $A'' = E_{1i_0}^T(1) \cdot A \cdot \overline{E_{1i_0}}(1)$ instead of the matrix A'. Thus the matrix Q is the product of transpose matrices of elementary matrices by which the matrix A is multiplied on the left. Taking

$$Q_1 = \prod_{i=1}^{r} E_i\left(\frac{1}{\sqrt{|\beta_i|}}\right)$$

we get (3) for $P = Q \cdot Q_1$.
It is easy to see that $\varepsilon_i = \text{sign}(\beta_i)(i = 1,\ldots,r)$ which completes the proof of Theorem 2.

It is well-known fact that the signature of a Hermitian matrix may be introduced after the proof of the law of inertia. That is also the case with positive definite matrices. We are in position here to prove, without the law of inertia, that $\beta_i > 0$ $(i = 1,2,\ldots,n)$ is necessary and sufficient condition that the matrix A be positive definite. From this we give an elementary proof of Silvester's criterion.

Theorem 3. *Let $A = (a_{ij})_{n\,x\,n}$ be a Hermitian matrix, $n = rang\ A$, then the matrix A is positive definite if and only if, $\beta_i > 0$, (β_i being as in (2), $i = 1, 2, \ldots, n$).*
Proof. This is obvious, since A is positive definite if and only if the matrix B in (2) is positive definite i.e. if and only if $\beta_i > 0$ $(i=1,2,\ldots,n)$.

Theorem 4. (Silvester's criterion) *A Hermitian matrix $A = (a_{ij})_{nxn}$ is positive definite if and only $D_k > 0\,(k = 1,\ldots,n)$ where D_k are the principal minors of the order k of A.*
Proof. Suppose that $D_k > 0$ $(k = 1,\ldots,n)$ .Theorem 2 implies that $D_k = \beta_1 \cdot \beta_2 \cdots \beta_k$ $(k = 1,2,\ldots,n)$ and thus we have $D_1 = \beta_1 > 0$ and

$$\beta_k = \frac{D_k}{D_{k-1}} > 0 \ (k = 2,3,\ldots,n)$$

and the matrix A is positive definite by Theorem 3.

Conversely, let a matrix A be a positive definite. In the view of Theorem 3 this means that $\beta_i > 0 (i = 1,2,\dots,n)$ in the relation (2). Since A is positive definite, it is obvious that, for every k, $A_k = (a_{ij})_{k \times k}$ is positive definite, hence $D_k \neq 0$. Therefore, by Theorem 2, $D_k = \beta_1 \cdot \beta_2 \cdots \beta_k \ (k = 1,2,\dots,n)$.

Since for every i =1,2,…n; A is positive definite if and only if $E_{1i} \cdot A \cdot \overline{E_{1i}^T}$ is positive definite we also have the following

Corollary 2. *Each diagonal element of a positive definite Hermitian matrix is positive.*

We shall finally give a criterion for positive definite matrices which is derived from a property of diagonal elements of these matrices. For a Hermitian matrix $A = (a_{ij})_{n \times n}$ we shall denote by $a_{(k-1)}$ (k = 2,…,n) the k-1-dimensional vector $(a_{k1}, a_{k2}, \dots, a_{k,k-1})$ and by P_{k-1} the matrix P which correspond to the matrices $A_{k-1} = (a_{ij})_{(k-1) \times (k-1)}$ in Theorem 2.

Theorem 5. *A Hermitian matrix is positive definite if and only if*

(4) $$a_{11} > 0,$$

$$(\forall k = 2,3,\dots,n; \forall X \in C^{(k-1)}, \|X\| = 1), \left|\left\langle a_{(k-1)} \cdot \overline{P_{k-1}}, X \right\rangle\right|^2 < a_{kk}$$

Proof. If the matrix A is positive definite then all matrices $A_k = (k = 1,\dots,n)$ are positive definite by Corollary 2. It also means that quadratic forms whose matrices are A_k are positive definite. Let k be an integer such that $1 < k \leq n$, P_{k-1} and $a_{(k-1)}$ as above and $X = (x_1, x_2, \dots, x_n) = (X_{k-1}, x_k) \neq 0$ an arbitrary row vector. For the matrix $P = \left(\begin{array}{c|c} P_{k-1} & 0 \\ \hline 0 & 1 \end{array}\right)$ it holds $X \cdot P^T \cdot A_k \cdot \overline{P} \cdot \overline{X^T} > 0$.

From this we conclude that

$$(X_{k-1}, x_k) \cdot \left(\begin{array}{c|c} P_{k-1}^T \cdot A_{k-1} \cdot \overline{P_{k-1}} & P_{k-1}^T \cdot \overline{a_{(k-1)}^T} \\ \hline a_{(k-1)} \cdot \overline{P_{k-1}} & a_{kk} \end{array}\right) \cdot \left(\begin{array}{c} \overline{X_{k-1}^T} \\ \hline \overline{x_k} \end{array}\right) > 0.$$

It follows from Theorem 2 that

$$P_{k-1}^{T} \cdot A_{k-1} \cdot \overline{P_{k-1}} = E_{k-1}$$

and so we obtain

$$0 < \|X_{k-1}\|^2 + x_k \cdot \left\langle a_{(k-1)} \cdot \overline{P_{k-1}}, \overline{X_{k-1}} \right\rangle + \overline{x_k} \cdot \left\langle X_{k-1}, \overline{a_{(k-1)} \cdot P_{k-1}} \right\rangle + a_{kk} |x_k|^2 =$$

$$= \|X_{k-1}\|^2 + 2 \cdot \mathrm{Re}\left(x_k \cdot \left\langle a_{(k-1)} \cdot \overline{P_{k-1}}, \overline{X_{k-1}} \right\rangle \right) + a_{kk} |x_k|^2 \leq$$

$$\leq \|X_{k-1}\|^2 + 2 \cdot \mathrm{Re} |x_k| \left| \cdot \left\langle a_{(k-1)} \cdot \overline{P_{k-1}, X_{k-1}} \right\rangle \right| + a_{kk} |x_k|^2 .$$

Since this holds for every x_k we conclude that

$$\left| \left\langle a_{(k-1)} \cdot \overline{P_{k-1}}, \overline{X_{k-1}} \right\rangle \right|^2 - a_{kk} \cdot \|X_{k-1}\|^2 < 0$$

which gives (4).

Assume now that a Hermitian matrix A satisfies the condition (4). We shall prove by induction that each matrix A_1, A_2, ..., A_n is positive definite. This is true for k=1 since $A_1 = a_{11} > 0$ by (4). Suppose that the statement is true for k-1. The matrix

$$\left(\begin{array}{c|c} P_{k-1}^{T} \cdot A_{k-1} \cdot \overline{P_{k-1}} & P_{k-1}^{T} \cdot \overline{a_{(k-1)}^{T}} \\ \hline a_{(k-1)} \cdot \overline{P_{k-1}} & a_{kk} \end{array} \right)$$

has the form

$$\left(\begin{array}{c|c} E_{k-1} & P_{k-1}^{T} \cdot \overline{a_{(k-1)}^{T}} \\ \hline a_{(k-1)} \cdot \overline{P_{k-1}} & a_{kk} \end{array} \right)$$

Applying Theorem 2 to this matrix we get the matrix $T = \mathrm{diag}(1,1,\ldots,a_{kk} - \|c\|^2)$,
where $c = a_{(k-1)} \cdot \overline{P_{k-1}}$. Taking $X_{k-1} = c$ we get $a_{kk} - \|c^2\| > 0$ which, by Theorem 3, yields that the matrix A_k is positive definite and the theorem is proved.

Reference

[1] V. V. Voyevodin, *Linear Algebra*, **Mir Publishers,** Moscow, 1983