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ON COIDEALS OF PRODUCT OF COMMUTATIVE RINGS WITH APARTNESS

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Abstract. The structure of coideals of the product $\Pi_{t\in T}R_t$ of commutative rings with apartnesses is quite complicated. It depends on the structure of coideals of components R_t as well as on the indexing set T. In this paper we shall give two constructions of coideals of the product ΠR_t which depends on indexing set T.

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In the classical algebra, the structure of ideals of the product ΠR_t of rings R_t is quite complicated. It depends on the structure of ideals of componenets R_t as well as on the indexing set T. (See [2], [3], [5]) Given $r\in \Pi R_t$ put $Y(r) = \{t\in T : r(t) = 0\}$. It is not difficult to show that if $\boldsymbol{F}$ is a filter on T then $Y(\boldsymbol{F}) = \{r\in \Pi R: Y(r) \in \boldsymbol{F}\}$ is an ideal of ΠR_t ([2], [3]). Kochen proved ([2],[3]) that if R_t are division rings then the mapping $\boldsymbol{F} \rightarrow Y(\boldsymbol{F})$ is an isomorphism of the lattice of filters on T onto the lattice of ideals of ΠR_t .

In the constructive mathematics ([1], [4], [6], [8], [9]) the matter is more complicated than classical. Given $r \in \Pi R_t$ put $Z(r) = \{t\in T : r(t) \neq 0\}$. If Q is a coideal of the ring ΠR_t ,then $Z(Q) = \{Z(r) : r \in Q\}$ need not be an a-coideal of the set T. However, if Q is an inhabited subset of ΠR_t such that 0 # Q, we can construct an a-coideal $\boldsymbol{K}(Q) = \{A \subseteq T : (\exists r\in Q)(Z(r) \cap$

$A \neq \emptyset)\}$ of the set T and $Z(Q) \subseteq \boldsymbol{K}(Q)$ holds. It is not difficult to show that if $\boldsymbol{H}$ is a nonempty family of inhabited subsets of T, then $S(\boldsymbol{H}) = \{r \in \Pi R_t : (\exists A \in \boldsymbol{H})(Z(r) \cap A \neq \emptyset)\}$ is a coideal of ΠR_t and if $\boldsymbol{K}$ is an a-coideal on T then $C(\boldsymbol{K}) = \{r \in \Pi R_t : Z(r) \in \boldsymbol{K}\}$ is a coideal of ΠR_t and holds $C(\boldsymbol{K}) \subseteq S(\boldsymbol{K}) \subseteq (\Pi R_t)_0$. If R_t is an integrally domain, then the coideal $S(\boldsymbol{H})$ is the union of prime coideals of the ring ΠR_t.

For all notions and notations in constructive mathematics which we used here the reader is refered to the books [1, 4, 9] and to the papers [6, 7, 8].

Let $(R,=,\neq,+,0,\cdot,1)$ be a commutative ring with identity and with apartness. A subset S of R is called a ***coideal*** of R if and onli if:

(1) $S \# 0$;

(2) $-r \in S \Rightarrow r \in S$;

(3) $r+s \in S \Rightarrow r \in S \vee s \in S$;

(4) $rs \in S \Rightarrow r \in S \wedge s \in S$.

If S is an inhabitet coideal of r, then $1 \in S$. The coideal S od r is a *prime* coideal of R if and only if

(5) $r \in S \wedge s \in S \Rightarrow rs \in S$.

Recall that an ***a-coideal K*** on set $(T,=,\neq)$ such that

(a) $T \in \boldsymbol{K}, \emptyset \# \boldsymbol{K}$;

(b) $A \in \boldsymbol{K} \wedge A \subseteq B \Rightarrow B \in \boldsymbol{K}$;

(c) $A \cup B \in \boldsymbol{K} \Rightarrow A \in \boldsymbol{K} \vee B \in \boldsymbol{K}$.

The following two theorems have simply and technically proofs, but they have very interesting corollaries.

THEOREM 1. *Let H be a nonempty family of inhabited subsets of T. Then the set* $S(\boldsymbol{H}) = \{r \in \Pi_{t \in T} R_t : (\exists A \in \boldsymbol{H})(A \cap Z(r) \neq \emptyset)\}$ *is a coideal of the ring* ΠR_t.

Proof.

(1) $r \in S(\boldsymbol{H}) \Leftrightarrow (\exists A \in \boldsymbol{H})(A \cap Z(r) \neq \emptyset)$
$\Rightarrow Z(r) \neq \emptyset$
$\Rightarrow r \neq 0$.

(2) $-r \in S(\boldsymbol{H}) \Leftrightarrow (\exists A \in \boldsymbol{H})(A \cap Z(-r) \neq \emptyset)$

$\Leftrightarrow (\exists A\in \boldsymbol{H})(A\cap Z(r)\neq\emptyset)$
$\Leftrightarrow r\in S(\boldsymbol{H})$.

(3) $r+s\in S(\boldsymbol{H}) \Leftrightarrow (\exists A\in \boldsymbol{H})(A\cap Z(r+s)\neq\emptyset)$
$\Rightarrow (\exists A\in \boldsymbol{H})(A\cap(Z(r)\cup Z(s))\neq\emptyset)$
$\Rightarrow (\exists A\in \boldsymbol{H})(A\cap Z(r)\neq\emptyset \vee A\cap Z(s)\neq\emptyset)$
$\Rightarrow r\in S(\boldsymbol{H}) \vee s\in S(\boldsymbol{H})$.

(4) $rs\in S(\boldsymbol{H}) \Leftrightarrow (\exists A\in \boldsymbol{H})(Z(rs)\cap A\neq\emptyset)$
$\Rightarrow (\exists A\in \boldsymbol{H})(Z(r)\cap Z(s)\cap A\neq\emptyset)$
$\Rightarrow r\in S(\boldsymbol{H}) \wedge s\in S(\boldsymbol{H})$.

COROLLARY 1.1. *For every inhabited subset* A *of* T *we can construct the coideal* $S(\{A\})$.

COROLLARY 1.2. *For every element* t *in* T *the set* $S(\{\{t\}\}) = \{r\in\Pi R_u : t\in Z(r)\}$ *is a coideal of the ring* ΠR_u . *If* R_u *is an integral domain, then the coideal* $S(\{\{t\}\})$ *is a prime coideal of* ΠR_u .

Proof. Put $S(t) = S(\{\{t\}\})$. We have
$r\in S(t) \wedge s\in S(t) \Leftrightarrow t\in Z(r) \wedge t\in Z(s)$
$\Leftrightarrow t\in Z(r)\cap Z(s) = Z(rs)$
$\Leftrightarrow rs\in S(t)$.

COROLLARY 1.3. *Let* $\boldsymbol{H}$ *be a nonempty family of inhabited subsets of* T. *Then* $S(\boldsymbol{H}) = \bigcup_{A\in H}\bigcup_{t\in A} S(t)$. *If* R_t *is an integral domain, then the coideal* $S(\boldsymbol{H})$ *is the union of prime coideals of* ΠR_t .

COROLLARY 1.4. *Let* $\boldsymbol{H}_1$ *and* $\boldsymbol{H}_2$ *be nonempty families of inhabited subsets of the set* T. *Then:*

(1) $\boldsymbol{H}_1 \subseteq \boldsymbol{H}_2 \Rightarrow S(\boldsymbol{H}_1)\subseteq S(\boldsymbol{H}_2)$;

(2) $S(\boldsymbol{H}_1\cup \boldsymbol{H}_2) = S(\boldsymbol{H}_1)\cup S(\boldsymbol{H}_2)$;

(3) $S(\boldsymbol{H}_1\cap \boldsymbol{H}_2) \subseteq S(\boldsymbol{H}_1)\cap S(\boldsymbol{H}_2)$.

COROLLARY 1.5. *The mapping* $S:\boldsymbol{H} \rightarrow S(\boldsymbol{H})$ *is a strongly extensional function of the power-set* $\boldsymbol{P}(\boldsymbol{P}(T))$ *in the family of coideals of the ring* ΠR_t .

THEOREM 2. *Let* $\boldsymbol{K}$ *be an a-coideal of the set* T. *The the set* $C(\boldsymbol{K}) = \{r\in\Pi R_t : Z(r)\in \boldsymbol{K}\}$ *is a coideal of the ring* ΠR_t .

Proof.

(1) $r\in C(\mathbf{K}) \Leftrightarrow Z(r)\in \mathbf{K}$
$\Rightarrow Z(r) \neq \varnothing$
$\Leftrightarrow r \neq 0$.

(2) $-r\in C(\mathbf{K}) \Leftrightarrow Z(-r)\in \mathbf{K}$
$\Leftrightarrow Z(r)\in \mathbf{K}$
$\Leftrightarrow r\in C(\mathbf{K})$.

(3) $r+s\in C(\mathbf{K}) \Leftrightarrow Z(r+s)\in \mathbf{K}$
$\Rightarrow Z(r+s)\in \mathbf{K} \wedge Z(r+s) \subseteq Z(r)\cup Z(s)$
$\Rightarrow Z(r)\cup Z(s)\in \mathbf{K}$
$\Rightarrow Z(r)\in \mathbf{K} \vee Z(s)\in \mathbf{K}$
$\Leftrightarrow r\in C(\mathbf{K}) \vee s\in C(\mathbf{K})$.

(4) $rs\in C(\mathbf{K}) \Leftrightarrow Z(rs)\in \mathbf{K}$
$\Rightarrow Z(rs)\in \mathbf{K} \wedge Z(rs) \subseteq Z(r)\cap Z(s)$
$\Rightarrow Z(r)\cap Z(s)\in \mathbf{K}$
$\Rightarrow Z(r)\in \mathbf{K} \wedge Z(s)\in \mathbf{K}$
$\Leftrightarrow r\in C(\mathbf{K}) \wedge s\in C(\mathbf{K})$.

COROLLARY 2.1. *Let $\mathbf{K}$ be an a-coideal of the set* T *such that* $A\in \mathbf{K} \wedge B\in \mathbf{K} \Rightarrow A\cap B \in \mathbf{K}$ *and let* R_t $(t\in T)$ *be integral domains. Then the coideal* $C(\mathbf{K})$ *is a prime coideal of the ring* ΠR_t .

Proof.

$r\in C(\mathbf{K}) \wedge s\in C(\mathbf{K}) \Leftrightarrow Z(r)\in \mathbf{K} \wedge Z(s)\in \mathbf{K}$
$\Rightarrow Z(rs) = Z(r)\cap Z(s)\in \mathbf{K}$
$\Leftrightarrow rs\in C(\mathbf{K})$.

COROLLARY 2.2. *Let $\mathbf{K}$ be an a-coideal of the set* T. *Then*

$$C(\mathbf{K}) \subseteq S(\mathbf{K}) = (\Pi_{t\in T}R_t)_0 .$$

COROLLARY 2.3. *Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be a-coideals of the set* T. *Then:*

(1) $\mathbf{K}_1 \subseteq \mathbf{K}_2 \Rightarrow C(\mathbf{K}_1) \subseteq C(\mathbf{K}_2)$;

(2) $C(\mathbf{K}_1 \cup \mathbf{K}_2) = C(\mathbf{K}_1)\cup C(\mathbf{K}_2)$;

(3) $C(\mathbf{K}_1) \neq C(\mathbf{K}_2) \Rightarrow \mathbf{K}_1 \neq \mathbf{K}_2$.

COROLLARY 2.4. *The mapping* $C:\mathbf{K} \rightarrow C(\mathbf{K})$ *is strongly extendional function of the family of a-coideals on the set* T *in the family of coideals of the ring* ΠR_t .

In [5] Olszewski proved that if R_t ($t\in T$) are rings with identity then Y(J) is a filter on T for every ideal J of ΠR_t (in the classical algebra). In the next two theorems we analyse the symmetrical problem: Is Z(Q) a-coideal of the set T if Q is a coideal of the ring ΠR_t ? In the theorem 3 we give a construction of an a-coideal $\boldsymbol{K}$(Q) of T where Q is an inhabited subset of ΠR_t .

THEOREM 3. *Let Q be an inhabited subset of the ring* ΠR_t *such that* Q # 0. *Then the set* $\boldsymbol{K}(Q) = \{A \subseteq T : (\exists r\in Q)(Z(r)\cap A \neq \varnothing)\}$ *is an a-coideal of the set* T.

Proof.

(1) $A\in \boldsymbol{K}(Q) \Leftrightarrow (\exists r\in Q)(A\cap Z(r) \neq \varnothing)$
$\Leftrightarrow (\exists r\in Q)(\exists t\in Z(r))(t\in A)$
$\Rightarrow A \neq \varnothing$.

(2) $A \subseteq B \wedge A\in \boldsymbol{K}(Q) \Leftrightarrow A \subseteq B \wedge (\exists r\in Q)(A\cap Z(r) \neq \varnothing)$
$\Rightarrow (\exists r\in Q)(B\cap Z(r) \neq \varnothing)$
$\Leftrightarrow B\in \boldsymbol{K}(Q)$.

(3) $A\cup B\in \boldsymbol{K}(Q) \Leftrightarrow (\exists r\in Q)((A\cup B)\cap Z(r) \neq \varnothing)$
$\Rightarrow (\exists r\in Q)(A\cap Z(r) \neq \varnothing \vee B\cap Z(r) \neq \varnothing)$
$\Rightarrow A\in \boldsymbol{K}(Q) \vee B\in \boldsymbol{K}(Q)$.

(4) Q # 0 $\Rightarrow (\exists r\in Q)(r \neq 0)$
$\Leftrightarrow (\exists r\in Q)(Z(r) \neq \varnothing)$
$\Rightarrow (\exists r\in Q)(Z(r)\cap T \neq \varnothing)$
$\Leftrightarrow T\in \boldsymbol{K}(Q)$.

COROLLARY 3.1. *Let* Q *be an inhabited subset of the ring* ΠR_t *such that* 0 # Q. *Then* $Q \subseteq C(\boldsymbol{K}(Q)) \subseteq S(\boldsymbol{K}(Q)) = (\Pi R_t)_0$.

Proof.

$r\in Q \Rightarrow r \neq 0$
$\Leftrightarrow Z(r) \neq \varnothing$
$\Leftrightarrow Z(r)\cap Z(r) \neq \varnothing$
$\Leftrightarrow Z(r)\in \boldsymbol{K}(Q)$
$\Leftrightarrow r\in C(\boldsymbol{K}(Q))$.

COROLLARY 3.2. *Let* ΠR_t *be product of rings such that*

$$(\forall A \subseteq T)(\exists a\in \Pi R_t)(Z(a) = A).$$

Then, for every a-coideal $\boldsymbol{K}$ *on* T , $\boldsymbol{K} \subseteq \boldsymbol{K}(C(\boldsymbol{K}))$.

Proof.

$$\begin{aligned}\varnothing \neq A\in \boldsymbol{K} &\Rightarrow \{r\in \Pi R_t : Z(r) = A\} \subseteq C(\boldsymbol{K})\\ &\Rightarrow (r\in \Pi R_t)(Z(r)\cap Z(r) \neq \varnothing)\\ &\Rightarrow A\in \boldsymbol{K}(C(\boldsymbol{K}(Q)).\end{aligned}$$

In the next theorem we describe a case when Z(S) is an a-coideal of the set T where S is a coideal of the ring ΠR_t .

THEOREM 4. *Let* ΠR_t *be the product of discrete Heyting fields and let S be a coideal of* ΠR_t *. Then the set* Z(S) *is an a-coideal of* T.

Proof.

$$\begin{aligned}A\in Z(S) &\Leftrightarrow (\exists r\in S)(A = Z(r))\\ &\Rightarrow Z(r) \neq \varnothing .\end{aligned}$$

$$\begin{aligned}A \subseteq B \wedge A\in Z(S) &\Leftrightarrow (\exists a\in S)(A = Z(a)) \wedge (\exists b\in \Pi R_t)(B = Z(b) \supseteq Z(a))\\ &\Rightarrow (\exists x\in \Pi R_t)(xb = a \in S)\\ &\Rightarrow b \in S\\ &\Rightarrow Z(b) = B \in Z(S).\end{aligned}$$

$$\begin{aligned}A\cup B \in Z(S) &\Leftrightarrow (\exists c\in S)(A\cup B = Z(c))\\ &\Rightarrow (\exists a,b\in \Pi R_t)(A = Z(a) \wedge B = Z(b) \wedge a+b = c \in S)\\ &\Rightarrow a\in S \vee b\in S\\ &\Leftrightarrow A = Z(a) \in Z(S) \vee B = Z(b) \in Z(S) .\end{aligned}$$

COROLLARY 4.1. *If* S *is a prime coideal of the ring* ΠR_t *of discrete Heyting fields, then the a-coideal* Z(S) *has the property*

$$A\in Z(S) \wedge B\in Z(S) \Rightarrow A\cap B\in Z(S).$$

Proof.

$$\begin{aligned}A\in Z(S) \wedge B\in Z(S) &\Leftrightarrow (\exists a,b\in S)(A = Z(a) \wedge B = Z(b))\\ &\Rightarrow ab\in S\\ &\Rightarrow Z(ab) \in Z(S) \wedge Z(ab) = Z(a)\cap Z(b)\\ &\Rightarrow Z(ab)\in Z(S)\\ &\Rightarrow A\cap B\in Z(S).\end{aligned}$$

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