Bull. Soc. Math, Banja Luka
11(2004), 25-28

A RESULT ON EXPANSION OF DETERMINANTS

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ABSTRACT. A theorem will be proved which in a natural way generalizes Laplace's expansion theorems for determinants. When, in a special case, we restate the theorem in terms of permutations a known extension of Laplace's expansion theorem will be obtained.

Introduction

The determinants with entries in a commutative ring will be considered. If D is an n-th order determinant, then for subsets $I=\{i_1,\dots,i_r\}$ and $J=\{j_1,\dots,j_r\}$ of $\{1,\dots,n\}$ we shall denote by $D\begin{bmatrix} I\\ J\end{bmatrix}$ the r-th order minor lying in the intersection of i_1-th, i_2-th,...,i_r-th rows and j_1-th, j_2-th,...,j_r-th columns of D. By $\overline{D}\begin{bmatrix} I\\ J\end{bmatrix}$ will be denoted the complement minor, that is, the minor of the order $n-r$ lying in the intersection of the remaining $n-r$ rows and columns of D. If $n=r$ we then set $\overline{D}=1$.

If $I_1,I_2,\dots,I_r$ are mutually disjoint subsets of $\{1,\dots,n\}$ then by

$$\sigma(I_s,I_t),\ (s,t\in\{1,\dots,r\},\ s\neq t)$$

will be denoted the number of pairs $(i,j),\ \ (i\in I_s, j\in I_t)$ such that $i<j$. If there are no such pairs then we put $\sigma(I_s,I_t)=0$. It is convenient to denote $\sigma(I_0,I_s)=\sigma(\{1,\dots,n\}\setminus I_s,I_s),\ (s=1,\dots,r)$. In particular if $I_s=\{1,\dots,n\}$ then $\sigma(I_0,I_s)=0$.

If $I=\{i_1,\dots,i_s\}$ then the following holds

$$\sigma(I_0,I)=i_1+\cdots+i_s-\frac{s(s+1)}{2}. \tag{1}$$

2000 *Mathematics Subject Classification.* 15A15.

Key words and phrases. Determinants, Laplace's expansion theorem.

It is easily seen that if I, J, K are mutually disjoint then we have

$$\sigma(I, J\cup K)=\sigma(I,K)+\sigma(J,K) \tag{2}$$

and

$$\sigma(I\cup J, K)=\sigma(I,K)+\sigma(J,K). \tag{3}$$

In what follows the set $\{I_1,\ldots,I_r\}$ of mutually disjoint subsets of $\{1,\ldots,n\}$ will be fixed. For another set $\{J_1,\ldots,J_r\}$ of mutually disjoint subsets of $\{1,\ldots,n\}$ will be assumed that the sets I_k and J_k have the same number of elements, for every $k=1,\ldots,r$.

Note that if $I_k=\{i_1,\ldots,i_s\}$ and $J_k=\{j_1,\ldots,j_s\}$ then, by (1), one gets

$$\sigma(I_0,I_k)+\sigma(J_0,I_k)=i_1+\cdots+i_s+j_1+\cdots+j_s-s(s+1). \tag{4}$$

Furthermore, for $i\notin I,\ j\notin J$ the indices of the row and the column of the element a_{ij} of D in the minor $\overline{D}\left[\begin{array}{c} I\\ J\end{array}\right]$ are

$$i-\sigma(I,i) \text{ and } j-\sigma(J,j). \tag{5}$$

Finally we define

$$sgn(I_1,\ldots,I_r,J_1,\ldots,J_r)=\begin{cases} 1 & \text{if } \sum\limits_{0\le s<t\le r}[\sigma(I_s,I_t)+\sigma(J_s,J_t)] \text{ is even}\\ -1 & \text{if } \sum\limits_{0\le s<t\le r}[\sigma(I_s,I_t)+\sigma(J_s,J_t)] \text{ is odd.}\end{cases}$$

Theorem and proof

We shall now prove the following extension of Laplace's expansion theorem.

Theorem. *Let D be a determinant of the order n, and let $I_k,\ (k=1,\ldots,r)$ be mutually disjoint subset of $\{1,\ldots,n\}$ then*

$$D=\sum_{J_1,\ldots,J_r} sgn(I_1,\ldots,I_r,J_1,\ldots,J_r)\prod_{k=1}^{r} D\left[\begin{array}{c} I_k\\ J_k\end{array}\right]\cdot\overline{D}\left[\begin{array}{c} I_1\cup\cdots\cup I_r\\ J_1\cup\cdots\cup J_r\end{array}\right], \tag{6}$$

where $J_1,\ldots,J_r$ run trough all mutually disjoint subsets of $\{1,2,\ldots,n\}$.

Proof. We use induction on r. If $r=1$, $I_1=\{i_1,\ldots,i_s\}$ and $J_1=\{j_1,\ldots,j_s\}$ we have, by (4),

$$sgn(I_1,J_1)=\sigma(I_0,I_1)+\sigma(J_0,J_1)=i_1+\cdots+i_s+j_1+\cdots+j_s-s(s+1).$$

In this case (6) becomes

$$D=\sum_{J_1}(-1)^{i_1+\cdots+i_s+j_1+\cdots+j_s}D\left[\begin{array}{c} I_1\\ J_1\end{array}\right]\cdot\overline{D}\left[\begin{array}{c} I_1\\ J_1\end{array}\right],$$

which is well-known generalized Laplace's expansion theorem.

Suppose that the claim of the theorem is true for $r-1$. Then we have
(7)

$$D=\sum_{J_1,\dots,J_{r-1}} sgn(I_1,\dots,I_{r-1},J_1,\dots,J_{r-1})\prod_{k=1}^{r-1} D\begin{bmatrix} I_k \\ J_k \end{bmatrix}\cdot\overline{D}\begin{bmatrix} I_1\cup\dots\cup I_{r-1} \\ J_1\cup\dots\cup J_{r-1}\end{bmatrix},$$

where $J_1,\dots,J_{r-1}$ run over all mutually disjoint subset of $\{1,\dots,n\}$, providing that J_k has the same number of elements as I_k for all $k=1,\dots,r-1$.

By generalized Laplace's theorem we may expand $\overline{D}\begin{bmatrix} I_1\cup\dots\cup I_{r-1} \\ J_1\cup\dots\cup J_{r-1}\end{bmatrix}$ along the rows of D which indices belong to I_r. Take J_r to be disjoint by each $J_1,\dots,J_{r-1}$ and suppose that $i\in I_r,\ j\in J_r$. Then the indices of the row and the column of the element a_{ij} in $\overline{D}\begin{bmatrix} I_1\cup\dots\cup I_{r-1} \\ J_1\cup\dots\cup J_{r-1}\end{bmatrix}$, according to (5), are

$$i-\sigma(I_1\cup\dots\cup I_{r-1},i) \text{ and } j-\sigma(J_1\cup\dots\cup J_{r-1},j).$$

The sums of these indices, when i runs over I_r, and j runs over J_r, by (2), are

$$\sigma(I_0,I_r)+\frac{r(r+1)}{2}-\sigma(I_1\cup\dots\cup I_{r-1},I_r),\ \sigma(J_0,J_r)+\frac{r(r+1)}{2}-\sigma(J_1\cup\dots\cup J_{r-1},J_r).$$

Adding these two numbers and then using (3) we obtain

$$\sigma(I_0,I_r)-\sigma(I_1\cup\dots\cup I_{r-1},I_r)+\sigma(J_0,J_r)-\sigma(J_1\cup\dots\cup J_{r-1},J_r)+r(r+1)=$$
$$=\sigma(I_0,I_r)-\sigma(I_1,I_r)-\dots-\sigma(I_{r-1},I_r)+\sigma(J_0,J_r)-\sigma(J_1,J_r)-\dots-\sigma(J_{r-1},J_r)+$$
$$+r(r+1),$$

hence this number and

$$\sum_{k=0}^{r-1}[\sigma(I_k,I_r)+\sigma(J_k,J_r)]$$

are of the same parity. It follows that

$$\overline{D}\begin{bmatrix} I_1\cup\dots\cup I_{r-1} \\ J_1\cup\dots\cup J_{r-1}\end{bmatrix}=$$
$$=\sum_{J_r}(-1)^{\sum_{0\le k\le r-1}[\sigma(I_k,I_r)+\sigma(J_k,J_r)]}D\begin{bmatrix} I_r \\ J_r\end{bmatrix}\overline{D}\begin{bmatrix} I_1\cup\dots\cup I_r \\ J_1\cup\dots\cup J_r\end{bmatrix},$$

where J_r runs over all subsets of $\{1,\dots,n\}\setminus(J_1\cup\dots\cup J_{r-1})$ (with the same number of elements as I_r).

Replacing this in the right side of (7), according to the fact that

$$sgn(I_1,\dots,I_{r-1},J_1,\dots,J_{r-1})\cdot(-1)^{\sum_{0\le k\le r-1}[\sigma(I_k,I_r)+\sigma(J_k,J_r)]}=$$
$$=(-1)^{\sum_{0\le s<t\le r-1}[\sigma(I_s,I_t)+\sigma(J_s,J_t)]}\cdot(-1)^{\sum_{0\le k\le r-1}[\sigma(I_k,I_r)+\sigma(J_k,J_r)]}=$$
$$=(-1)^{\sum_{0\le s<t\le r}[\sigma(I_s,I_t)+\sigma(J_s,J_t)]}=sgn(I_1,\dots,I_r,J_1,\dots,J_r),$$

we get (6) and the theorem is proved.

A consequence

Suppose, in particular, that $I_1,\dots,I_r$ make a partition of $\{1,\dots,n\}$. Then $J_1,\dots,J_r$ also make a partition of $\{1,\dots,n\}$. In this case we have $\overline{D}\left[\begin{array}{c} I_1\cup\cdots\cup I_r\\ J_1\cup\cdots\cup J_r\end{array}\right]=1$ so that our theorem takes the form

$$(8)\qquad D=\sum_{J_1,\dots,J_r} sgn(I_1,\dots,I_r,J_1,\dots,J_r)\prod_{s=1}^{r} D\left[\begin{array}{c} I_s\\ J_s\end{array}\right].$$

Define a permutation $\nu\in S_n$ in the following way. If $I_k=\{i_{k,1},\dots,i_{k,m_k}\}$, where $1\le i_{k,1}<\cdots<i_{k,m_k}\le n$ and $J_k=\{j_{k,1},\dots,j_{k,m_k}\}$, where $1\le j_{k,1}<\cdots<j_{k,m_k}\le n$, then we set $\nu(i_{k,s})=j_{k,s},\ (k=1,\dots,r,\ s=1,\dots,m_k)$.

We want to prove that $sgn(I_1,\dots,I_r,J_1,\dots,J_r)$ is equal to the sign of ν, where the sign of ν is as usual the number of inversion of ν. Being $I_1\cup\cdots\cup I_r=\{1,\dots,n\}$, and $J_1\cup\cdots\cup J_r=\{1,\dots,n\}$, (4) implies that the sum $\sum_{s=1}^{r}[\sigma(I_0,I_s)+\sigma(J_0,J_s)]$ has the form $2(1+\cdots+n)+2p$, for some integer p, hence this sum is an even number. It follows that $sgn(I_1,\dots,I_r,J_1,\dots,J_r)$ depends only of the parity of the number

$$(9)\qquad \sum_{0<s<t\le r}[\sigma(I_s,I_t)+\sigma(J_s,J_t)].$$

Let $i,j\in\{1,\dots,n\},(i<j)$ be arbitrary. If $i,j\in I_s$ for some s, then by the definition of ν the pair (i,j) does not make an inversion of ν. On the other hand, the pairs (i,j), and $(\nu(i),\nu(j))$ does not change the number (9).
Suppose that $i\in I_s,\ j\in I_t,\ (s\ne t)$. The sum (9) now contains either the term $\sigma(I_s,I_t)+\sigma(J_s,J_t)$ if $s<t$ or the term $\sigma(I_t,I_s)+\sigma(J_t,J_s)$ if $t<s$. If $\nu(i)<\nu(j)$ then, in the first case the pairs (i,j), and $(\nu(i),\nu(j))$ increase the sum $\sigma(I_s,I_t)+\sigma(J_s,J_t)$ for 2, while in the second case the sum $\sigma(I_t,I_s)+\sigma(J_t,J_s)$ remains unchanged.
It follows that the pairs $(i,j),\ (\nu(i),\nu(j))$ add to the sum (9) either 0 or 2, if (i,j) does not make an inversion of ν. If, on the other hand, the pair (i,j) makes an inversion of ν then either $\sigma(I_s,I_t)+\sigma(J_s,J_t)$ or $\sigma(I_t,I_s)+\sigma(J_t,J_s)$ is increased for 1, so that the pairs $(i,j),\ (\nu(i),\nu(j))$ add 1 to the number (9).
From the preceding we conclude that the number (9) has the equal parity as the sign of ν.

Let $I_1,\dots,I_r$ be a fixed partition of $\{1,\dots,n\}$. For any $\nu\in S_n$ denote $\nu(I_k)=J_k,\ (k=1,\dots,r)$. From the preceding and (8) we obtain

$$D=\sum_{\nu\in S_n} sgn(\nu)\prod_{s=1}^{r} D\left[\begin{array}{c} I_s\\ J_s\end{array}\right],$$

which is a well-known extension of Laplace's expansion theorem in [1].

References

[1] I. V. Proskuryakov: Problems in Linear Algebra Mir, Publisher, Moscow, 1978.