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Computing The Scattering Number and The Toughness for Gear Graphs

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Abstract

In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. To measure the vulnerability we have some parameters which are *connectivity, toughness, scattering number, integrity, tenacity and their edge analogues*. This paper includes several results on the toughness and the scattering number of a gear graph as a communication network. Firstly, we compute the scattering number and toughness of a gear graph. In addition, the scattering number and toughness of the complement of a gear graph, the cartesian product of two gear graphs and the sequential join of gear graphs are computed. Finally, we compare the results for the scattering number and toughness.

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1 Introduction

In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. To measure the vulnerability we have some parameters which are connectivity [3], toughness [4], scattering number [6], integrity [1], tenacity [5] and their edge-analogues. In this paper we discuss the scattering number and the toughness of a graph. The **toughness** of a graph G, denoted t(G), was defined by Chvátal [4]. For the complete graph K_n we have $t(K_n) = \infty$; if G is not complete, then

$$t(G) = min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G) \text{ and } \omega(G-S) > 1\}$$

where $\omega(G-S)$ denotes the number of components in $G-S$.

The **scattering number** of a graph G, denoted sc(G), was introduced in 1978 by Jung [6]. For the complete graph K_n we have $sc(K_n) = 2-n$. If G is not complete, then

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$$sc(G) = max\{\omega(G - S) - \mid S \mid : S \subseteq V(G) \text{ and } \omega(G - S) \neq 1\}$$

where $\omega(G - S)$ denotes the number of components in $G - S$.

The scattering number of a graph is closely related to the toughness of a graph. Moreover Jung calls the scattering number the "additive dual" of the toughness. From the definitions of the toughness and the scattering number, it is clear that these two parameters are very similar.

Geared systems are used in dynamic modelling. These are graph theoretic models that are obtained by using gear graphs. Similarly the complement of a gear graph, the cartesian product of gear graphs and the sequential join of gear graphs can be used to design a gear network.

Consequently these considerations motivated us to investigate the vulnerability of gear graphs by using the scattering number and the toughness. Now we give the following definitions.

Definition 1.1 *The* wheel graph *with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle. In Figure* 1 *we display W_6.*

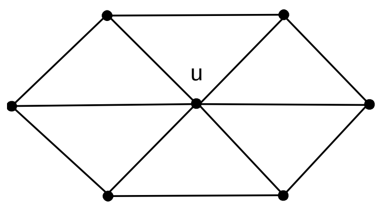

Figure 1: W_6 Wheel graph

Definition 1.2 ([2]) *The* gear graph *is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G_n has $2n + 1$ vertices and $3n$ edges. In Figure* 2 *we display G_6.*

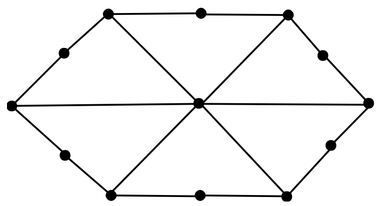

Figure 2: G_6 Gear graph

Let G = (V,E) be a graph. By $\kappa(G)$ we denote the connectivity of G. $\alpha(G)$ and $\beta(G)$, respectively, denotes the independence number and the covering number of G.

Definition 1.3 *The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.*

Definition 1.4 *Let G_1 and G_2 be two graphs. The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The join is denoted $V(G_1) + V(G_2)$ and consists of $V(G_1) \cup V(G_2)$ and all edges joining $V(G_1)$ with $V(G_2)$. For three or more disjoint graphs $G_1, G_2, ..., G_n$, the sequential join $G_1 + G_2 + ... + G_n$ is $(G_1 + G_2) \cup (G_2 + G_3) \cup ... \cup (G_{n-1} + G_n)$.*

Next we give some lower and upper bounds for the scattering number and the toughness in terms of well known graph parameters. To have an idea about minimum and maximum values of the scattering number and the toughness for any graph G, we can use Theorems $1.1 - 1.3$.

Theorem 1.1 ([4]) *Let G be a graph of order n. Then $t(G) \leq \frac{n-\alpha(G)}{\alpha(G)}$.*

Theorem 1.2 ([11]) *Let G be a graph of order n. Then*

$$2\alpha(G) - n \leq sc(G) \leq \alpha(G) - \kappa(G).$$

Theorem 1.3 ([10]) *Let G be a graph of order n. Then $t(G) \geq \frac{\kappa(G)}{\kappa(G)+sc(G)}$.*

In Section 2 we compute the toughness of a gear graph. Also we give some results about the toughness of graphs obtained from graph operations *between gear graphs*. In Section 3 we first compute the scattering number of a gear graph. Moreover we give some results connecting scattering number and graph operations *between gear graphs*. In Section 4, we compare the results in Sections 2 and 3.

2 Toughness

In this section we first calculate the toughness of a gear graph. In addition we consider the graph operations that are complement, cartesian product and sequential join. So we give several results about gear graphs and graph operations.

We begin with the toughness of a gear graph.

Theorem 2.1 *Let G_n be a gear graph of order $n \geq 4$. Then $t(G_n) = \frac{n}{n+1}$.*

Proof. The graph G_n has a subgraph $K_{1,n}$. Let $x_1, x_2, ..., x_{n+1}$ be the vertices of $K_{1,n}$ such that $deg(x_1) = n$. Hence we have two cases:
Case 1: Suppose that $S = \{x_i | x_i \in K_{1,n}, 2 \leq i \leq n+1\}$, that is, S must contain all the vertices of $K_{1,n}$ except x_1. If we remove the vertices in S from G_n, then we have exactly n+1 components. Hence

$$t(G_n) = \tfrac{n}{n+1} \tag{1}$$

Case 2: Let S be a vertex cut of G_n such that $S \neq \{x_i | x_i \in K_{1,n}, 2 \leq i \leq n+1\}$. If we remove $|S| = r$ vertices from G_n where $1 \leq r \leq 2n$, then we have at most r components. Hence

$$t(G_n) \geq \min_r \{\frac{r}{r}\} = 1 \tag{2}$$

By (1) and (2) we have $t(G_n) = \min\{\frac{n}{n+1}, 1\} = \frac{n}{n+1}$.

□

The following theorem gives the toughness of a complement of a gear graph.

Theorem 2.2 *If $\overline{G}_n$ be a complement of a gear graph G_n. Then $t(\overline{G}_n) = \frac{n}{2}$.*

Proof. Let S be a vertex cut of $\overline{G}_n$. Hence if we remove $|S| = r$ vertices from $\overline{G}_n$ where $n \leq r \leq 2n-1$ then we have exactly 2 components. Then

$$t(\overline{G}_n) = \min_S \{\frac{|S|}{\omega(G-S)}\} = \min_r \{\frac{r}{2}\}.$$

Let $f(r) = \frac{r}{2}$. The function f(r) takes its minimum value at $r = n$ and

$$t(\overline{G}_n) = \tfrac{n}{2}.$$

□

Now we consider the cartesian product. Firstly we give the toughness of graph $K_2 \times G_n$. Moreover we calculate the toughness of cartesian product of two gear graphs.

Theorem 2.3 *Let $n \geq 4$ be a positive integer. Then $t(K_2 \times G_n) = 1$.*

Proof. The graph $K_2 \times G_n$ has $4n+2$ vertices and has two subgraphs, namely G_{n1} and G_{n2} . Gear graph contains vertices set of whell graph. Now we define S_1 and S_2 as follows.

$$S_1 = \{x_i | x_i \in V(W_{n1}) \ \ and \ \ deg(x_i) \neq n\}$$

and

$$S_2 = \{x_i | x_i \in V(G_{n2} - V(W_{n2})\} \cup \{x_i | x_i \in V(W_{n2}) \ \ and \ \ deg(x_i) = n\}.$$

Case 1: Suppose that $S = S_1 \cup S_2$. If we remove the vertices in S from $K_2 \times G_n$ then $|S| = 2n+1$ and $\omega((K_2 \times G_n) - S) = 2n+1$. Hence we have

$$t(K_2 \times G_n) = 1 \tag{3}$$

Case 2: Suppose that $S \neq S_1 \cup S_2$. Hence if we remove $|S| = r$ vertices from $K_2 \times G_n$ where $1 \leq r \leq 4n+2$ then we have $\omega((K_2 \times G_n) - S) < r$. Then

$$t(K_2 \times G_n) > 1 \tag{4}$$

By (3) and (4) we have $t(K_2 \times G_n) = 1$. □

Theorem 2.4 *Let $m \geq 3$ and $n \geq 3$ be positive integers. Then*

$$t(G_m \times G_n) = \tfrac{2mn+m+n}{2mn+m+n+1}.$$

.

Proof. It is obvious that $\alpha(G_m \times G_n) = 2mn+m+n+1$ and $\beta(G_m \times G_n) = 2mn+m+n$. To prove this theorem we have two cases.
Case 1: By Theorem 1.1 we have

$$\begin{aligned} t(G_m \times G_n) &\leq \tfrac{(2n+1)(2m+1)-(2mn+m+n+1)}{2mn+m+n+1} \\ &= \tfrac{2mn+m+n}{2mn+m+n+1} \end{aligned} \tag{5}$$

Case 2: Since $\omega(G-S) \leq \alpha(G)$ for any graph G, we have

$$t(G_m \times G_n) \geq \min_S \{\frac{|S|}{\alpha(G_m \times G_n)}\}$$

- Let $|S| = \beta(G_m \times G_n)$. Since $\beta(G_m \times G_n) = 2mn+m+n$ we have
  $$t(G_m \times G_n) \geq \min_S \{\frac{|S|}{\alpha(G_m \times G_n)}\} = \frac{2mn+m+n}{2mn+m+n+1} \tag{6}$$
- If $|S| \neq \beta(G_m \times G_n)$ and $|S| = r$ then $\omega((G_m \times G_n) - S) \leq r$. Then
  $$t(G_m \times G_n) \geq \min_r \{\frac{r}{r}\} = 1 \tag{7}$$

By (6) and (7) we have $t(G_m \times G_n) \geq \tfrac{2mn+m+n}{2mn+m+n+1}$ (8)

Consequently, by (5) and (8) we have

$$t(G_m \times G_n) = \tfrac{2mn+m+n}{2mn+m+n+1}.$$ □

Let $G_3, G_4, ..., G_n$ be gear graphs. In the following theorems, the toughness of graph $G_3 + G_4 + ... + G_n$, which is obtained sequential join operation, is calculated when n is odd and when n is even.

Theorem 2.5 *If n is an even number, then*

$$t(G_3 + G_4 + ... + G_n) = \tfrac{3n+10}{n+6}.$$

Proof. To prove this theorem we have two cases.
Case 1: If we remove all the vertices of graphs $G_3, G_5, ..., G_{n-1}$, then the remaining components are $G_4, G_6, ..., G_n$ and the number of removing vertices is $\sum_{i=1}^{\frac{n}{2}-1} |V(G_{2i+1})| = \sum_{i=1}^{\frac{n}{2}-1}(4i+3)$. Moreover, we must remove $2i$ more vertices from each G_{2i} where $2 \leq i \leq \frac{n}{2}$. Hence, $2i+1$ components are obtained from each G_{2i} where $2 \leq i \leq \frac{n}{2}$. Then the number of removed vertices is exactly

$$|S| = \sum_{i=1}^{\frac{n}{2}-1}(4i+3) + \sum_{i=2}^{\frac{n}{2}} 2i$$

and the number of components is exactly

$$\omega((G_3 + G_4 + ... + G_n) - S) = \sum_{i=2}^{\frac{n}{2}}(2i+1)$$

Therefore, we have

$$t(G_3 + G_4 + ... + G_n) = \tfrac{3(-1+\frac{n}{2})+\frac{1}{2}(-2+n)n+\frac{1}{4}(-2+n)(4+n)}{-1+\frac{n}{2}+\frac{1}{4}(-2+n)(4+n)}$$

$$t(G_3 + G_4 + ... + G_n) = \tfrac{3n+10}{n+6} \tag{9}$$

Case 2: If we remove all the vertices of graphs $G_4, G_6, ..., G_n$, then the remaining components are $G_3, G_5, ..., G_{n-1}$ and the number of removed vertices is $\sum_{i=2}^{\frac{n}{2}} |V(G_{2i})| = \sum_{i=2}^{\frac{n}{2}}(4i+1)$. Moreover, we must remove $2i-1$ more vertices from each G_{2i-1} where $2 \leq i \leq \frac{n}{2}$. Hence, $2i$ components are obtained from each G_{2i} where $2 \leq i \leq \frac{n}{2}$. Then the number of removed vertices is exactly

$$|S| = \sum_{i=2}^{\frac{n}{2}}(4i+1) + \sum_{i=2}^{\frac{n}{2}}(2i-1)$$

and the number of components is exactly

$$\omega((G_3 + G_4 + ... + G_n) - S) = \sum_{i=2}^{\frac{n}{2}} 2i$$

So

$$t(G_3 + G_4 + ... + G_n) = \tfrac{\frac{3}{4}(n-2)(n+4)}{\frac{1}{4}(n-2)(n+4)} = 3 \tag{10}$$

By (9) and (10) we have

$$t(G_3 + G_4 + ... + G_n) = \tfrac{3n+10}{n+6}. \qquad \square$$

Theorem 2.6 *If n is an odd number, then*

$$t(G_3 + G_4 + ... + G_n) = \tfrac{3n^2+4n-27}{(n-1)(n+5)}.$$

Proof. The proof follows directly from theorem 2.5. □

3 Scattering number

In this section we first calculate the scattering number of a gear graph. Also we calculate the scattering number of some graphs which are obtained by using gear graphs and graph operations.

Now we give the scattering number of a gear graph.

Theorem 3.1 *Let G_n be a gear graph of order n. Then $sc(G_n) = 1$.*

Proof. Since $\alpha(G_n) = n + 1$, then we have

$$sc(G_n) \geq 1 \tag{11}$$

by Theorem 1.2.

On the other hand, let S be a vertex cut of G_n and $|S| = r$. If we remove r vertices from G_n, then $\omega(G_n - S) \leq r + 1$. Since $\omega(G_n - S) - |S| \leq r + 1 - r$, we have

$$sc(G_n) \leq 1 \tag{12}$$

By (11) and (12) we have $sc(G_n) = 1$. □

The following theorem gives the scattering number of a complement of a gear graph.

Theorem 3.2 *Let $\overline{G}_n$ be a complement graph of a gear graph G_n. Then*

$$sc(\overline{G}_n) = 2 - n.$$

it Proof. The graph $\overline{G}_n$ has two complete subgraphs, namely K_{n1} and K_{n2}. Each vertices of K_{n1} is joined to the vertices of K_{n2} with $(n-2)$ edges. Let S be a vertex cut of $\overline{G}_n$ and so $n \leq |S| \leq 2n - 1$. If we remove all the vertices of S from $\overline{G}_n$, then the number of remaining components is exactly 2. Then $sc(\overline{G}_n) = \max\limits_{S}\{2 - |S|\}$. The function $2 - |S|$ takes its maximum value at $|S| = n$ and

$$sc(\overline{G}_n) = 2 - n.$$ □

Next we concentrate on the scattering number and cartesian product. Hence we calculate the scattering number of graphs $K_2 \times G_n$ and $G_m \times G_n$.

Theorem 3.3 *Let G_n be a gear graph. Then $sc(K_2 \times G_n) = 0$.*

Proof. Since $t(K_2 \times G_n) = 1$ by Theorem 2.3, then we have

$$\frac{\kappa(K_2 \times G_n)}{\kappa(K_2 \times G_n) + sc(K_2 \times G_n)} \leq 1$$

by Theorem 1.3. So

$$sc(K_2 \times G_n) \geq 0 \tag{13}$$

On the other hand let S be a vertex cut of $K_2 \times G_n$ and $|S| = r$. If we remove r vertices from $K_2 \times G_n$ then $\omega((K_2 \times G_n) - S) \leq r$ and $\omega((K_2 \times G_n) - S) - |S| \leq r - r$. So

$$sc(K_2 \times G_n) \leq 0 \tag{14}$$

By (13) and (14) we have $sc(K_2 \times G_n) = 0$. □

Theorem 3.4 *Let $m \geq 3$ and $n \geq 3$ be positive integers. Then*

$$sc(G_m \times G_n) = 1.$$

Proof. Since $\alpha(G_m \times G_n) = 2mn + m + n + 1$ and $|V(G_m \times G_n)| = (2m+1)(2n+1)$, then we have

$$sc(G_m \times G_n) \geq 1 \tag{15}$$

by Theorem 1.2.

Now let S be a vertex cut of $G_m \times G_n$ and $|S| = r$. If we remove r vertices from $G_m \times G_n$, then $\omega((G_m \times G_n) - S) \leq r + 1$. Hence we have

$$sc(G_m \times G_n) \leq 1 \tag{16}$$

By (15) and (16) we have $sc(G_m \times G_n) = 1$. □

The following theorems give some results on the scattering number and sequential join operation.

Theorem 3.5 *Let $n \geq 5$ be a positive integer. Then*

$$sc(G_3 + G_4 + ... + G_n) = -7.$$

Proof. Let S be a vertex cut of graph $G_3 + G_4 + ... + G_n$ and set $|S| = r$. Since $|S| \geq \kappa(G)$ for any graph G and $\kappa(G_3 + G_4 + ... + G_n) = 9$, we have two cases:
Case 1: If we remove r vertices from $G_3 + G_4 + ... + G_n$, then $\omega((G_3 + G_4 + ... + G_n) - S) \leq r - 7$. So

$$sc(G_3 + G_4 + ... + G_n) \leq -7 \tag{17}$$

Case 2: Since the set S is a vertex cut, we have $\omega((G_3 + G_4 + ... + G_n) - S) \geq 2$. So

$$\omega((G_3 + G_4 + ... + G_n) - S) - |S| \geq 2 - r$$

and

$$sc(G_3 + G_4 + ... + G_n) \geq \max_r \{2 - r\}$$

The function 2-r takes its maximum value at r=9 and

$$sc(G_3 + G_4 + ... + G_n) \geq -7 \tag{18}$$

By (17) and (18) we have $sc(G_3 + G_4 + ... + G_n) = -7$. □

Remark 3.1 *One can easily show that* $sc(G_3 + G_4) = -6$.

4 Conclusion

A network has often as considerable an impact on network's performance as the vertices themselves. Performance measures for the networks are essential to guide the designer in choosing an appropriate topology. In order to measure the performance we are interested the following performance metrics:
1. The number of the components of the remaining network,
2. The diameter of the network,
3. The average distance between node pairs,
4. The probability that the network becomes disconnected,
They measure the extent to which the network can withstand the failure of links and vertices while still remaining functional [8, 9].

If the network does get disconnected, then remaining components should continue to function with reduced capacity. We would prefer a network which would disconnect in such a way that its capacity is almost seem as before. That is, we have the fundamental question: "How difficult is it to reconstruct the network?". This question is analyzed by considering the number of components of the remaining graph. Therefore, we are concerned with the toughness and scattering number of a graph as a measure of graph vulnerability.

In order to reconstruct a disrupted network easily, the number of connected components, formed after the vertices deleted, should be possibly small. In the following Table 1, if we consider the graphs G_n and $G_m \times G_n$ for both measures the number of components is always more than the number of deleted vertices. But if the difference is only one, this case shows that the graphs G_n and $G_m \times G_n$ are neither very strong nor very weak. If we examine the graphs $\overline{G}_n$ and $G_3 + G_4 + ... + G_n$, regardless of the number of deleted vertices we can say that the number of components is quite small and according to these results the reconstruction of these two graphs is very easy.

	toughness	scattering number
G_n	$\frac{n}{n+1}$	1
$\overline{G}_n$	$\frac{n}{2}$	$2-n$
$G_m \times G_n$	$\frac{2mn+m+n}{2mn+m+n+1}$	1
$G_3 + G_4 + \cdots + G_n$	$\frac{3n+10}{n+6}$	-7

Table 1

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