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A REVIEW OF SOME COEQUALITY RELATIONS ON SEMIGROUPS WITH APARTNESS

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ABSTRACT. In this note we shall expose a review of some relations and subsets of a semigroup with apartness.

Introduction

In the classical mathematics an object x exists if we can prove the impossibility of its nonexistence. The constructive mathematician must be with have an algorithm that constructs the object x before he recognizes that x exists. In the Bishop's constructive mathematics ([1], [4], [9]) and in the Brouwer's intuitionism ([5], [24]) the notion of an algorithm is taken as primitive. A set $(S, =)$ is defined when we describe haw to construct its members from objects that have been construct prior to S, and describe what it means for two members of S to be equal. Following Bishop we regard the equality relation on a set as conventional. We regard the relation of inequality $\neq$ as conventional. Every set admits this denial inequality. Some sets admit other natural inequalities.

A consistent, symmetric relation is a diversity relation ([9]) on $(S, =)$ if it is extensional. [1] A diversity relation is an apartness iff $\forall(a, b, cS)(a \neq\Rightarrow a \neq b \vee b \neq c)$. If S and T are sets, then a function from S to T is a rule that assigns to each element a of S an element $f(a)$ of T, and is extensional in sense $x = y \Rightarrow f(x) = f(y)$. A function f is strongly extensional ([9], [23]) iff $f(x) \neq f(y)$ implies $x \neq y$

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and f is an embedding ([23], [24]) iff $x \neq y \Rightarrow f(x) \neq f(y)$. Let Y a subset of the set $(S, =, \neq)$ and let x be an element of S. We define $x \# Y \Leftrightarrow (\forall y \in Y)(x \neq y)$, and $Y' = x \in X : x \# Y$.

In a series of several papers [8], [15]-[21] this author argues of semigroup with apartness. Particularly, he described some strongly extensional consistent subsets of semigroup and some anticongruence on semigroup with apartness In this paper it is made retrospective some relations and strongly extensional consistent subsets of semigroup with apartness that they are been under attention of this author in the several past years. Almost each of them satisfies the following conditions:

$(*) q \subseteq \neq,\ q * q \subset = q, q \cap q^{-1} = \emptyset.$

Some of them and another condition

$(**) \neq \subset = q \cup q^{-1.}$

It is seems that there exist reasons to research those relations.

Coequality relation

Let us remind ourselves how we come to the idea of coequality relation on set with apartness and anticongruence on algebraic structures with apartness. Let $(R, =, not\ =, +, 0, \cdot, 1)$ be a commutative ring in the sense of books [5], [9], [23], [24] and papers [7], [10], [11], [21]. A subset Q of R is a coideal of R iff $0 \# Q, -x \in Q \Rightarrow x \in Q, x + y \in Q \Rightarrow x \in Q \vee y \in Q, xy \in Q \Rightarrow x \in Q \wedge yQ$. Coideal of commutative ring with apartness were first defined and studied by William Ruitenburg in 1982.([23]) After that, coideals (anti-ideals) of ring were studied by A.S.Troelstra and D. van Dalen in their very famous monograph [24]. Coideals of commutative ring studied by this author in his several papers (for example: [11],[14],[15], [21]). He proved, in his paper [11], if Q is a coideal of R, then the relation q on R, defined by $(x, y) \in q \Leftrightarrow x - y \in Q$, satisfies the following conditions:

(1) $(\forall a \in R)((a, a) \# q)$, (consistent)
(2) $(\forall a, b \in R)((a, b) \in q \Rightarrow (b, a) \in q)$, (symmetric)
(3) $(\forall a, b, c \in R)((a, c) \in q \Rightarrow (a, b) \in q \vee (b, c) \in q)$, (cotransitive)
(4) $(\forall a, b, c, d \in R)((a + c, b + d) \in q \Rightarrow (a, b) \in q \vee (c, d) \in q)$,
(5) $(\forall a, b, c, dR)((ac, bd) \in q \Rightarrow (a, b) \in q \vee (c, d)q)$.

The relation q on R, which satisfies the above conditions, is called anticongruence on R ([11],[14],[21]). Conversely, if q is an anticongruence on a ring R, then the set $Q = a \in R : (a, o) \in q$ is a coideal of R ([11]). Let J be an ideal of R and let Q be a coideal of R. Ruitenburg, in his dissertation ([23]), first stated a demand that it must be $J \subseteq \neg Q$. This condition is equivalent with the following condition

$$(\forall a, b \in R)(a \in J \wedge b \in Q \Rightarrow a + b \in Q).$$

In this case, we say that J and Q are compatible ([11]). More information on commutative ring with apartness readers can find in book [5], [9] and [23] and in the paper [7], [10], [11], [15] and [21]. If e is the congruence on the ring R, determined by the ideal J, and if q is the anticongruence on R, determined by Q, then J and Q are compatible iff

$$(\forall a, b, c \in R)((a, b) \in e \wedge (b, c) \in q \Rightarrow (a, c) \in q).$$

In this case, we say that relation e and q are also compatible. We can reduce number of demand (1)-(5) in the definition of anticongruence on the ring R. A relation q on a set $(R, =, \neq)$ that satisfies the condition (1)-(3), is called a coequality relation on R ([3],[12]). Coequality relation was first defined and studied M.Bozic and D.A.Romano in his article [3]. More information on coequality relation readers can find in the article [12], [14], [19] and [22].

The filled product of relation f and relation g on a set $(S, =, \neq)$ is the relation $g * f$, defined in the articles ([12]) on this way:

$$g * f = \{(x, z) \in S \times S : (\forall y \in S)((x, y) \in f \vee (y, z) \in g)\}.$$

The filled product is associative and $(g * f)^{-1} = f^{-1} * g^{-1}$ holds. For n (≥ 2) let $n^f = f * f * ... * f(n \text{ factors})$. Put $1^f = f$. By $c(f)$ we denote the intersection $\cap_{n \in N} n^f$. For this notation we have a very important construction given by the following result:

Theorem 1. *.([12],[14]) Let f be a relation on a set (S,=,). The relation c(f) is a cotransitive relation on S.*

The relation $c(f)$ is called the cotransitive fulfillment of the relation f. As consequences of the above mentioned theorem, we have the following results:

Corollary 1. *([8],[12]) Let f be a relation on a set S. Then the relation $c(\neq \cap(f \cup f^{-1)})$ is a coequality relation on S.*

Corollary 2. *([8],[12],[14]) Let e be an equality relation on a set S. The relation $c(e')$ is a maximal coequality relation on S compatible with e.*

Remark 1. *Note that a relation $q \subseteq S \times S$ on a set $(S, =, \neq)$ is a coequality relation on S iff*

$$q \subseteq \neq, \ q^{-1} = q, q \subseteq q * q.$$

Anticongruence on semigroup with apartness

Let $S = (S, =, \neq, \cdot, 1)$ be a semigroup with apartness, where the semigroup operation is strongly exstensional in the following sense

$$(\forall a, b, x, y \in S)(ax \neq by \Rightarrow a \neq b \vee x \neq y),$$

and let q be a coequality relation on S.

We say that it is:

(1) a *right anticongruence) on S iff:*

$$(\forall a, b, x \in S)((xa, xb) \in q \Rightarrow (a, b) \in q);$$

(2) a left anticongruence) on S iff:

$$(\forall a, b, y \in S)((ay, by) \in q \Rightarrow (a, b) \in q).$$

The coequality relation q on a semigroup S is *anticongruence* on S iff it is right and left anticongruence on S. The notion of (left, right) anticongruence was first defined and studied by the author in several his papers ([11] - [18]). Semigroup with apartness was first defined and studied by Arend Heyting. After that, several authors have worked on this important topic as for example Johnstone ([7]), Mulvey ([10]), Ruitenburg ([23]), Troelstra and van Dalen ([24]), and Romano.

We start with the following theorems in which we establish connection between anticongruence and congruence in semigroup:

Theorem 2. *([17])Let q be a left coequality relation on a semigroup S. Then the relation q' is a right congruence on S.*

Symmetrically, we have:

Theorem 3. *([17])Let q be a right anticongruence on a semigroup S. Then the relation q' is a left congruence on S.*

As corollary of these theorems we have the statements:

Theorem 4. *([17]) If q is an anticongruence on a semigroup S, then the relation q' is a congruence on S.*

Beside that, we have the assertion that family $S/(q', q) = aq' : a \in S$, where q is an anticongruence on a semigroup S, is a semigroup:

Theorem 5. *([8],[17], [22]) Let q be an anticongruence on a semigroup S. Then the set $S/(q', q)$ is a semigroup with*

$$aq' = bq' \Leftrightarrow (a, b) \# q \quad aq' \neq bq' \Leftrightarrow (a, b) \in q, \\ aq' \cdot bq' = abq'$$

Let q be an anticongruence on a semigroup S. The following theorem on the family $S/q = \{aq : a \in S\}$ is very interesting:

Theorem 6. *([8],[17],[22]) The family S/q is a semigroup with*

$$aq = bq \Leftrightarrow (a, b) \# q, aq \neq bq \Leftrightarrow (a, b) \in q, \\ aq \cdot bq = abq.$$

Corollary 3. *Let q be an anticongruence on a semigroup S. Then the mapping $p : S \rightarrow S/q$ defined by $p(a) = aq$, is a strongly extensional epimorphism of semigroups.*

Reversed result in some sense given by the following theorem:

Theorem 7. *([8], [17], [22]) If $f : S \rightarrow T$ be a strongly extensional homomorphism of semigroups with apartness, the relation $q = \{(x, y) \in S \times S : f(x) \neq f(y)\}$ is an anticongruence on S. Beside that, if the appartness on T is tight, then there is the strongly extensional embedding monomorphism $g : S/q \rightarrow T$, defined by $g(aq) = f(a)(aq \in S/q)$, such that the diagram*

$$\begin{array}{ccc} S & \rightarrow & T \\ \downarrow & \nearrow & \\ S/q & & \end{array}$$

commutes.

In the next statement we will give explanation about two coequalityies α and β on a semigroup $(S, =, \neq, \cdot)$ such that $\beta \subseteq \alpha$.

Theorem 8. *([19]) Let α and β be anticongruences on a semigroup $(S, =, \neq, \cdot)$ with apartness such that $\beta \subseteq \alpha$. . Then the relation β/α on X/α, defined by $\beta/\alpha = \{(xq, yq) \in X/\alpha \times X/\alpha : (x, y) \in \beta$, is an anticongruence on $(S, =, \neq, \cdot)/\alpha$ and there exists the strongly extensional and embedding bijection $f : ((S, =, \neq, \cdot)/\alpha)/(\beta/\alpha) \to (S, =, \neq, \cdot)/\beta$.*

This section we finish with the following theorem on a construction of anticongruence on semigroup based by given coequality relation.

Theorem 9. *([17], [22]) Let q be a coequality relation on a semigroup S. The relation $q^* = \{(a, b) \in S \times S : (\exists x \in S)(xa, xb) \in q\}$ is a right anticongruence on S such that $q \subseteq q^*$. That relation q^* is minimal extension of q.*

Similarly, we have

Theorem 10. *Let q be a coequality relation on a semigroup S. The relation ${}^*q = \{(a, b) \in S \times S : (\exists y \in S)(ay, by) \in q\}$ is a left anticongruence on S such that $q \subseteq^* q$. That relation is minimal extension of q.*

As corollary of theorem 9 and 10, we have

Theorem 11. *([11]) Let q be a coequality relation on a semigroup S. Then the relation $q^{**} = \{(a, b) \in S \times S : (\exists x, y \in S)(xay, xby) \in q\}$ is an anticongruence on S such that $q \subseteq q^{**}$. That relation q^{**} is minimal extension of q.*

In this section we introduce the notions of principal (left, right) consistent subsets of semigroup S.
Let T be a subset of semigroup S. We say that:
T is a right consistent subset of S ([2], [6]) iff

$$(\forall x, y \in S)(xy \in T \Rightarrow y \in T);$$

T is a left consistent subset of S ([2], [6]) iff

$$(\forall x, y \in S)(xy \in T \Rightarrow x \in T);$$

T is a consistent subset of S ([2], [6]) (or a coideal of S) iff

$$(\forall x, y \in S)(xy \in T \Rightarrow x \in T \vee y \in T);$$

Consistent subsets of semigroup are very important in semigroup theory. They are studied in many books on semigroups, on example, in book [2] and [6]. There exists interesting connection between consistent subsets and anticongruence on semigroup: If a set T is a strongly extensional consistent subset of a semigroup S, then the relation q on S, defined by $(x, y) \in q \Leftrightarrow (x \neq y) \wedge (x \in T \vee y \in T)$ is a anticongruence on S ([22]).

Our first statements in this section are on construction strongly extensional subsets of a semigroup. We can construct (lefr, right) consistent subset in a given semigroup S. Those ideas are presented in the following statements:

Theorem 12. *([8], [17], [22]) Let a be an element of a semigroup S. Then set $L_{(a)} = \{x \in S : x \# Sa\}$ is a right consistent subset of S such that*

(1) $a \# L_{(a)}$;
(2) $L_{(a)} \neq \emptyset \Rightarrow 1 \in L_{(a)}$;
(3) *If the element a is invertible, then* $L_{(a)} = \emptyset$;
(4) $(\forall x \in S)(L_{(a)} \subseteq L_{(xa)})$;
(5) $(\forall x \in S)\neg(xa \in L_a)$.

Symmetrically, we have

Theorem 13. *([8], [17], [22]) Let a be an element of a semigroup S. Then the set $R_{(a)} = \{x \in S : x \# aS\}$ is a left consistent subset of S such that*

(1) $a \# R_{(a)}$;
(2) $R_{(a)} \neq \emptyset \Rightarrow 1 \in R_{(a)}$;
(3) *If the element a is invertible, then* $R_{(a)} = \emptyset$;
(4) $(\forall x \in S)(R_{(a)} \subseteq R_{(xa)})$;
(5) $(\forall y \in S)\neg(ay \in R_a)$.

Theorem 14. *([8], [17], [22]) Let a and b be elements of a semigroup S. Then the set $C_{(a)} = \{x \in S : x \# SaS\}$ is a consistent subset of S such that*

(1) $a \# C_{(a)}$;
(2) $C_{(a)} \neq \emptyset \Rightarrow 1 \in C_{(a)}$;
(3) *If the element a is invertible, then* $C_{(a)} = \emptyset$;
(4) $C_{(a)} \cup C_{(b)} \subseteq C_{(ab)}$.

The consistent subsets $L_{(a)}, R_{(a)}, C_{(a)}$ are called *principal (right, left) consistent subsets* ((right, left) coideal) of S generated by the element a. Another construction of strongly extensional consistent subset in a given semigroup is presented in the paper [17] and [22].

Some quasi-antiorder relations on semigroup

Now, we need some notions in the semigroup theory. We took them from the book [2] and [6]. Element a of S is an idempotents of S iff $a^2 = a$. $E(S)$ will denote the set of all idempotents of S. The semigroup S is a band iff $E(S) = S$. A band S is a right (left) zero band iff

$$(\forall a, b \in S)(ab = b)((\forall a, b \in S)(ab = a)).$$

Commutative band is semilattices. Let q be an anticongruence on S. We say that q is (right, left zero band anticongruence) *semilattices anticongruence on S iff $S/(q', q)$ is a (right, left zero band) semilattice. If q' is a matrix congruence on S, i.e. if*

$$(\forall a, b \in S)((a, a^2) \# q \wedge (aba, a) \# q)$$

we say that q is a matrix anticongruence on S.

We introduce the following relations

$$(a,b) \in l \Leftrightarrow b \in L_{(a)}, \quad (a,b) \in r \Leftrightarrow b \in R_{(a)}, \quad (a,b) \in s \Leftrightarrow b \in C_{(a)},$$

and

$$L = l \cup l^{-1},\ R = r \cup r^{-1},\ S = s \cup s^{-1},$$
$$p = l \cap l^{-1}\ q = r \cap r^{-1},\ h = s \cap s^{-1},\ m = p \cap q.$$

In the following theorem we describe these relations and their constructive fulfillment

Theorem 15. *([15]) The relation $c(l)$ is a consistent and contransitive on S and the relation $q_1 = c(l) \cup c(l^{-1})$ is a left anticongruence on S.*

Dually, we have

Theorem 16. *([18]) The relation $c(r)$ is a consistent and contransitive on S and the relation $q_2 = c(r) \cup c(r^{-1})$ is a right anticongruence on S.*

Analogously, we have the following results on relation $c(s)$.

Theorem 17. *([13])*

(1) *The relation $q_3 = c(s) \cup c(s^{-1})$ is a coequality relation on S.*

(2) *([18])] The relation $c(h)$ is an anticongruence on S such that*

$$(\forall a,b,x,y \in S)((a,b) \in c(h) \Rightarrow (xa,xb) \in c(h) \wedge (ay,by) \in c(h))$$

and

$$(\forall a \in S)(\forall n \in \mathbb{N})((a,a^n)\ \#\ c(h)).$$

For relation p and m we have statements:

Theorem 18. (1) ([17]) *The relation $c(p)$ is a right zero band anticongruence on S.*

(2) *The relation $c(q)$ is a left zero band anticongruence on S.*

Theorem 19. *([16]) The relation $c(m)$ is a matrix anticongruence on S.*

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