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A SPECTRAL PROPERTY OF REGULAR AND SEMIREGULAR GRAPHS

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ABSTRACT. Let G be a connected graph on n vertices $v_1, v_2, \ldots, v_n$ and let $\delta(v_i)$ be the degree of the vertex v_i. If $\left(\sqrt{\delta(v_1)}, \sqrt{\delta(v_2)}, \ldots, \sqrt{\delta(v_n)}\right)^t$ is an eigenvector of the $(0,1)$-adjacency matrix of G, then G is either regular or (bipartite) semiregular.

Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The number of first neighbors of the vertex v_i is the degree of this vertex and is denoted by $\delta(v_i)$. The column–vector $\left(\sqrt{\delta(v_1)}, \sqrt{\delta(v_2)}, \ldots, \sqrt{\delta(v_n)}\right)^t$ is denoted by $\vec{\delta}(G)$.

A graph G is said to be regular (or, more precisely: r-regular) if there exists an integer r, such that for all $v \in V(G)$, $\delta(v) = r$. A bipartite graph G with bipartition $V(G) = V_1(G) \cup V_2(G)$ is said to be semiregular (or more precisely: (r_1, r_2)-semiregular) if there exist integers r_1 and r_2, such that for all $v \in V_1(G)$, $\delta(v) = r_1$ and for all $v \in V_2(G)$, $\delta(v) = r_2$.

Some time ago graphs possessing a peculiar spectral property were examined [2].

Definition. A graph G is said to be SQR if $\vec{\delta}(G)$ is one of its eigenvectors, i. e., if the equality $A(G)\,\vec{\delta}(G) = \lambda\,\vec{\delta}(G)$ is obeyed for some λ, where $A(G)$ is the $(0,1)$-adjacency matrix of G.

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Recall that the adjacency matrix is defined as

$$A(G) = ||A(G)_{ij}|| \quad ; \quad A(G)_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G) \\ 0 & \text{otherwise} . \end{cases}$$

It is immediately verified that regular and semiregular graphs are SQR. In the paper [2] it was conjectured that no other connected graph is SQR. We now show that, indeed, this conjecture is true, i. e., we prove the following:

Theorem. *Let G be a connected graph. Then $\vec{\delta}(G)$ is an eigenvector of G if and only if G is either regular or (biparitite) semiregular.*

Proof of the Theorem

From the Definition it immediately follows [2] that a graph G is SQR if the equality

$$\lambda \sqrt{\delta(v_i)} = \sum_{(v_i,v_j)\in E(G)} \sqrt{\delta(v_j)} \tag{1}$$

holds for all $i = 1, 2, \ldots, n$. (Recall [1] that λ is the element of the graph spectrum, corresponding to the eigenvector $\vec{\delta}(G)$.)

By direct checking it can be verified [2] that all regular and all semiregular graphs satisfy Eq. (1). This implies the "if" part of Theorem 1.

Consider an arbitrary connected graph G. Let x be a vertex of G, such that for all $v \in G$, $\delta(x) \leq \delta(v)$. Denote $\delta(x)$ by a, $a \geq 1$.

Let $y_1, \ldots, y_a$ be the first neighbors of x, labelled so that for all $i = 1, \ldots, a$, $\delta(v_1) \geq \delta(v_i)$. Denote $\delta(v_1)$ by b, $b \geq 1$.

If $b > 1$, let the first neighbors of y_1 other than x be $z_1, \ldots, z_{b-1}$.

With the above specified notation, the application of Eq. (1) to the vertices x and y (i. e., choosing in Eq. (1) $v_i = x$ and $v_i = y$) yields:

$$\lambda \sqrt{a} \;=\; \sqrt{b} + \sum_{i=2}^{a} \sqrt{\delta(v_i)} \tag{2}$$

$$\lambda \sqrt{b} \;=\; \sqrt{a} + \sum_{i=1}^{b-1} \sqrt{\delta(z_i)} \,. \tag{3}$$

If $a = 1$, then the sum on the right–hand side of (2) does not exist and has, formally, to be set equal to zero. Similarly, if $b = 1$, then the sum on the right–hand side of (3) is zero.

Expressing λ from (2) and substituting it into (3) one obtains:

$$(b-1) + \sum_{i=2}^{a} \left[\sqrt{b\,\delta(y_i)} - 1 \right] = \sum_{i=1}^{b-1} \sqrt{a\,\delta(z_i)} \,. \tag{4}$$

Consider first the left–hand side of (4) and observe that because of $\delta(y_i) \leq b$, the term $\sqrt{b\,\delta(y_i)}$ is less than or equal to b. Then, assuming $a > 1$, we have

$\sum_{i=2}^{a}\left[\sqrt{b\,\delta(y_i)}-1\right] \leq (a-1)(b-1)$ and, consequently,

$$(b-1)+\sum_{i=2}^{a}\left[\sqrt{b\,\delta(y_i)}-1\right] \leq a(b-1)\,. \tag{5}$$

Clearly, relation (5) holds, as an equality, also if $a=1$. If $a>1$ then equality in (5) will occur if and only if $\delta(y_1)=\delta(y_2)=\cdots=\delta(y_a)=b$.
Consider now the right–hand side of (4). Because of $\delta(z_i)\geq a$, the term $\sqrt{a\,\delta(z_i)}$ is greater than or equal to a. Then, assuming $b>1$, we have

$$\sum_{i=1}^{b-1}\sqrt{a\,\delta(z_i)} \geq a(b-1)\,. \tag{6}$$

Again, relation (6) holds, as equality, also if $b=1$. If $b>1$ then equality in (6) will occur if and only if $\delta(z_1)=\cdots=\delta(z_{b-1})=\delta(x)=a$.
Comparing (5) and (6) we see that the relation (4) will be obeyed if and only if equality holds in both (5) and (6). This happens in the following four cases:
Case 1. $a=1$, $b=1$;
Case 2. $a=1$, $b>1$ and $\delta(z_1)=\cdots=\delta(z_{b-1})=\delta(x)=1$;
Case 3. $a>1$, $b=1$;
Case 4. $a>1$, $b>1$ and $\delta(y_1)=\delta(y_2)=\cdots=\delta(y_a)=b$ and $\delta(z_1)=\cdots=\delta(z_{b-1})=\delta(x)=a$.
In Case 1 G is the 1-regular graph (possessing two vertices). In Case 2 G is the $(b+1)$-vertex star, i. e., the connected $(1,b)$–semiregular graph. Similarly, in Case 3 G is the $(a+1)$-vertex star, a $(1,a)$-semiregular graph. Thus in Cases 1–3 the proof of the "only if" part of Theorem is done. Remains the Case 4.
In Case 4 all the vertices $y_1,y_2,\ldots,y_a$ adjacent to the vertex x have equal degrees. In view of this, whatever condition must be obeyed by y_1 (in order that Eq. (1) be satisfied), must also be obeyed by $y_2,\ldots,y_a$. All first neighbors of y_1 must have equal degrees. Therefore, all first neighbors of y_i , $i=1,2,\ldots,a$, must have equal degrees, equal to $\delta(x)$. Because x was an arbitrarily chosen vertex of G, whatever holds for x and its first neighbors, must hold for all vertices of G. This, in particular, implies that all vertices of G are either of degree a or of degree b.
If $a=b$ then all vertices of G have equal degrees. Hence G is regular. If $a\neq b$ then any vertex of degree a is adjacent to vertices of degree b and vice versa. Hence G is bipartite, (a,b)-semiregular.
This completes the proof of the Theorem. □
Denote by $\mathcal{N}_0$ the set of all non-negative integers.
Corollary 1. *Let G be SQR. Then the eigenvalue associated with the eigenvector $\vec{\delta}(G)$ is of the form $\lambda=\sqrt{k}$, $k\in\mathcal{N}_0$.*
Corollary 2. *Let G be SQR, but not connected. If $\lambda\in\mathcal{N}_0$, then each component of G is a λ-regular graph or an (r_1,r_2)-semiregular graph, such that $r_1 r_2=\lambda^2$, or an isolated vertex. If $\lambda\notin\mathcal{N}_0$, then each component of G is an (r_1,r_2)-semiregular graph, such that $r_1 r_2=\lambda^2$, or an isolated vertex.*

References

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