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A REMARK ON APPROXIMATION PROCEDURE FOR FIXED POINTS

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ABSTRACT. In this paper we give iterative algorithm for fixed points of sum of two operators on a nonempty closed convex subsets of a Hilbert spaces. Our result generalizes the iterative procedure of Ram U. Verma.

Introduction and Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$ and let K be a nonempty closed convex subset of K. In the paper [4] Ram U. Verma gave the following iterative procedure of fixed points of an operator of the form $S = T + U$,

(1) $$x_{n+1} = (1-a_n)x_n + a_n[(1-t)x_n + tSx_n] \text{ for all } n \geq 0,$$

where $t > 0$ is arbitrary, the sequence $\{a_n\}$ of elements os $[0,1]$ is such that $\sum_{n=0}^{+\infty} a_n$ diverges, T is strongly Lipschitz and Lipschitz continuous and U is Lipschitz continuous on K.

In this paper the following iterative procedure is given

(2) $$x_{n+1} = I(a_{1n}, x_n, I(a_{2n}, x_n, \ldots, I(a_{in}, x_n, S)) \cdots) \text{ for all } n \geq 0,$$

where

$$I(a_{kn}, x_n, S) = (1-a_{kn})x_n + a_{kn}Sx_n,$$

and the sequences $\{a_{kn}\}$, $k = 1, \ldots, i$, have its elements in $[0,1]$.

If $i = 1$ and $a_{1n} = a_n$ we obtain (see examples $[1-3]$),

(3) $$x_{n+1} = (1-a_n)x_n + a_nSx_n \text{ for all } n \geq 0.$$

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If $i = 2,\ a_{1n} = a_n$ and $a_{2n} = t$ then (1) is obtained.

An operator $T : H \to H$ is said to be strongly Lipschitz [4] if, for all u, v in H, there exists a real number $r \geq 0$ such that

$$\langle Tu - Tv, u - v\rangle \leq -r||u - v||^2.$$

The operator T satisfies the Lipschitz condition if there exists a real number $s > 0$ such that

$$||Tu - Tv|| \leq s||u - v|| \text{ for all } u, v \text{ in } H.$$

Main Result

In this section, we give sufficient conditions for the convergence of the sequence $\{x_n\}$ defined by (2).

Theorem 1. *Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $T : K \to K$ be a strongly Lipschitz and Lipschitz continuous with respective real numbers $r \geq 0$ and $s \geq 1$, and $U : K \to K$ be a Lipschitz continuous with a real number $m > 0$. Let F be a nonempty set of fixed points of $S = T + U$, and let $\{a_{kn}\}, k = 1, \dots, i$ be sequences in $[0, 1]$. Define sequences $\{t_{kn}\}, k = 1, \dots, i$ by*

$$t_{kn} = [(1 - a_{kn})^2 - 2a_{kn}(1 - a_{kn})r + a_{kn}^2 s^2]^{\frac{1}{2}} + a_{kn}m, \text{ for all } n \geq 0, i \geq 1$$
$$(k = 1, \dots, i).$$

If there exists $i_0 \in \{1, \dots, i\}$ such that $max\{t_{i_0 n} : n \geq 0\} \leq k < 1$ and $\sum_{n=0}^{+\infty} \prod_{k=1}^{i_0 - 1} a_{kn}$ diverges for $i_0 \geq 2$ then, for any x_0 in K, the sequence $\{x_n\}$ generated by

$$x_{n+1} = I(a_{1n}, x_n, I(a_{2n}, x_n, \dots, I(a_{i_0 n}, x_n, S)) \cdots) \text{ for all } n \geq 0,$$

converges to a fixed point of $S = T + U$.

Proof. Suppose for $i_0 \in \{1, \dots, i\}$ holds $max\{t_{i_0 n} : n \geq 0\} \leq k < 1$ and suppose that $\sum_{n=0}^{+\infty} \prod_{k=1}^{i_0 - 1} a_{kn}$ diverges for $i_0 \geq 2$. If $i_0 = 1$ then for an element z in F holds

$$||z - I(a_{1n}, x_n, S)|| \leq ||(1 - a_{1n})(x_n - z) + a_{1n}(Tx_n - Tz)|| + a_{1n}m||x_n - z||.$$

Since

$$||(1 - a_{1n})(x_n - z) + a_{1n}(Tx_n - Tz)||^2 =$$
$$(1 - a_{1n})^2||x_n - z||^2 + 2a_{1n}(1 - a_{1n})\langle Tx_n - Tz, x_n - z\rangle + a_{1n}^2||Tx_n - Tz||^2$$

and T is strongly Lipschitz and Lipschitz continuous we get

$$||z - I(a_{1n}, x_n, S)|| \leq t_{1n}||x_n - z||,$$

so that,

$$||z - I(a_{1n}, x_n, S)|| \leq k^{n+1}||x_0 - z||$$

and in this case the proof is complete.
Similarly we obtain

$$||z - I(a_{1n}, x_n, I(a_{2n}, x_n, S))|| \leq (1 - (1 - t_{2n})a_{1n})||x_n - z||,$$

and

$$||z - I(a_{1n}, x_n, I(a_{2n}, x_n, \ldots, I(a_{i_0 n}, x_n, S))\cdots)|| \leq$$
$$\leq (1 - a_{1n})||x_n - z|| + a_{1n}||z - I(a_{2n}, x_n, \ldots, I(a_{i_0 n}, x_n, S)\cdots)||.$$

This gives

$$||z - x_{n+1}|| \leq \left[1 - (1 - t_{i_0 n}) \prod_{k=1}^{i_0 - 1} a_{kn}\right] ||x_n - z||,$$

and

$$||z - x_{n+1}|| \leq \prod_{j=0}^{n} \left[1 - (1 - k) \prod_{k=1}^{i_0 - 1} a_{kj}\right] ||x_0 - z||.$$

Since $\sum_{n=0}^{+\infty} \prod_{k=1}^{i_0 - 1} a_{kn}$ diverges and $k < 1$, it follows that

$$\lim_{n \to +\infty} \prod_{j=0}^{n} \left[1 - (1 - k) \prod_{k=1}^{i_0 - 1} a_{kj}\right] = 0,$$

and $\{x_n\}$ converges to a fixed point z of $S = T + U$. □

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