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THE ab-INDEX OF POSET OF INTERVALS

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ABSTRACT. We show that if a graded poset P admits an R-labelling, then there exists an R-labelling for its interval poset $I(P)$. This motivate us to define a linear operator $\mathcal{I}$, which express ab-index of interval poset via ab-index of P.

Throughout this paper, we will consider graded posets with rank function r. We refer to [9] as a good general reference for the poset terminology. For a poset P of rank $n+1$ and $S \subseteq [n] = \{1, 2, ..., n\}$ we define $f_S(P)$ as the number of chains $x_1 < x_2 < \cdots < x_{|S|}$ such that $S = \{r(x_1), r(x_2), \cdots, r(x_{|S|})\}$. The sequence $(f_S(P))_{S\subseteq[n]}$ is called *the flag f-vector* of P. The flag f-vector of P can be encoded as a non-commutative polynomial in variables a and b. Let P be a poset of rank $n+1$. To every chain c:$\hat{0} < x_1 < x_2 < \cdots < x_k < \hat{1}$ of P we associate a *weight* $wt(c) = w_1 w_2 \cdots w_n$ where

$$w_i = \begin{cases} \mathrm{b}, & \text{if } i \in \{r(x_1), r(x_2), \ldots, r(x_k)\}; \\ \mathrm{a} - \mathrm{b}, & \text{otherwise.} \end{cases}$$

Now, *the* ab-*index* of P is defined as

$$\Psi_P = \sum_{c\text{-chain in} P} wt(c) \tag{1}$$

The flag h-vector of a poset P of rank $n+1$ is the sequence $(h_S)_{S\subseteq[n]}$, defined by

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P).$$

Using the above transformation of flag f-vector, we can note that

$$\Psi_P = \sum_{S \subseteq [n]} h_S u_S$$

where $u_S = u_1 u_2 \cdots u_n$, $u_i = \mathrm{a}$ if $i \notin S$, $u_i = \mathrm{b}$ if $i \in S$.

The star involution $*$ is defined on **ab**-monomials by $(u_1 u_2 \cdots u_k)^* = u_k u_{k-1} \cdots u_1$, and extended linearly on the polynomial algebra $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$. For **ab**-index of dual poset P^* holds $\Psi_{P^*} = (\Psi_P)^*$ ([8]).

For a poset P let $\mathcal{E}(P)$ denote its covering relation, $\mathcal{E}(P) = \{(x, y) \in P \times P : x \prec y\}$. An *edge-labelling* of P is a map $\lambda : \mathcal{E}(P) \to \Lambda$, where Λ is some poset (usually integers). Given an edge labelling λ, each unrefinable chain $C : x = x_0 \prec x_1 \prec \cdots \prec x_{k-1} \prec x_k = y$ of length k can be associated with its *label* $\lambda(C) = (\lambda(x_0 \prec x_1), \lambda(x_1 \prec x_2), \ldots, \lambda(x_{k-1} \prec x_k))$. The edge labelling λ of P is said to be an *R-labelling* if in every interval $[x, y]$ of P there is a unique maximal chain where the labels are weakly increasing.

For a maximal chain $c : \widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \widehat{1}$ we define its *descent set* $D(c) = \{i \in [n] : \lambda(x_{i-1} \prec x_i) > \lambda(x_i \prec x_{i+1})\}$ and its *descent monomial* $u(c) = u_1 u_2 \cdots u_n$, where $u_i = \mathbf{a}$ if $i \notin D(c)$ and $u_i = \mathbf{b}$ otherwise. If a poset P admits an R-labelling then the following result (see [4],[9]) gives the combinatorial interpretation of the flag $h-$vectors.

Theorem 1 (R. Stanley). *Let P be a finite bounded graded poset of rank $n+1$ with an $R-$labelling λ. Then, for all $S \subseteq [n]$, $h_S(P)$ is equal to the number of maximal chains of P with descent set S.*

As a corollary we obtain that if a poset P has an R-labelling, then

$$\Psi_P = \sum_c u(c) \tag{2}$$

where the sum is over all maximal chains c.

The *interval poset* $I(P)$ of a poset P is the set of all closed intervals of P ordered by containment:

$$[x, y] \le [x', y'] \text{ in } I(P) \text{ if and only if } x' \le x \le y \le y' \text{ in } P.$$

By the convention we also adjoin the empty interval to $I(P)$ as the minimal element.

In [7] it is noted that for all $n \in \mathbb{N}$ the interval poset of the Boolean lattices B_n (the face lattice of an $(n-1)$-simplex) is the face lattice of n-cube, i.e. $I(L(\Delta_{n-1})) \cong L(C_n)$. Also, in [7] is asked wether it is true for every polytope P such that there exists a polytope Q with $I(L(P)) \cong L(Q)$.

The next proposition follows directly from the definition of interval posets and from the basic properties of operations Pyr and $Prism$ over posets and polytopes (see [1], [5]).

Proposition 2.

(i) *For any poset P we have $I(P) \cong I(P^*)$.*
(ii) *Intervals in the poset $I(P)$ have the following form*

$$[[x, y], [x', y']]_{I(P)} \cong [x', x]^* \times [y, y'] \;, \quad [\hat{0}, [x, y]]_{I(P)} \cong I([x, y]). \tag{3}$$

(iii) *Let P be a graded poset of rank n. Then $I(P)$ is a graded poset of rank $n+1$ and from (3) we obtain that $r[x,y] = r(y) - r(x) + 1$. Also, from (3) it follows that*

$$f_i(I(P)) = \sum_{j=0}^{n-i+1} f_{j,j+i-1}(P) \tag{4}$$

(iv) *For any poset P holds*

$$I(P \times B_1) \cong I(P) \diamond B_2. \tag{5}$$

If P and Q are polytopes such that $I(L(P)) \cong L(Q)$, then (as a special case of (5)) we have $I(L(Pyr(P))) \cong L(Prism(Q))$.

Proposition 3. *If a poset P admits an R-labelling, then there exists an R-labelling for $I(P)$.*

Proof. Assume that $\lambda : \mathcal{E}(P) \to \Lambda$ is an R labelling for P. Let $-\Lambda$ denote a poset which is isomorphic with Λ^* and disjoint with Λ. Now, we can define

$$\overline{\lambda} : \mathcal{E}(I(P)) \to -\Lambda \oplus \{0\} \oplus \Lambda$$

(here $\oplus$ denotes the ordinal sum of posets) with $\overline{\lambda}(\hat{0}, [x,x]) = 0$ for all $x \in P$ and

$$\overline{\lambda}([x,y],[x',y']) = \begin{cases} \lambda(y,y'), & \text{if } x = x', y \prec y'; \\ -\lambda(x',x), & \text{if } x \prec x', y = y'. \end{cases}$$

We claim that $\overline{\lambda}$ is an R-labelling. If $x' = x_0 \prec x_1 \prec \cdots \prec x_r = x$ and $y = y_0 \prec y_1 \prec \cdots \prec y_s = y'$ are the unique rising chains in $[x',x]$ and $[y,y']$ then $[x,y] = [x_r, y_0] \prec [x_{r-1}, y_0] \prec \cdots \prec [x_0, y_0] \prec [x_0, y_1] \prec \cdots \prec [x_0, y_s] = [x', y']$ is the unique rising chain in $[[x,y],[x',y']]$.

Similarly, for the unique rising chain $x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ in $[x,y]$ we have that $\hat{0} \prec [x,x] = [x_0,x_0] \prec [x_0,x_1] \prec \cdots \prec [x_0,x_r] = [x,y]$ is the unique rising chain in $[\hat{0},[x,y]]$. □

Coalgebra techniques in studying **ab**-index were first applied in [5]. For a vector space W, a *coproduct* is a linear map $\Delta : W \to W \otimes W$. To denote the coproduct of an element $w \in W$, we use the Sweedler notation (see [10]) $\Delta(w) = \sum_w w_{(1)} \otimes w_{(2)}$. A coproduct Δ is *coassociative* if it satisfies the identity $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$.

The following coassociative coproduct Δ on $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ is defined in [5] by:

$$\Delta(u_1 \cdot u_2 \cdots u_n) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n$$

and extended linearly to $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$.

For a coassociative coproduct Δ on a coalgebra W we consider map $\Delta^k : W \to W^{\otimes k}$ defined by

$$\Delta^1 = id \text{ and } \Delta^{k+1} = (\Delta^{\otimes k} \otimes id) \circ \Delta.$$

The Sweedler notation for the map Δ^k is

$$\Delta^k(x) = \sum_x x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(k)}.$$

Maps Δ^k on coalgebra $\mathbb{Q}\langle \mathbf{a}, \mathbf{b}\rangle$ were used in [5] for the definition of mixing operator, which express $\Psi_{P\times Q}$ in the terms of Ψ_P and Ψ_Q. Our purpose is to define linear operator $\mathcal{I} : \mathbb{Q}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbb{Q}\langle \mathbf{a}, \mathbf{b}\rangle$ such that $\Psi_{I(P)} = \mathcal{I}(\Psi_P)$.

Definition 4. For $k = 2m$ we define an operator I_k on a monomial u by

$$I_k(u) = \sum_u \mathbf{a} \cdot u_{(m+1)} \cdot \mathbf{b} \cdot u^*_{(m)} \cdot \mathbf{a} \cdot u_{(m+2)} \cdots \mathbf{b} \cdot u^*_{(1)} + \\ +\mathbf{b} \cdot u^*_{(m)} \cdot \mathbf{a} \cdot u_{(m+1)} \cdot \mathbf{b} \cdot u^*_{(m-1)} \cdots \mathbf{a} \cdot u_{(2m)}$$

and for $k = 2m + 1$

$$I_k(u) = \sum_u \mathbf{a} \cdot u_{(m+1)} \cdot \mathbf{b} \cdot u^*_{(m)} \cdot \mathbf{a} \cdot u_{(m+2)} \cdots \mathbf{a} \cdot u_{(2m+1)} + \\ +\mathbf{b} \cdot u^*_{(m+1)} \cdot \mathbf{a} \cdot u_{(m+2)} \cdot \mathbf{b} \cdot u^*_{(m)} \cdots \mathbf{b} \cdot u^*_{(1)}$$

where $u_{(1)} \otimes u_{(2)} \otimes \cdots \otimes u_{(k)}$ is a summand in $\Delta^k(u)$ and $*$ is the star involution. Now, we define an operator $\mathcal{I}$ on **ab**-monomials by

$$\mathcal{I}(u) = \sum_{k\geq 1} I_k(u)$$

and extend it linearly to $\mathbb{Q}\langle \mathbf{a}, \mathbf{b}\rangle$.

For example, $\sum_k \Delta^k(\mathbf{a}^2) = \mathbf{a}^2 + 1 \otimes \mathbf{a} + \mathbf{a} \otimes 1 + 1 \otimes 1 \otimes 1$ and $\mathcal{I}(\mathbf{a}^2) = \mathbf{a}^3 + \mathbf{b}\mathbf{a}^2 + \mathbf{a}^2\mathbf{b} + \mathbf{b}\mathbf{a}^2 + \mathbf{aba} + \mathbf{b}\mathbf{a}^2 + \mathbf{aba} + \mathbf{bab}$.

Theorem 5. *For any graded poset P we have that $\Psi_{I(P)} = \mathcal{I}(\Psi_P)$.*

Proof. In order to motivate the definition of operator $\mathcal{I}$, we shall first prove this theorem in the case when the poset P has an R-labelling λ.

Assume that $r(P) = n$ and let $\overline{\lambda}$ be the R-labelling of $I(P)$ defined as in the proof of proposition 3. Let $c{:}\hat{0}_P = t_0 \prec t_1 \prec \cdots \prec t_{n-1} \prec t_n = \hat{1}_P$ be a maximal chain in P, $\lambda(c) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ its label and $u(c) = u_1 u_2 \cdots u_{n-1}$ its descent monomial.

For any maximal chain $C{:}\hat{0} = \emptyset \prec [x_0, y_0] \prec [x_1, y_1] \prec \cdots \prec [x_{n-1}, y_{n-1}] \prec [\hat{0}_P, \hat{1}_P] = \hat{1}$ in $I(P)$ we can consider the multichain

(6) $\quad \hat{0}_P \leq x_{n-1} \leq x_{n-2} \leq \cdots x_0 = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_{n-1} \leq \hat{1}_P$

in P. We say that C corresponds with c if all elements of c appear in (6). Note that if $x_i = x_{i-1}$ then $y_{i-1} \prec y_i$, and if $y_{i-1} = y_i$ then $x_i \prec x_{i-1}$. So, for any maximal chain C in $I(P)$ there exists the unique C-corresponding maximal chain in P.

Now, we will show that for a fixed maximal chain c in P, the contribution of all its corresponding maximal chains to $\Psi_{I(P)}$ is exactly $\mathcal{I}(u(c))$. Assume that C corresponds with c. If in (6) holds $x_0 = y_0 = t_i$, $x_0 = x_1 = \cdots =$

$x_{j_1} \succ x_{j_1+1} \succ \cdots \succ x_{j_2} = x_{j_2+1} = \cdots = x_{j_3} \succ x_{j_3+1} \succ \cdots$, then (according with the definition of labelling $\overline{\lambda}$) we have

$$\overline{\lambda}(C) =$$

$$= (0, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{i+j_1}, -\lambda_{i-1}, \ldots, -\lambda_{i-j_2}, \lambda_{i+j_1+1}, \ldots \lambda_{i+j_3}, -\lambda_{i-j_2-1}, \cdots).$$

So, the descent monomial of C

$$u(C) = \mathbf{a}u_i \cdots u_{i+j_1-1}\mathbf{b}u_{i-2} \cdots u_{i-j_2}\mathbf{a}u_{i+j_1+1} \cdots u_{i+j_3-1}\mathbf{b} \cdots$$

is a monomial that appears as a summand in $\mathcal{I}(u(c))$.

Similarly, if $x_0 \succ x_1 \succ \cdots \succ x_{j_1} = x_{j_1+1} = \cdots = x_{j_2} \succ x_{j_2+1} \succ \cdots \succ x_{j_3} = x_{j_3+1} = \cdots$ we obtain that the descent monomial $u(C)$ is a summand in $\mathcal{I}(u(c))$ which begin with $\mathbf{b}$.

Two different c-corresponding maximal chains in $I(P)$ (in the above construction) will be associated with different summands of $\mathcal{I}(u(c))$. For a given c, there are exactly 2^n of its corresponding maximal chains in $I(P)$ and 2^n summands in $\mathcal{I}(u(c))$.

So, we obtain that $\mathcal{I}(u(c)) = \sum_C u(C)$, where C ranged over all corresponding maximal chains of $I(P)$.

From linearity of $\mathcal{I}$ and (2) it follows that $\Psi_{I(P)} = \mathcal{I}(\Psi_P)$.

Proof of the theorem in general case: Let P be a graded poset of rank n. Let $c{:}\hat{0}_P < t_1 < \cdots < t_k < \hat{1}_P$ be an arbitrary chain in P and let r_i denote $r(t_i)$. The contribution of c to Ψ_P is

$$wt(c) = (\mathbf{a}-\mathbf{b})^{r_1-1}\mathbf{b}(\mathbf{a}-\mathbf{b})^{r_2-r_1-1}\mathbf{b} \cdots \mathbf{b}(\mathbf{a}-\mathbf{b})^{n-r_k-1}.$$

We say that a chain $C{:}\hat{0} = \emptyset < [x_0, y_0] < [x_1, y_1] < \cdots < [x_r, y_r] < [\hat{0}_P, \hat{1}_P] = \hat{1}$ in $I(P)$ corresponds with c iff the multichain

$$\hat{0}_P \leq x_r \leq x_{r-1} \leq \cdots \leq x_0 \leq y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_r \leq \hat{1}_P$$

in P contains exactly those elements which appear in c. Note that for every chain C in $I(P)$ there exists the unique chain in P which corresponds with C.

If we denote by a_k the number of chains in $I(P)$ which correspond with a given chain c in P whose length is k, then (considering all possibilities for the greatest element of C) we have

$$a_{k+1} = 2a_k + a_{k-1} \text{ where } a_0 = 3, a_1 = 7.$$

Now, we shall show that

$$\mathcal{I}(wt(c)) = \sum_C wt(C)$$

where C ranges over all chains in $I(P)$ which correspond with c.

Summands that appear in $\Delta^{l+1}(wt(c))$ are indexed by all l-elements subsets of $\{1, 2, \ldots, n-1\}$. For a given set $S = \{i_1, i_2, \ldots, i_l\}$ and $w = wt(c)$ we put

$$w_S = w_1 \cdots w_{i_1-1} \otimes w_{i_1+1} \cdots w_{i_2-1} \otimes \cdots \otimes w_{i_l+1} \cdots w_{n-1} =$$

$$= w_{S(1)} \otimes w_{S(2)} \otimes \cdots \otimes w_{S(l+1)}$$

(if $i_j = i_{j-1} + 1$ then $w_{i_{j-1}+1} \cdots w_{i_j-1} = 1$).

Note that summands which correspond with $S \not\subseteq \{r_1, r_2, \ldots, r_k\}$ will vanish in $\Delta^{|S|+1}(wt(c))$ and in $\mathcal{I}(wt(c))$ too.

For $S = \{i_1, i_2, \ldots, i_l\} \subseteq \{r_1, r_2, \ldots, r_k\}$ and $l = 2m$ we consider chains

$$C^+:[t_{i_m}, t_{i_m}] < [t_{i_m}, t_{i_m+1}] < \ldots < [t_{i_m}, t_{i_{m+1}}] <$$
$$< [t_{i_m-1}, t_{i_{m+1}}] < \ldots < [t_{i_{m-1}}, t_{i_{m+1}}] < [t_{i_{m-1}}, t_{i_{m+1}+1}] \ldots <$$
$$< [t_{i_{m-1}}, t_{i_{m+2}}] < [t_{i_{m-1}-1}, t_{i_{m+2}}] \ldots < [\hat{0}_P, t_{i_l}] < \ldots < [\hat{0}_P, \hat{1}_P]$$

and

$$C^-[t_{i_{m+1}}, t_{i_{m+1}}] < [t_{i_{m+1}-1}, t_{i_{m+1}}] < \ldots < [t_{i_m}, t_{i_{m+1}}] < [t_{i_m}, t_{i_{m+1}+1}] <$$
$$< \ldots < [t_{i_m}, t_{i_{m+2}}] < [t_{i_m-1}, t_{i_{m+2}}] \ldots < [t_{i_{m-1}}, t_{i_{m+2}}] <$$
$$< [t_{i_{m-1}}, t_{i_{m+2}+1}] \ldots < [t_{i_1}, \hat{1}_P] < \ldots < [\hat{0}_P, \hat{1}_P].$$

The contribution of C^+ and all of its 2^{m+1} subchains (which correspond with c and may not contain elements $[t_{i_m}, t_{i_m}], [t_{i_{m-1}}, t_{i_{m+1}}], \cdots, [t_{i_1}, t_{i_{l-1}}]$, $[\hat{0}_P, t_{i_l}]$) to $\Psi_{I(P)}$ is

$$\mathrm{a} \cdot w_{S(m+1)} \cdot \mathrm{b} \cdot w^*_{S(m)} \cdot \mathrm{a} \cdot w_{S(m+2)} \cdots \mathrm{a} \cdot w_{S(l+1)}.$$

Similarly, the contribution of C^- and all of its 2^m subchains (which correspond with c and may not contain elements $[t_{i_m}, t_{i_{m+1}}], [t_{i_{m-1}}, t_{i_{m+2}}], \cdots$ $[t_{i_1}, t_{i_l}]$) to $\Psi_{I(P)}$ is

$$\mathrm{b} \cdot w^*_{S(m+1)} \cdot \mathrm{a} \cdot w_{S(m+2)} \cdot \mathrm{b} \cdot w^*_{S(m)} \cdots \mathrm{b} \cdot w^*_{S(1)}.$$

We use the same reasoning in the case when $l = 2m + 1$. The number of the used c-corresponding chains (for all $S \subseteq \{r_1, \ldots r_k\}$) is

$$\sum_{i=0}^{k} \binom{k}{i} (2^{\lfloor \frac{i+1}{2} \rfloor} + 2^{\lfloor \frac{i+2}{2} \rfloor}),$$

which is exactly a_k. So, we have $\mathcal{I}(wt(c)) = \sum_C wt(C)$, where C ranges over all chains in $I(P)$ that correspond with c. From linearity of $\mathcal{I}$ and (1) we obtain $\mathcal{I}(\Psi_P) = \Psi_{I(P)}$ for any graded poset P. □

A derivation f on an algebra A is a linear map satisfying the product rule $f(xy) = f(x)y + xf(y)$, and $f(1) = 0$. In [5] are defined the derivations G, G' and D on $\mathbb{Q}\langle \mathrm{a}, \mathrm{b} \rangle$ by

$$G(\mathrm{a}) = \mathrm{ba},\ G(\mathrm{b}) = \mathrm{ab};\ G'(\mathrm{a}) = \mathrm{ab},\ G'(\mathrm{b}) = \mathrm{ba};\ D = G + G'.$$

In [5] are also defined operators Pyr and $Prism$ on $\mathbb{Q}\langle \mathrm{a}, \mathrm{b} \rangle$ as

$$Pyr(u) = u \cdot (\mathrm{a} + \mathrm{b}) + G(u), Prism(u) = u \cdot (\mathrm{a} + \mathrm{b}) + D(u).$$

For any graded poset P holds (theorem 4.4. in [5])

$$\Psi_{P \times B_1} = Pyr(\Psi_P)\ , \Psi_{P \diamond B_2} = Prism(\Psi_P).$$

As **ab**-index is surjective (lemma 3.4. in [6]), from (i) and (iv) of Proposition 2 we obtain

Corollary 6. *The operator* $\mathcal{I}$ *satisfies*

$$\mathcal{I}(u^*) = \mathcal{I}(u) \ , \ \mathcal{I}(Pyr(u)) = Prism(\mathcal{I}(u))$$

for any **ab**-*polynomial* u.

A finite graded poset is *Eulerian* if every interval whose rank is at least one contains as many elements of even rank as of odd rank, i.e. for every interval in P the Euler-Poincaré formula holds.

M. Bayer and A. Klapper showed in [3] that the **ab**-index of an Eulerian poset P may be written uniquely as a polynomial Φ_P in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called *the* **cd**-*index* of P. It is easy to see (from (4) and generalized Dehn-Sommerville relations, see [2]) that if a poset P is Eulerian, then $I(P)$ is Eulerian, and so is $\Psi_{I(P)} \in \mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$. As in [5], [6], we can ask: Do there exist formulae for $\Psi_{I(P)}$ where the computation is inside algebra $\mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$?

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