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SPECTRAL INVERSES AND JORDAN CANONICAL FORM

by

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ABSTRACT. In this paper we continue the work of Giurescu and Gabriel [4], and solve the corresponding set of matrix equations, which determine the weak spectral inverses, using the Jordan canonical representation. In this way, we obtain more detailed and effective representation of this weak spectral inverses of a square matrix.

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1. Introduction and preliminaries

The set of square complex matrices of the order n is denoted by $\mathbf{C}^{n\times n}$, the zero block of an appropriate size is denoted by O, and ind(A) denotes the index of matrix A.

We begin by listing several known facts and definitions.

A. If $A\in\mathbf{C}^{n\times n}$, then the left weak spectral inverses [12] are the solutions of the following system in X :

$$(1)\ AXA = A \qquad (2)\ XAX = X \qquad (1^k)\ XA^{k+1} = A^k$$

and the right weak spectral inverses [12] are solutions of the following system in X :

$$(1)\ AXA = A \qquad (2)\ XAX = X \qquad (2^k)\ A^{k+1}X = A^k.$$

Any left weak spectral inverse is denoted by A^W, and the right weak spectral inverse is denoted by A_W.

B. Let $A\in C^{n\times n}$ and $A = TJT^{-1}$ be its Jordan canonical representation. then J can be represented in the form

$$J=\begin{pmatrix} J_1 \dots O & O \dots O \\ \dots\dots\dots\dots & \\ O\dots J_q & O\dots O \\ O\dots O & J_{q+1}\dots O \\ \dots\dots\dots\dots & \\ O\dots O & O\dots J_p \end{pmatrix} = J_1\oplus\dots\oplus J_q\oplus J_{q+1}\oplus\dots\oplus J_p,$$

where $J_1, \dots, J_q$ are Jordan and invertible, and $J_{q+1}, \dots, J_p$ are Jordan and noninvertible [6],[7],[11].

C. The following results are obtained in [4]: if $A = TJT^{-1}$ is the Jordan canonical representation of $A\in C^{n\times n}$ and a matrix $X\in C^{n\times n}$ satisfies the Penrose's equations (1),(2) , then the matrix $Z = T^{-1}XT$ satisfies the following matrix equations :

$$\text{(J1) } ZJZ = Z \qquad \text{(J2) } JZJ = J.$$

Also, the matrix $Z\in J\{1\}$ can be represented in the form

$$\text{(1.1) } Z=\begin{pmatrix} J_1^{-1} & O\dots & O & Z_{1,q+1}\dots & Z_{1p} \\ O & J_2^{-1} & O & Z_{2,q+1}\dots & Z_{2p} \\ \dots & \dots & \dots & \dots & \dots \\ O & O & J_q^{-1} & Z_{q,q+1} & Z_{q,p} \\ Z_{q+1,1} & Z_{q+1,2} & Z_{q+1,q} & Z_{q=1,q+1} & Z_{q+1,p} \\ \dots & \dots & \dots & \dots & \dots \\ Z_{p,1} & Z_{p,2} & Z_{p,q} & Z_{p,q+1} & Z_{p,p} \end{pmatrix}$$

where the blocks $Z_{\alpha,\beta}$ satisfies the following conditions:

$$(1.2)\begin{cases}(Z1) & J_iZ_{ii}J_i = J_i \quad (i=q+1,\dots,p)\\ (Z2) & Z_{ij}J_j = O \quad (i=1,\dots,q;\ j=q+1,\dots,p)\\ (Z3) & J_iZ_{ij} = O \quad (i=q+1,\dots,p;\ j=1,\dots,q)\\ (Z4) & J_iZ_{ij}J_j = O \quad (i=q+1,\dots,p;\ j=q+1,\dots,p;\ i\neq j)\end{cases}$$

$J_\alpha \in \mathbf{C}^{m_\alpha\times m_\alpha}$, $(\alpha=1,\dots,p)$, $Z_{\alpha,\beta}\in \mathbf{C}^{m_\alpha\times m_\beta}$, $(\alpha=1,\dots,p;\ \beta=1,\dots,p)$.

Besides, in the same paper [4], by solving the equations (Z1)-(Z4), the following forms of the blocks $Z_{\alpha,\beta}$ are obtained (1.3):

$$(C1)\qquad Z_{ij}=\begin{pmatrix}0 & \dots & 0 & z^{ij}_{1,m_j}\\ 0 & \dots & 0 & z^{ij}_{2,m_j}\\ \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & z^{ij}_{m_i,m_j}\end{pmatrix}\ (i=1,\dots,q;\ j=q+1,\dots,p);$$

(1.3)

$$(C2)\qquad Z_{ij}=\begin{pmatrix}z^{ij}_{1,1} & z^{ij}_{1,2} & \dots & z^{ij}_{1,m_j}\\ 0 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0\end{pmatrix}\ (i=q+1,\dots,p;\ j=1,\dots,q);$$

$$(C3)\qquad Z_{q+i,q+j}=\begin{cases}\begin{pmatrix}0\,0\dots 0\,u_{ij}\\ 0\,0\dots 0\,0\\ \dots\dots\\ 0\,0\dots 0\,0\end{pmatrix}, i\neq j\\[2ex] \begin{pmatrix}0\,0\dots 0\,u_{ii}\\ 1\,0\dots 0\,0\\ 0\,1\dots 0\,0\\ \dots\dots\\ 0\,0\dots 1\,0\end{pmatrix}, i=j\end{cases}\quad (i=1,\dots,p-q;\ j=1,\dots,p-q).$$

where $z^{\gamma\delta}_{\alpha\beta}$ represents the (α,β)element in the cell $Z_{\gamma,\delta}, \lambda_c$, is the corresponding eigenvalue of *A*, and

$$u_{ij} = \sum_{c=1}^{q}\left(z^{q+i,c}_{1,m_j-1} + \lambda_c z^{q+i,c}_{1,m_j}\right) z^{c,q+j}_{1,m_j}, (i = 1,\dots,p-q;\, j = 1,\dots,p-q).$$

In this paper we use described results from [4] and the principles analogous to the corresponding in [10]. We find a partition $Z_{\alpha,\beta}$ of the weak spectral inverse Z of J, convenient to the partition in J, obtained from the Jordan canonical representation $A = TJT^{-1}$. After that, relations between the bloks J_γ and the bloks $Z_{\alpha,\beta}$ are investigated. Explicit representations of the blocks $Z_{\alpha,\beta}$ are developed, using these relations. Finally, from the transformation of similarity $X = TZT^{-1}$, we obtain explicit representation for the weak spectral inverse X of A.

2. Representation of the weak spectral inverses

In the following lemma we find equivalent and useful transformations of the equations (1^k) and (2^k).

Lemma 2.1. *If $A = TJT^{-1}$ is the Jordan canonical representation of $A \in C^{n\times n}$, then:*

(i) The equation $XA^{k+1} = A^k$ is equivalent to the equation

(J3) $$ZJ^{k+1} = J^k.$$

(ii) The equation $A X = A$ is equivalent to

(J4) $$J^{k+1} Z = J^k.$$

Using this lemma and the known results (1.2) and (1.3), we obtain relation s between the corresponding blocks in the matrices J and Z.

Lemma 2.2. *Let $A = TJT^{-1}$ be the Jordan canonical representation of $A \in C^{n\times n}$, $A^w \in C^{n\times n}$ be a left weak spectral inverse of A. Then $Z = T^{-1}A^{w}T$ if and only if the blocks $Z_{\alpha,\beta}$ of Z and Jordan cells J_γ satisfy the Following conditions :*

(D1) $Z_{\alpha,\beta}J^{k+1}_{\beta} = O(\alpha = q+1,\dots,p;\beta = 1,\dots,q);$

(D2) $Z_{\alpha,\beta}$satisfyonly(C1)and(C3)$(\alpha = 1,\dots,q;\beta = q+1,\dots,p)$.

Proof. ($\Rightarrow$) Suppose that matrix $X = A^{w} \in C^{n\times n}$ satisfies the equations (1^k) and (2^k). Using the known fact [12, Theorem 1.2]: the systems (1),(2),(1^k) and (1),(2),(2^k) posses at least one solution each if and only if $k \geq ind(A)$, we immediately conclude $J^i_\alpha = O$, $i \geq k$, $\alpha = q+1,\dots,p$. Now, starting from the equation $ZJ^{k+1} = J^k$ of lemma 2.1 , and using the form (1.2) for the matrix Z, we obtain

$$\begin{pmatrix} J_1^k \dots & OO\dots O \\ \dots\dots & \dots\dots \\ O\dots & J_q^k O\dots O \\ Z_{q+1,1}\ J_1^{k+1} \dots Z_{q+1,q} & J_q^{k+1} O\dots O \\ \dots\dots & \dots\dots \\ Z_{p,1}\ J_1^{k+1} \dots Z_{p,q} & J_q^{k+1} O\dots O \end{pmatrix} = \begin{pmatrix} J_1^k & \dots OO\dots O \\ \dots\dots & \\ O\dots J_q^k O\dots O & \\ O\dots OO\dots O & \\ \dots\dots & \\ O\dots OO\ O & \end{pmatrix}.$$

Comparing the corresonding blocks in the last equation we get the equality (D1). The results (D2) are implied from [4].

($\Leftarrow$) From the other hand, if the matrix $X = TZT^{-1}$ satisfies conditions of lemma 2.2, for the matrix $X = TZT^{-1}$ the equations (1), (2) and (1^k) can be easily verified.

Simaiarly, the following lemma can be proved:

Lemma 2.3 *Let $A = TJT^{-1}$ be the Jordan canonical representation of $A \in C^{n\times n}$ and $A_w \in C^{n\times n}$ be a right weak spectral inverse of A. Then $U = T^{-1}A_wT$ if and only if the clocks $U_{\alpha\beta}$ of U and Jordan cells satisfy*

(D3) $J_\alpha^{k+1} U_{\alpha,\beta} = O \quad (\alpha = 1,\dots,q;\beta = q+1,\dots,p)$;

(D4) $U_{\alpha,\beta}$, $(\alpha=q+1,\dots,p;\ \beta=1,\dots,q)$ *satisfy only (C2) and (C3).*

Now we are ready for main theorem, where we obtain an explicit block representation for the class of weak spectral inverses.

Theorem 2.1. *If $A = TJT^{-1}$ be the Jordan canonical representation of $A \in C^{n\times n}$, then $Z = T^{-1}A^wT$ and $U = T^{-1}A_wT$ if and only if*

$$(2.1)\ Z=\begin{pmatrix} J_1^{-1} & O \ldots O & Z_{1,q+1} & \cdots & Z_{1,p} \\ \cdots & \cdots & \cdots & & \\ O & O \ldots J_q^{-1} & Z_{q,q+1} & \cdots & Z_{q,p} \\ O & O \ldots O & Z_{q+1,q+1} & \cdots & Z_{q+1,p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O \ldots O & Z_{p,q+1} & \cdots & Z_{p,p} \end{pmatrix},$$

$$(2.2)\ U=\begin{pmatrix} J_1^{-1} & \ldots O & O & \ldots O \\ \cdots & \cdots & \cdots & \cdots \\ O & \ldots J_q^{-1} & O & \ldots O \\ U_{q+1,1} \ldots & U_{q+1,q} & U_{q+1,q+1} & \cdots U_{q+1,p} \\ \cdots & \cdots & \cdots & \cdots \\ U_{p,1} \ldots & U_{p,q} & U_{p,q+1} & \cdots U_{p,p} \end{pmatrix}$$

where;

(i) the blocks $Z_{\alpha\beta}$ ($\alpha = 1,\ldots,q$; $\beta = q+1,\ldots,p$) *possess the form (C1);*

(ii) the blocks $U_{\alpha\beta}$ ($\alpha = q+1,\ldots,p$; $\beta = 1,\ldots,q$) *are of the form (C2);*

(iii) the blocks $U_{\alpha\beta}$ *and* $Z_{\alpha\beta}$ ($\alpha = q+1,\ldots,p$; $\beta = q+1,\ldots,p$) *both have the form (C3).*

Proof. ($\Rightarrow$) In Lemma 2.2 it is proved that the block $U_{\alpha,\beta}$ ($\alpha=q+1,\ldots,p$; $\beta=1,\ldots,p$) satisfy only relations (C2), (C3), valid for an arbitrary {1,2} inverse of A. For ($\alpha=1,\ldots,q$; $\beta=q+1,\ldots,p$), from (D3) and (C1) we get

$$\begin{pmatrix} \lambda_\alpha^{k+1} & \binom{k+1}{1}\lambda_\alpha^{k} & \cdots & \binom{k+1}{m_\alpha-1}\lambda_\alpha^{k-m_\alpha+2} \\ 0 & \lambda_\alpha^{k} & \cdots & \binom{k+1}{m_\alpha-2}\lambda_\alpha^{k-m_\alpha+3} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_\alpha^{k+1} \end{pmatrix} \begin{pmatrix} 0 & \ldots & 0 & u_{1,m_\beta}^{\alpha,\beta} \\ 0 & \ldots & 0 & u_{2,m_\beta}^{\alpha,\beta} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & 0 & u_{m_\alpha,m_\beta}^{\alpha,\beta} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \ldots 0 & \sum_{i=0}^{m_\alpha-1}\binom{k+1}{i}\lambda_\alpha^{k+1-i}u_{i+1,m_\beta}^{\alpha,\beta} \\ 0 \ldots 0 & \sum_{i=0}^{m_\alpha-2}\binom{k+1}{i}\lambda_\alpha^{k+1-i}u_{i+2,m_\beta}^{\alpha,\beta} \\ \cdots & \cdots \\ 0 \ldots 0 & \lambda_\alpha^{k+1}u_{m_\alpha,m_\beta}^{\alpha,\beta} \end{pmatrix} = O.$$

Therefore, we obtain the following system of linear equations

$$\sum_{i=0}^{m_\alpha-c}\binom{k+1}{i}\lambda_\alpha^{k+1-i}u_{i+c,m_\beta}^{\alpha,\beta} = 0, \quad c = 1,\ldots,m_\alpha$$

According to $\lambda_\alpha \neq 0$, $\alpha = 1,\ldots,q$, unique solution of this system is

(2.3) $$u_{1,m_\beta}^{\alpha,\beta} = \ldots = u_{m_\alpha,m_\beta}^{\alpha,\beta} = 0.$$

Now (2.3) and (C1) imply $U_{\alpha,\beta} = O$ $(\alpha = 1,\ldots,q;\ \beta = q+1,\ldots,p)$.

Similarly, for $Z = T^{-1}AT$, an application of (C2) leads to

$$\begin{pmatrix} z_{11}^{\alpha\beta} & z_{12}^{\alpha\beta} & \ldots & z_{1,m_\beta}^{\alpha\beta} \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} \lambda_\beta^{k+1} & \binom{k+1}{1}\lambda_\beta^{k} & \ldots & \binom{k+1}{m_\beta-1}\lambda_\beta^{k-m_\beta+2} \\ 0 & \lambda_\beta^{k+1} & \ldots & \binom{k+1}{m_\beta-2}\lambda_\beta^{k-m_\beta+3} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_\beta^{k+1} \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_\beta^{k+1} z_{11}^{\alpha\beta} & 0 \ldots 0 \\ \sum_{i=0}^{1}\binom{k+1}{i}\lambda_\beta^{k+1-i} z_{1,i+1}^{\alpha\beta} & 0 \ldots 0 \\ \ldots & \ldots \\ \sum_{i=0}^{m_\beta-1}\binom{k+1}{i}\lambda_\beta^{k+1-i} z_{1,i+1}^{\alpha\beta} & 0 \ldots 0 \end{pmatrix} = O$$

Therefore, we obtain the following system of linear equations

$$\sum_{i=0}^{m_\beta-c}\binom{k+1}{i}\lambda_\beta^{k+1-i} z_{1,i+1}^{\alpha,\beta} = 0, \quad c = 1,\ldots,m_\beta .$$

According to $\lambda_\beta \neq 0$, $\beta = 1,\ldots,q$, we get

(2.4) $$z_{1,m_\beta}^{\alpha,\beta} = \ldots = u_{m_\alpha,m_\beta}^{\alpha,\beta} = 0 .$$

Appling (2.4) and (C2) we obtain

$$Z_{\alpha,\beta} = O \ (\alpha = q+1,\ldots,p;\ \beta = 1,\ldots.q).$$

($\Leftarrow$) From the other hand, if U and Z satisfy (2.1) and (2.2), it is easily to verify that TZT^{-1} and TUT^{-1} are left and the right weak spectral inverses of A, respectively.

The results of Theorem 2.1 are generalizations of the following results, obtained in [9] and [2] : if $A = T\begin{pmatrix} R & O \\ O & S \end{pmatrix}T^{-1}$, then :

$GA^{k+1} = A^k$ is equivalent to $G = T\begin{pmatrix} R^{-1} & O \\ O & S \end{pmatrix}T^{-1}$, and

$GA^{k+1} = A^k$ is equivalent to $G = T\begin{pmatrix} R^{-1} & O \\ L & S \end{pmatrix} T^{-1}$,
where L, M and S are arbitrary matrices of adequate sizes.

As an consecuence of Theorem 2.1 , and the following fact [12]: any spectral inverse of A is both the right and the left weak spectral inverse of A, we obtain

Corollary 2.1. *Let $A \in C^{n \times n}$, $A = TJT^{-1}$ be the Jordan canonical representation of A and X be spectral inverse. Then the matrix $Z = T^{-1}XT$ posseses the form*

$$Z = \begin{pmatrix} J_1^{-1} & \dots & O & O & \dots & O \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O & \dots & J_q^{-1} & O & \dots & O \\ O & \dots & O & Z_{q+1,q+1} & \dots & Z_{q+1,p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O & \dots & O & Z_{p,q+1} & \dots & Z_{p,p} \end{pmatrix},$$

where the blocks $Z_{\alpha,\beta}$ (α = q+1,...,p; β = q+1,...,p) *satisfy the conditions* (C3).

Consequently, the list of generalized inverses represented by means of the Jordan canonical form in [2, Theorem 9.7.1], can be extended by the representation of *the weak spectral inverses*, which follows from Theorem 2.1 and Corollary 2.1:

Corollary 2.2. *Suppose that $A \in C^{n \times n}$, $k = ind(A)$. Suppose that $T \in C^{n \times n}$ is a non-singular matrix such that $T^{-1}AT = \begin{pmatrix} R & O \\ O & N \end{pmatrix}$, R is non-singular, $N^k = O$. Then:*

- *X is the left weak spectral inverse of A if and only if* $X = T\begin{pmatrix} R^{-1} & M \\ O & Y \end{pmatrix}T^{-1}$, *where the blocks in M have the form (C1), $YNY = Y$, $NYN = N$ and blocks in Y have the form (C3).*
- *X is the right weak spectral inverse of A if and only if* $X = T\begin{pmatrix} R^{-1} & O \\ L & Y \end{pmatrix}T^{-1}$, *blocks in L have the form (C2), $YNY = Y$, $NYN = N$ and blocks in Y have the form (C3).*
- *X is the spectral inverse of A if and only if* $X = T\begin{pmatrix} R^{-1} & O \\ O & Y \end{pmatrix}T^{-1}$, *$YNY = Y$, $NYN = N$ and blocks in Y have the form (C3).*

3. Example

Example 3.1. The Jordan representation of $A = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix}$ is

$A = TJT^{-1}$, where

$$J = \begin{pmatrix} [2] & 0 & 0 & 0 \\ 0 & [2] & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

According to Theorem 2.1 we ger $A^w = TJ^wT^{-1} = TZT^{-1}$. The set of $\{1,1^k\}$-inverses of J is

$$J\{1,1^k\} = \begin{pmatrix} J_1^{-1} & 0 & Z_{13} \\ 0 & J_2^{-1} & Z_{23} \\ O & O & Z_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & z_{11}^{13} \\ 0 & \frac{1}{2} & 0 & z_{11}^{23} \\ 0 & 0 & z_{11}^{33} & z_{12}^{33} \\ 0 & 0 & 1 & z_{22}^{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & z_{14} \\ 0 & \frac{1}{2} & 0 & z_{24} \\ 0 & 0 & z_{33} & z_{34} \\ 0 & 0 & 1 & z_{44} \end{pmatrix}, \; z_{ij} \text{ arbitrary.}$$

From the equation $ZJZ = Z$ we get $z_{33}\, z_{44} = z_{34}$, so that

$$A^{w}=T\begin{pmatrix}\frac{1}{2} & 0 & 0 & z_{14}\\ 0 & \frac{1}{2} & 0 & z_{24}\\ 0 & 0 & z_{33} & z_{33}z_{44}\\ 0 & 0 & 1 & z_{44}\end{pmatrix}T^{-1}.\quad \text{Similarly}\quad A_{w}=T\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0\\ z_{31} & z_{32} & z_{33} & z_{33}z_{44}\\ 0 & 0 & 1 & z_{44}\end{pmatrix}T^{-1}.$$

According to Corollary 2.1 the weak spectral inverses of A are represented by $T\begin{pmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & z_{33} & z_{33}z_{44}\\ 0 & 0 & 1 & z_{44}\end{pmatrix}T^{-1}$.

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