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On Some Algebraic and Differential Equation in the Space of Generalized Functions

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Abstract

Here we consider existence of distributional solutions for some algebraic equations and find general solution for two ordinary differential equations in the space of generalized functions.

1. Introduction

Denote $\mathcal{D}'(\mathbb{R})$ to be the space of generalized functions (or distributions), $\mathcal{D}(\mathbb{R})$ the test function space, and $C^\infty(\mathbb{R})$ the space of infinitely differentiable functions on $\mathbb{R}$. It is well-known that for $m \in \mathbb{N}$, the equation

$$x^m u = 1 \tag{1}$$

has a solution in $\mathcal{D}'(\mathbb{R})$(see [1]); also, for differential equation

$$u' + xu = 0 \tag{2}$$

it is easy to derive the general solution, since it is an elliptic–type equation with infinitely differentiable coeffitients (see [2]); nevertheless, we will derive general solution for this equation in Example 2. However, the algebraic equation $\alpha u = 1$, where

$$\alpha(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{3}$$

intrinsicly differs from the equation (1), because $(\forall k \in \mathbb{N} \cup \{0\})\, \alpha^{(k)}(0) = 0$. Also, the equation $x^3u' + 2u = 0$ has quite different nature compared to the equation (2). In this paper we will consider the last two equations. Throughout the whole paper, α will denote the function defined by (3). As usual, $< \cdot, \cdot >$ stands for the duality relation betweeen $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$.

2. Algebraic equations

Proposition 1 *The equation*

$$\alpha u = 1 + \sum_{i=0}^{n} C_i \delta^{(i)} \tag{4}$$

has no solution in $\mathcal{D}'(\mathbb{R})$, *where* $C_i \in \mathbb{R}$ $(i = 0, \ldots, n)$. *Thereby* δ *and* $\delta^{(i)}$ $(i = 0, \ldots, n)$ *are Dirac's delta distribution and its derivatives.*

Proof. For $\varepsilon > 0$ introduce

$$f_\varepsilon(x) = \begin{cases} \varepsilon e^{-1/10(x-\varepsilon)^2}, & x > \varepsilon \\ 0, & x \leq \varepsilon. \end{cases}$$

Obviously, $f_\varepsilon \in C^\infty(\mathbb{R})$. Let $\eta \in \mathcal{D}(\mathbb{R})$ be a function such that $\eta \geq 0$ on $\mathbb{R}$ and $\eta = 1$ in a neighborhood of 0. For example, we can use $\eta(x) = 1 + \omega(x)$, where

$$\omega(x) = \begin{cases} e^{-1/(x^2-1)}, & |x| > 1 \\ 0, & |x| \leq 1. \end{cases}$$

Let $\varphi_\varepsilon = \eta f_\varepsilon$. Taking into account that all derivatives of function $f_\varepsilon / \varepsilon$ are uniformly bounded with respect to $\varepsilon \in (0, 1)$ and $x \in \mathbb{R}$, we easily conclude that $\varphi_\varepsilon \to 0$ in $\mathcal{D}(\mathbb{R})$ as $\varepsilon \to 0$. If u satisfies (4), then $< u, \varphi_\varepsilon > \to < u, 0 > = 0$, and, on the other hand

$$< u, \varphi_\varepsilon > = < u, e^{1/x^2} \alpha \varphi_\varepsilon > = < \alpha u, e^{1/x^2} \varphi_\varepsilon > =$$

$$= < 1 + \sum_{i=0}^{n} C_i \delta^{(i)}, e^{1/x^2} \varphi_\varepsilon > = < 1, e^{1/x^2} \varphi_\varepsilon > = \int_{-\infty}^{\infty} e^{1/x^2} \varphi_\varepsilon \, dx \geq$$

$$\geq \int_{2\varepsilon}^{3\varepsilon} \varepsilon e^{1/x^2} e^{-1/10(x-\varepsilon)^2} dx \geq \int_{2\varepsilon}^{3\varepsilon} \varepsilon e^{1/9\varepsilon^2} e^{-1/10\varepsilon^2} dx = \varepsilon^2 e^{1/90\varepsilon^2} \to +\infty,$$

which is a contridiction. □

Corolary 1 *The function* $f(x) = e^{1/x^2}$ *can't be extended to a distribution in* $\mathcal{D}'(\mathbb{R})$.

Proof. The support of $\alpha f - 1$ is $\{0\}$. According to the Schwartz's theorem (see [3]), we have $\alpha f = 1 + \sum_{i=0}^{n} C_i \delta^{(i)}$ for some $n \in N$ and $C_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$. Hence, Proposition 1 yields the result. □

Proposition 2 *If* $\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ *(the Heaviside's function), then the equation*

$$\alpha u = C_0 + C_1 \theta,$$

has a solution in $\mathcal{D}'(\mathbb{R})$ *only for* $C_0 = C_1 = 0$.

Proof. Similarly as we have done in the previous proposition, we have

$$< u, \varphi_\varepsilon >= \int_{-\infty}^{\infty} (C_0 + C_1\theta(x))e^{1/x^2} \varphi_\varepsilon(x)\, dx = (C_0 + C_1) \int_0^{\infty} e^{1/x^2} \varphi_\varepsilon(x)\, dx.$$

But, since $\int_0^\infty e^{1/x^2}\varphi_\varepsilon(x)\,dx \to +\infty$ as $\varepsilon \to 0$, it must be $C_0 + C_1 = 0$. On the other hand, putting $f_\varepsilon(x) = \begin{cases} \varepsilon e^{-1/10(x+\varepsilon)^2}, & x < -\varepsilon \\ 0, & x \geq -\varepsilon. \end{cases}$, $\varphi_\varepsilon = \eta f_\varepsilon$ (for η as in Proposition 1), we obtain

$$< u, \varphi_\varepsilon >= \int_{-\infty}^{\infty} (C_0 + C_1\theta(x))e^{1/x^2} \varphi_\varepsilon(x)\, dx = C_0 \int_{-\infty}^{0} e^{1/x^2} \varphi_\varepsilon(x)\, dx.$$

As above, $\int_{-\infty}^0 e^{1/x^2}\varphi_\varepsilon(x)\,dx \to +\infty$, and we conclude that $C_0 = 0$, i.e. $C_0 = C_1 = 0$. □

Lemma 1 *Define for* $\varepsilon > 0$, $\alpha_\varepsilon(x) = \begin{cases} e^{-1/(|x|-\varepsilon)^2}, & |x| > \varepsilon \\ 0, & |x| \leq \varepsilon \end{cases}$. *Then*

$$(\forall k \in N \cup \{0\}) \quad \alpha_\varepsilon^{(k)} \to \alpha^{(k)} \quad \textit{uniformly on } \mathbb{R},$$

as $\varepsilon \to 0+$.

Proof. Let $\alpha_+(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases}$, $\alpha_-(x) = \begin{cases} e^{-1/x^2}, & x < 0 \\ 0, & x \geq 0 \end{cases}$. Then, for every $k \in \mathbb{N}$ and $x \in \mathbb{R}$, the esitimates

$$\begin{aligned} |\alpha_+^{(k)}(x) - \alpha_+^{(k)}(x-\varepsilon)| &\leq \varepsilon M_{k+1}^+, \\ |\alpha_-^{(k)}(x) - \alpha_-^{(k)}(x+\varepsilon)| &\leq \varepsilon M_{k+1}^+, \end{aligned} \tag{5}$$

where $M_k^+ = \max_{x\in\mathbb{R}} |\alpha_+^{(k)}(x)|$, hold. Clearly, $\alpha = \alpha_+ + \alpha_-$. The fact that $\alpha_\varepsilon(x) = \alpha_+(x-\varepsilon) + \alpha_-(x+\varepsilon)$ for $\varepsilon > 0$, $x \in \mathbb{R}$ and the estimates (5) imply the assertion of the Lemma. □

Let's consider another interesting equation:

Example 1 *The equation* $\alpha u = \delta$ *has no solution in* $\mathcal{D}'(\mathbb{R})$.

Indeed, if we introduce $\varphi_\varepsilon = \eta\alpha_\varepsilon$, $\varphi = \eta\alpha$, where α_ε was defined in Lemma 1 and η in Proposition 1, we conclude, according to Lemma 1, that $\varphi_\varepsilon \to \varphi$ in $\mathcal{D}(\mathbb{R})$ as $\varepsilon \to 0+$. If such $u \in \mathcal{D}'(\mathbb{R})$ exists then it satisfies $< u, \varphi_\varepsilon > \to < u, \varphi >$, which is impossible, because

$$< u, \varphi_\varepsilon > = < u, \alpha e^{1/x^2}\varphi_\varepsilon > = < \alpha u, e^{1/x^2}\varphi_\varepsilon > = < \delta, e^{1/x^2}\varphi_\varepsilon > = 0,$$

and, on the other hand,

$$< u, \varphi > = < u, \alpha\eta > = < \alpha u, \eta > = < \delta, \eta > = \eta(0) \neq 0. \ \square$$

More generally, we have

Theorem 1 *The equation $\alpha u = C_0 + C_1\theta + \sum_{i=2}^{n} C_i\delta^{(i-2)}$ has a solution only for $C_i = 0$, $i = 0, 1, \ldots, n$.*

Proof. First we need

Lemma 2 *For all $x \in \mathbb{R}$, $\quad 0 < \varepsilon_1 < \varepsilon_2 \Rightarrow \alpha_{\varepsilon_1}(x) \geq \alpha_{\varepsilon_2}(x)$.*

Proof. Obvious.

We can now start proving Theorem 1.

We define for $k \in \{0, 1, \ldots, n-2\}$, $\eta_k(x) = \frac{x^k}{k!}\eta$, where η was defined in Proposition 1, and $\varphi_\varepsilon^k = \eta_k\alpha_\varepsilon$, $\varphi^k = \eta_k\alpha$. As we concluded in the previous example, we have

$$< u, \varphi_\varepsilon^k > \to < u, \varphi^k >, \tag{6}$$

and

$$< u, \varphi_\varepsilon^k > = < C_0 + C_1\theta + \sum_{i=2}^{n} C_i\delta^{(i-2)}, e^{1/x^2}\varphi_\varepsilon^k > = < C_0 + C_1\theta, e^{1/x^2}\varphi_\varepsilon^k > =$$

$$= C_0 \int_{-\infty}^{\infty} e^{1/x^2}\eta_k(x)\alpha_\varepsilon(x)\,dx + C_1 \int_0^{\infty} e^{1/x^2}\eta_k(x)\alpha_\varepsilon(x)\,dx.$$

According to Lemma 2, we can apply the Lebesgue's monotone convergence theorem in the last two integrals. Thus,

$$< u, \varphi_\varepsilon^k > \to C_0 \int_{-\infty}^{\infty} \eta_k(x)\,dx + C_1 \int_0^{\infty} \eta_k(x)\,dx, \quad \text{as } \varepsilon \to 0. \tag{7}$$

On the other hand,

$$< u, \varphi^k > = < u, \alpha\eta_k > = < \alpha u, \eta_k > = < C_0 + C_1\theta + \sum_{i=2}^{n} C_i\delta^{(i-2)}, \eta_k > =$$

$$= C_0 \int_{-\infty}^{\infty} \eta_k(x)\,dx + C_1 \int_0^{\infty} \eta_k(x)\,dx + \sum_{i=2}^{n} (-1)^{i-2} C_i \eta_k^{(i-2)}(0).$$

The last equality, (6) and (7) read for $k \in \{0, 1, \ldots, n-2\}$,

$$\sum_{i=2}^{n} (-1)^{i-2} C_i \eta_k^{(i-2)}(0) = 0.$$

Applying $\eta_k^{(i-2)}(0) = \delta_{i-2,k}$ to the last equation, where $\delta_{i,j}$ is the Kronecker's delta symbol, we obtain that $C_i = 0$ for $i = 2, 3, \ldots, n$. Therefore, u satisfies the equation $\alpha u = C_0 + C_1\theta$ which has a solution only for $C_0 = C_1 = 0$ (see Proposition 2). □

3. Differential equations

Lemma 3 *Let $a \in C^\infty(\mathbb{R})$ such that $a > 0$ on $\mathbb{R}$. Then for each $f, g \in \mathcal{D}'(\mathbb{R})$, the relation*

$$af = g \Longleftrightarrow f = \frac{1}{a}g$$

holds.

Proof. Trivial. □

Example 2 *Let's solve the equation $u' + xu = 0$ in $\mathcal{D}'(\mathbb{R})$. Multiplying this equation by $a(x) = e^{x^2/2}$, we have $(e^{x^2/2}u)' = 0$, i.e. $e^{x^2/2}u = C$. Hence, according to Lemma 3, the general solution has the form $u = Ce^{-x^2/2}$.*

remark 1 *$u' + xu = 0$ is elliptic on $\mathbb{R}$, hence all distributional solutions of this equation are in fact classical solutions, hence the result.*

However, the procedure showed above can't be applyed to the equation

$$x^3u' + 2u = 0, \tag{8}$$

because the term x^3 vanishes at 0. Also, the equation is not elliptic at $x = 0$. The following assertion holds:

Theorem 2 *The equation (8) has only trivial solution in $\mathcal{D}'(\mathbb{R})$.*

Multiplying (8) by α, we have

$$x^3\alpha u' + 2\alpha u = 0 \Rightarrow x^3(\alpha u' + \frac{2\alpha}{x^3}u) = 0 \Rightarrow x^3(\alpha u)' = 0.$$

Hence, $(\alpha u)' = C_1\delta + C_2\delta' + C_3\delta''$ and, finally, using uniqueness of the solution of $w' = f$ up to an additive constant,

$$\alpha u = C_0 + C_1\theta + C_2\delta + C_3\delta'.$$

Applying Theorem 1 to the last equation, we obtain that $\alpha u = 0$. Obviously, $\{0\}$ is the support of the distribution u. Then, the Schwartz's theorem yields that u has the form

$$u = \sum_{i=0}^{n} A_i\delta^{(i)}, \tag{9}$$

for some $n \in \mathbb{N}$ and $A_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n)$. Let's prove that $A_i = 0$ $(i = 0, \ldots, n)$. Indeed, if $n < 2$, (8) and (9) imply $x^3u' = 0$ and $x^3u' + 2u = 0$, i.e. $u = 0$. If $n \geq 2$, we have

$$0 = x^3u' + 2u = x^3\Big(\sum_{i=0}^{n} A_i\delta^{(i+1)}\Big) + 2\sum_{i=0}^{n} A_i\delta^{(i)} = -6\sum_{i=2}^{n} A_i\delta^{(i-2)} + 2\sum_{i=0}^{n} A_i\delta^{(i)} =$$

$$= -6\sum_{i=0}^{n-2} A_{i+2}\delta^{(i)} + 2\sum_{i=0}^{n} A_i\delta^{(i)} = 2\sum_{i=0}^{n-2}(A_i - 3A_{i+2})\delta^{(i)} + 2A_{n-1} + 2A_n.$$

Since $\delta, \delta', \ldots, \delta^{(n)}$ are linearly independent, we have $A_{n-1} = A_n = 0$, $A_i - 3A_{i+2} = 0$ for $i = 0, 1, \ldots, n-2$. From the last equations follows $A_i = 0$ $(i = 0, 1, \ldots, n)$. □

References

[1] Gel'fand I.M.,Shilov G.E.,*Generalized functions*, Fizmatgiz.,Moscow, 1958

[2] Hörmander L.,*Linear partial differential equations*, Academic Press, New York, 1963

[3] Schwartz L.,*Théorie des distributions*, Hermann, Paris, 1950

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