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Anti-Topological construction of $\mathbb{R}$

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Abstract

Hereby we reveal a construction of $\mathbb{R}$ dual to the topological one, due to Cantor [1], to complete the spectrum of all structurally possible completions of $\mathbb{Q}$. This construction can be considered as anti-topological because of the horistological structures involved, while in technical details it follows the classical one. [1]

A. Introduction

1. General overview. As a classical background we all learn two ways to obtain $\mathbb{R}$ from $\mathbb{Q}$: The first uses the order properties of $\mathbb{Q}$ exclusively, to realize an order completion based on Dedekind's cuts (see [2]); the other is a topological completion of $\mathbb{Q}$, being based on topological properties of elements like classes of fundamental sequences, continuous fractions, decimal approximations, and so on (see [1],[3], etc.). By this paper we'll show that "this service, usually done by a topology, can also be done by horistology", i.e. the same result can be obtained in third (structurally distinct) way.

The horistological structures (introduced by T. Balan in [7], II, and also presented in [11]) appear as dual, or, if we prefer, even "anti" or opposite to the topological ones. To offer now a relative independence from these papers we'll select the necessary elements in this introductory part. In particular it is easy to see that $\mathbb{Q}$ is one of the simplest examples of uniform horistological space, naturally endowed with a super-additive (briefly S.a.) norm, respectively a S.a. metric, generating a horistological structure which allows to obtain $\mathbb{R}$ in a way similar to the topological constructions. More intuitively, i.e. reflecting some geometric images outside $\mathbb{Q}$ and $\mathbb{R}$, we may distinguish three (manifestely exhaustive!) types of contructions of $\mathbb{R}$, namely elliptic, parabolic and hyperbolic, upon the starting structure we consider on $\mathbb{Q}$: topology, order, recpectively horistology.

2. Remarks. (on Cantor's construction). To justify some features of our construction it is useful to point out the role of symmetry in the topological

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construction. Differently from the Euclidean (uniform) topology of $\mathbb{Q}$, which is symmetric, the topologies to the right, respectively to the left, are essentialy nonsymmetric (see [4],[5]) but they lead to the same completion. The lack of symmetry however imposes a lot of profound changes in the classical construction. For example, if we take the filter $\mathcal{B} = /V_\varepsilon : \varepsilon > 0/$ of entourages $V_\varepsilon = \{(x, y) \in \mathbb{Q}^2 : 0 \le y - x < \varepsilon\}$ to represent a base of the uniform topology to the right on Q, then the Cauchy's condition on a sequence (x_n), namely

$$\forall \varepsilon > 0 \exists n_0 \in N \text{ such that } (\forall)\ m, n \ge n_0 \Rightarrow (x_m, x_n) \in V_\varepsilon$$

makes no sense without restrictions concerning monotony. Similarly, propositions like "convergent sequences are fundamental" cannot be obtained by "introducing the limit between the terms x_m and x_n" any longer. Even definitions, e.g. the equivalence of the Cauchy sequences, based on the relation $(x_m, x_n) \in V_\varepsilon$, must be correspondingly changed. Because the horistologies are always connected to an order (in particular $\le$ on $\mathbb{Q}$), they are strongly nonsymmetric. Consequently it is natural to expect that the *horistological completion* of $\mathbb{Q}$ shall face similar difficulties as using nonsymmetric topologies.

3. Starting facts. In order for us to facilitate the reading in absence of a complete bibliography, we recall the meaning of some basic notations and notions that will be used later on. Firstly, the *natural order* of $\mathbb{Q}$ will be alternatively noted $\le$ and $K \subset \mathbb{Q}^2$; the corresponding *strict order* $<$ may therefore be written K^s. The *section* of K at x is defined by

$$K[x] = \{y \in \mathbb{Q} : (x, y) \in K\}.$$

In particular, $K[0] = \mathbb{Q}_+$ is the *cone* of positive rational numbers. A function $\lceil\cdot\rceil : K[0] \to \mathbb{Q}_+$ is said to be a *supper additive norm* (briefly S.a. norm, see [7] I) if the following conditions hold:

(N_1) $\lceil x \rceil = 0$ if only if $x = 0$;
(N_2) $\lceil \lambda x \rceil = \lambda \lceil x \rceil$ at any $\lambda \in \mathbb{Q}$, and $x \in K[0]$;
(N_3) $\lceil x + y \rceil \ge \lceil x \rceil + \lceil y \rceil$ for any $x, y \in K[0]$.

Every S.a. norm generates a *super-additive metric* $d : K \to \mathbb{Q}_+$ by the usual rule

$$d(x, y) = \lceil y - x \rceil,$$

and this one further genreates a (uniform) *horistology* via the ideal bases of *hyperbolic perspectives* (corresponding in topology to spherical neighbourhoods, see [7] II) defined by

$$H_\varepsilon(x) = \{y \in K[x] : d(x, y) > \varepsilon\}$$

where $\varepsilon > 0$.

In the particular case of $\mathbb{Q}$, the restriction $|\cdot|_-$ of the usual norm to $\mathbb{Q}_+$ reduces to identity, i.e. $|x|_+ = |x| = x$ for any $x \ge 0$. Consequently $|\cdot|_- : \mathbb{Q}_+ \to \mathbb{Q}_+$ is *additive*, that is

$$|x + y|_+ = |x|_+ + |y|_+ \text{ for any } x, y \ge 0,$$

hence taking here $\leq$ instead of $=$, as for a usual sub-additive norm $\|\cdot\|$, or $\geq$, as for a sub-additive norm, say $\lceil\cdot\rceil$, is a question of choice. Choosing sub-aditivity is nowadays a hard tradition, since it is the way to classical (i.e. topological) structures, but hereby we see that the other option is still operative, and it leads to horistology.

Later on we'll refer to the *natural uniform horistology* of $\mathbb{Q}$ in the following sense: Starting with the *natural S.a. norm* $\lceil\cdot\rceil = |\cdot|_+$ we obtain the *natural S.a. metric* defined by $d(x,y) = y - x$ whenever $x \leq y$. Using d we construct the *prospects of size* $\varepsilon(>0)$,

$$\Pi_\varepsilon = \{(x,y) \in K \subset \mathbb{Q}^2 : d(x,y) > \varepsilon\}.$$

The *natural uniform horistology* (briefly u.h.) of $\mathbb{Q}$ is defined by the ideal $\mathcal{H}$ of *prospects*,

$$\mathcal{H} = \{\Pi \subset K : \exists\varepsilon > 0 \text{ s.t.} \Pi \subseteq \Pi_\varepsilon\}.$$

The *hyperbolic perspectives* of radius $\varepsilon(>0)$ and vertex $x(\in\mathbb{Q})$ can be alternatively presented as

$$H_\varepsilon(x) = H(x,\varepsilon) = \Pi_\varepsilon[x].$$

Finally, the *natural horistology* of $\mathbb{Q}$, expressed by the function

$$\chi : \mathbb{Q} \to \mathcal{P}(\mathcal{P}(\mathbb{Q}))$$

which attaches an ideal of *perspectives* to each $x \in \mathbb{Q}$, is defined by the (horistological) uniformity from above, according to the formula

$$\chi(x) = \{P \subset \mathbb{Q} : \exists\, \Pi \in \mathcal{H} \text{ such that } P \subseteq \Pi[x]\}.$$

More explicitly, the assertion "P is a perspective of x" means, in the natural horistology of $\mathbb{Q}$, that

$$\exists\varepsilon > 0 \text{ such that } P \subseteq (x+\varepsilon, \to).$$

In the general framework of horistological spaces we deal with *discreteness, emergence,* and *germs* (see [7] II) as dual notions to *continuity, convergence* , and *limit,* which are specific to topological structures. The efficiency of these notions in studying horistological structures is proved by the fact that any horistology is definable by specifying the emergent nets (or ideals, see [8] and [9]).

In particular, a sequence $\xi : \mathbb{N} \to \mathbb{Q}$ of rational terms $\xi(n) = x_n$ is said to be *emergent* from $x(\in\mathbb{Q})$ iff $\forall n \in N^* \exists\varepsilon > 0$ such that $(\forall) m \leq n \Rightarrow d(x, x_m) > \varepsilon$. If so, we say that x is a *germ* of ξ, and we note $x \to x_n, x = germ\xi$, etc. Because the germs are not unique , we speak of a *set of germs,* which is noted $Germ(\xi)$. The analysis of the structure of this set (see [10]) shows that it is always *negatively conical,* i.e. $x \in Germ(\xi) \Rightarrow K^{-1}[x] \subseteq Germ(\xi)$.

We may distinguish several types of germs. In this respect we remind that $x* \in Germ(\xi)$ is called *proper germ* iff $\{x_n : n \in N\} \notin \chi(x*)$. The set of all proper germs is noted $G(\xi)$; simple examples show that this set can also be large enough. In the particular case of $\mathbb{Q}$, endowed with its natural horistology, the property of the orders of $\mathbb{Q}$ and $\mathbb{N}$ of being total makes things much easier. Thus $Germ(\xi) \neq \emptyset$ whenever the sequence ξ is bounded from below, while $G(\xi)$ is either void or a singleton. The completion of $\mathbb{Q}$ is justified because the voidness of $G(\xi)$ is not acceptable for decreasing and bounded sequences.

B. Constructing R

Our starting point is horistological notion equivalent to the classical Cauchy's condition: **1. Definition.** A sequence is called *horistologically fundamental* (briefly h-fundamental) if

$$\forall n_0 \in N^* \exists \varepsilon > 0 (in \mathbb{Q}) \text{ such that } (\forall)\ m, n \in \mathbb{N},\ n < m \leq n_0 \Rightarrow x_n - x_m > \varepsilon.$$

2. Proposition. A sequence ξ is h-fundamental if and only if it is K^s-decreasing.

Proof. If we suppose that ξ is h-fundamental it is enough to take $m = n_0$ since $x_n > x_m + \varepsilon$ implies $x_n > x_m$.
Conversely, if ξ is strictly decreasing, then, for each $n_0 \in \mathbb{N}$, the above condition is proved with $\varepsilon = \frac{1}{2} min\{x_n - x_m : n < m \leq n_0$. □

3. Proposition. Let $\xi : \mathbb{N} \to Q$ be an h-fundamental sequence. The set $Germ(\xi)$ is nonvoid iff ξ is bounded from below (i.e. the set of values $\xi(N)$ has a lower bound).

Proof. Any germ is a lower bound of the set $\xi(N)$.
Conversely, let μ be a lower bound of $\xi(\mathbb{N}) in \mathbb{Q}$. We claim that μ is a germ of ξ. In fact, for any $n_0 \in N^*$ the number $\varepsilon = d(\mu, x_{n_0})$ is strictly positive. Consequently for any $n \leq n_0$ we have

$$d(\mu, x_n) \geq d(\mu, x_{n_0}) + d(x_{n_0}, x_n) > \varepsilon$$

since d is super-additive.

4. Remark. The germ μ in the above proof is not unique since the set $Germ$ is negatively conical. Emergent sequences are not necessarily h-fundamental because emergence is possible without monotony. On the other hand, bounded and h-fundamental sequences may have proper germs, so it is useful to distinguish different cases by an adequate terminology.

5. Definition. Any K^s-decreasing and bounded sequence $\xi : \mathbb{N} \to Q$ is called *emission*. The proper germ of an emission (if any) is named *emitter*. The set of all emissions in $\mathbb{Q}$ will be noted E, and called *emission power* of Q (since $E \subset \mathbb{Q}^N$).
We say that emission ξ *precedes* another emission η, and we note $\xi \preceq \eta$, if the following condition holds:

$$\forall m \in \mathbb{N} \exists j(m) \in \mathbb{N} \text{ such that } \xi(j(m)) \leq \eta(m).$$

If both $\xi \preceq \eta$ and $\eta \preceq \xi$ hold, then we say that ξ and η are *equivalent*, and we note $\xi \approx \eta$. Finally, if $\xi \preceq \eta$, but $\xi \not\approx \eta$, we say that ξ strictly preceeds η, and we note $\xi \prec \eta$.

Using these terms for the above relations is justified by the following properties:

6. Proposition

(*i*) $\preceq$ is a preorder on E;

(*ii*) $\approx$ is an equivalence on E;

(*iii*) $\prec$ is the strict preorder of $\preceq$ (i.e. $\prec=\preceq^s$).

Proof. (i) Because each sequence ξ from E is K^s-decreasing, we have $x_{n+1} < x_n$ at any $n \in \mathbb{N}$. Consequently $\xi \preceq \xi$, i.e. relation $\preceq$ is reflexive. The transitivity of $\preceq$ follows from that of K^s. The proof of (ii) is routine.
Property (iii) expresses the fact that $\xi \preceq \eta$ is the contrary of $\eta \prec \xi$. To prove (iii) in this form we write the negation of $\xi \preceq \eta$ as:

$$\exists m \in \mathbb{N} \text{ such that } (\forall)n \in \mathbb{N} \Rightarrow \xi(n) > \eta(m).$$

Then obviously $\eta \preceq \xi$, but $\eta \not\approx \xi$.

Passing from emissions to their set of germs is monotonous, i.e.

7. Proposition. For any $\xi, \eta \in E$ we have:

(*i*) If $\xi \preceq \eta$, then $Germ(\xi) \subseteq Germ(\eta)$;

(*ii*) If $\xi \prec \eta$, then $Germ(\xi) \subset Germ(\eta)$;

(*iii*) $Germ(\xi) = Germ(\eta)$ iff $\xi \approx \eta$
(and this is further equivalent to $G(\xi) = G(\eta)$ whenever $G(\xi) \neq \emptyset$ holds).

Proof. (i) Let $x^* \in Germ(\xi)$, and let $m \in \mathbb{N}$ be fixed. Then there is some $j(m) \in \mathbb{N}$ such that $\xi(j(m)) < \eta(m)$, and correspondingly an $\varepsilon > 0$, such that $d(x*, \xi(j(m))) > \varepsilon$. Because d is a S.a. metric, it follows that $d(x*, \eta(m)) \geq d(x*, \xi(j(m))) + d(\xi(j(m)), \eta(m)) > \varepsilon$.

(ii) If $\xi \prec \eta$, then there is some $m \in \mathbb{N}$ such that $\xi(m) < \eta(n)$ for all n in $\mathbb{N}$ (see also (iii) in the previous proposition), hence $\xi(m) \in Germ(\eta) \setminus Germ(\xi)$.

(iii) is a standard consequence of (i) and (ii).

Now we can introduce the *real numbers* as follows:

8. Definition. For any $\xi \in E$, its class of equivalence $\hat{\xi} = \{\eta \in E : \eta \approx \xi\}$ is called *real number*. The set of all real numbers is called *real line*, and we note $E/\approx = \mathbb{R}$.

In particular, for any x from $\mathbb{Q}$, the equivalence class that contains the emission ξ of terms $\xi(n) = x + \frac{1}{n}$, where $n \in \mathbb{N}^*$, is noted $\hat{x}$, and it is called the real number *associated* with x. The function $e : \mathbb{Q} \to \mathbb{R}$, defined by $e(x) = \hat{x}$ is called *canonical embedding*.

C. The Structure of $\mathbb{R}$

In this section we'll extend the order and the algebraic operations from $\mathbb{Q}$ to the above constructed $\mathbb{R}$, to justify that it is a totally and completely ordered field.

1. Definition. If $\xi \approx \eta$ then we consider that the generated real numbers coincide, and we note $\hat{\xi} = \hat{\eta}$. If $\xi \preceq \eta$, then we say that $\hat{\xi}$ is *less than* $\hat{\eta}$, and we note $\hat{\xi} \leq \hat{\eta}$. Similarly, if $\xi \prec \eta$, then $\hat{\xi}$ is said to be *strictly smaller than* $\hat{\eta}$, and we note $\hat{\xi} < \hat{\eta}$.

2. Proposition.

(i) The relations $=$, $\leq$, and $<$ are well defined (i.e. they are stable relative to any change of representatives in the equivalence classes);

(ii) $=$ is an equivalence, and $\leq$ is a total order on $\mathbb{R}$;

(iii) $<$ is the strict order asociated with $\leq$ (i.e. $\leq^s = <$).

Proof. (i) It is easy to see that $\alpha \approx \xi \preceq\approx \omega$ implies $\alpha \preceq \omega$.

(ii) $\leq$ is obviously reflexive, anti-symmetric and transitive. It is a total order on $\mathbb{R}$ because $\preceq$ on E is so.

(iii) ¡ is $(\leq \setminus =)$ because $\prec = (\preceq \setminus \approx)$ on E.

The *sum* of two emissions $\theta = \xi + \eta$ is defined, as usually, by $\theta(n) = \xi(n) + \eta(n)$ at any $n \in \mathbb{N}$.

3. Definition. If $\vartheta = \xi + \eta$, then $\hat{\vartheta}$ is called *sum of the real numbers* $\hat{\xi}$ and $\hat{\eta}$, and it is noted $\hat{\vartheta} = \hat{\xi} + \hat{\eta}$. The resulting binary operation is called *addition of real numbers*.

4. Proposition. The addition on $\mathbb{R}$ has the following properties:

(i) $+$ is well defined on $\mathbb{R}$;

(ii) $(\mathbb{R}, +)$ is a commutative group;

(iii) the order $\leq$ is compatible with $+$.

Proof. (i) Changing the representatives in the classes $\hat{\xi}$ and $\hat{\eta}$ doesn't affect the result $\hat{\xi} + \hat{\eta}$.

(ii) Proving associativity and commutativity is routine. The *null element* of the group is $\hat{0} = e(0)$, i.e. the class containing the emission ϑ of terms $\vartheta(n) = \frac{1}{n}, n \in \mathbb{N}^*$.

To define the *opposite* of a real number $\hat{\xi}$ let us note that taking $\widehat{(-\xi)}$, where $(-\xi)(n) = -\xi(n)$ at any n in N, makes no sense since $-\xi$ isn't emission any more. Therefore we have to imagine a different way to obtain $-\xi$ as for example using the germs of ξ. In fact, we may fix $g_0 \in Germ(\xi)$ and then divide the segment $[g_0, x_0]$ into 2,3, etc., say k equal parts, stopping when

$$g_0 + \frac{1}{k}(x_0 - g_0) \in Germ(\xi).$$

We note the resulting germ g_1. Applying the same construction to $[g_1, x_1]$, we obtain g_2 and so on. Now we can define a sequence η by taking $\eta(n) = -g_n$ at any n in N. It is easy to see that $\eta \in E$ and $\hat{\eta}$ is the opposite of $\hat{\xi}$, i.e. $\hat{\xi} + \hat{\eta} = \hat{\eta} + \hat{\xi} = \hat{0}$.

(iii) If $\hat{\xi} \leq \hat{\eta}$, then $\hat{\xi} + \hat{\vartheta} \leq \hat{\eta} + \hat{\vartheta}$ for arbitrary $\hat{\vartheta} \in \mathbb{R}$, since a similar property holds for their representatives.

As customarily, if $\hat{0} < \hat{\xi}$, we say that $\hat{\xi}$ is (strictly) *positive*. Similar conditions introduce the notions of (strictly) *negative, non-positive,* and *nonnegative* real numbers. About these terms we may mention the following properties:

5. Proposition.

(*i*) $\hat{\xi}$ is nonnegative iff $0 \in Germ(\xi)$; the contrary (i.e. $\hat{\xi}$ is strictly negative) holds iff $\xi(n) \leq 0$ at some $n \in \mathbb{N}$;

(*ii*) $\hat{\xi}$ is positive iff $-\hat{\xi}$ is negative;

(*iii*) $\hat{\xi} \leq \hat{\eta}$ iff $\hat{\eta} - \hat{\xi}$ is positive.

The proof is routine.

Finally, in order to define the *multiplication* of real numbers, we shall start with the usual multiplication of positive emissions, that is $\zeta = \xi \cdot \eta$, meaning $\zeta(n) = \xi(n)\eta(n)$ at any $n \in \mathbb{N}$.

6. Definition. If $\hat{\xi}$ and $\hat{\eta}$ are positive real numbers, then $\hat{\zeta}$ is said to be their *product*, noted $\hat{\zeta} = \hat{\xi} \cdot \hat{\eta}$, iff $\zeta = \xi \cdot \eta$. Otherwise, i.e. when one of the factors is non-positive, we define their product by the convention:

$$\hat{\xi} \cdot \hat{\eta} = \begin{cases} -(-\hat{\xi}) \cdot \hat{\eta} & \text{if } \hat{\xi} \leq \hat{0} \leq \hat{\eta} \\ -\hat{\xi} \cdot (-\hat{\eta}) & \text{if } \hat{\eta} \leq \hat{0} \leq \hat{\xi} \\ (-\hat{\xi}) \cdot (-\hat{\eta}) & \text{if } \hat{\xi}, \hat{\eta} \leq \hat{0} \end{cases} .$$

7. Proposition. The multiplication has the following properties:

(*i*) It is well defined;

(*ii*) $(\mathbb{R}, +, \cdot)$ is a commutative field;

(*iii*) the order $\leq$ is compatible with it.

Proof. The most part of the proof is routine. We only mention that the unit real number is $\hat{1} \notin E$, and the inverse $\hat{\xi}^{-1}$ is not correctly defined by $\xi^{-1}(n) = [\xi(n)]^{-1}$ because $\xi^{-1} \notin E$. Therefore we have to distinguish two cases: $\hat{\xi}$ strictly positive, and $\hat{\xi}$ strictly negative. In the first case, all the representatives have a strictly positive germ. Let ξ be strictly positive, and let (g_n) with $g_0 > 0$ be the sequence associated with ξ as in the proof of the proposition 4 (ii). If we note $\tau(n) = [g_n]^{-1}$ at any $n \in N$, it is easy to see that τ is an emission, and $\tau \cdot \xi \in \hat{1}$, that is $\hat{\tau} = \hat{\xi}^{-1}$. In the second case we define the inverse by $-[(-\hat{\xi})^{-1}]$. □

So far we can say that $\mathbb{R}$ reproduces the properties of $\mathbb{Q}$. A copy of $\mathbb{Q}$ is recognized as a part $e(\mathbb{Q}) \subset \mathbb{R}$. The following property is still essentially new, and justifies the whole construction:

8. Theorem. $\mathbb{R}$ is completely ordered.

Proof. Let $A \neq \emptyset$ be a bounded set of real numbers. We fix $\hat{\xi}_0 \in A$ and a lower bound $\hat{b}_0$ of A. If $\hat{b}_0 \in A$, or $\hat{c} < \hat{b}_0$ for any other lower bound $\hat{c}$, then $\hat{b}_0 = infA$. If not, we consider another point

$$\frac{1}{2}(\hat{b}_0 + \hat{\xi}_0)$$

If this is a member of A, then we note it $\hat{\xi}_1$; if not, we note it $\hat{b}_1$ and we repeat the construction from above by putting it instead of $\hat{b}_0$. Except the case $\hat{b}_1 = infA$, we continue the process, until eventually $\hat{b}_k = infA$ at some step $k \in \mathbb{N}$. In the contrary case the above process never ends, and it produces a decreasing and bounded sequence $(\hat{\xi}_n)$ in A. If for every n in $\mathbb{N}$ we select representative emission $\xi_n : \mathbb{N} \to \mathbb{Q}$ in the class $\hat{\xi}_n$, then the sequence $(\hat{\xi}_n)_{n\in\mathbb{N}}$ of these emissions is strictly decreasing in the order $\preceq$, while the sequence $(\hat{\xi}_n)_{n\in\mathbb{N}}$ of classes is strictly decreasing relative to the order $\leq$. Now we start another construction: We note $\xi_0(0) = x_0$; because $\xi_1 \prec \xi_0$, for $m = 0$ there exists $m_0 = j(0) \in N$ such that $\xi_1(m_0) < \xi_0(0)$, hence $\xi_1(m_0) < x_0$; we note $\xi_1(m_0) = x_1$. Similarly, since $\xi_2 \prec \xi_1$, for $m_0 \in \mathbb{N}$, there exists $m_1 = j(m_0) \in N$ such that $\xi_2(m_1) < \xi_1(m_0)$. We note $x_2 = \xi_2(m_1)$. So far we have $x_2 < x_1 < x_0$. By induction, using the term $x_k = \xi_k(m_{k-1})$, and the hypothesis $\xi_{k+1} \prec \xi_k$, it follows that for $m_{k-1} \in \mathbb{N}$ there exists $m_k = j(m_{k-1})$ in $\mathbb{N}$ such that $\xi_{k+1}(m_k) < \xi_k(m_{k-1})$. So we obtained the next term $x_{k+1} = \xi_{k+1}(m_k)$, satisfying obviously $x_{k+1} < x_k$.

The result of this construction is an emission $\alpha : \mathbb{N} \to \mathbb{Q}$ of terms $\alpha(m) = x_m$. We claim that $\hat{\alpha} = infA$, which is justified by a direct proof.

As generally, $sup(A) = -(inf(-A))$.

Summarizing the properties of the above constructed set $\mathbb{R}$, we may conclude that:

9. Corollary. $\mathbb{R}$ is a totally ordered commutative field that is complete in its order.

10. Remark. It is generally known (see [3], etc.) that any two fields possessing these properties are algebraically and order isomorphic. Consequently the above *horistological* $\mathbb{R}$ represents a copy of the previously constructed $\mathbb{R}'s$.

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