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On V-Coincidence Point

Zoran D. Mitrović

In this paper we prove some results of existence V- coincidence point for multivalued quasiconvex mapping. [1]

1 Introduction and Preliminaries

Using the methods of the KKM theory [8], by the new notion of the V-coincidence point, we prove in this paper some results for quasiconvex multivalued mapping. As corollaries some results of almost fixed point are obtained.
Let X be a nonempty subset of a linear space. We denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, by $\mathcal{P}_{co}(X)$, the family of all convex members of $\mathcal{P}(X)$. In the next *co* is the convex hull operation. For the given multivalued mapping $F : X \to \mathcal{P}(Y)$ and sets $A \subset X$ and $B \subset Y$ we define the sets:

$$F(A) = \bigcup_{x \in A} F(x),$$

$$F^-(B) = \{x \in X : F(x) \cap \ B \neq \emptyset\},$$

$$F^+(B) = \{x \in X : F(x) \subset B\}.$$

Definition 1 *[7] Let X and Y be real vector spaces, C be a convex nonempty subset of X and K be a cone with zero in Y. A mapping $F : C \to \mathcal{P}(Y)$ is called K-quasiconvex if and only if it satisfies the condition*

$$F(x_i) \cap (S - K) \neq \emptyset, \ i = 1, 2 \Rightarrow F(\lambda x_1 + (1 - \lambda)x_2) \cap (S - K) \neq \emptyset,$$

for all convex sets $S \subset Y$, $x_1, \ x_2 \in C$ and $\lambda \in [0, 1]$.

If $K = \{0\}$ then F is called quasiconvex (quasiconcave).

In 1961, Ky Fan [2], proved that the following generalized form of the Knaster-Kuratowski-Mazurkiewicz Theorem holds:

[1] Key Words : Paranormed space; KKM mapping; Quasiconvex multivalued mapping. MSC 2000: 47H10

Theorem 2 *[2] Let E be a topological vector space, X be a nonempty subset of E and $T : X \to \mathcal{P}(E)$ a mapping with closed values and KKM mapping. If $T(x)$ is compact for at least one $x \in D$ then $\bigcap_{x\in X} T(x) \neq \emptyset$.*

Remark 3 *Let E be a topological vector space and X be a nonempty subset of E. A mapping $T : X \to \mathcal{P}(E)$ is called a KKM mapping if*

$$co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} T(x_i).$$

for each finite subset $\{x_1, \ldots, x_n\}$ of X.

Theorem 4 *[6] Let (X, d) be a complete metric space and $\{F_i\}_{i\in I}$ family closed subset of X with finite intersection property . If*

$$\inf_{i\in I} \alpha(F_i) = 0$$

then $\bigcap_{i\in I} F_i$ nonempty and compact set.

2 V-Coincidence Point

Definition 5 *[3] Let E be linear topological space, $K \subset E$ and $\mathcal{U}$ fundamental system neighbourhood zero in E. For set K is called Zima's type if for each $V \in \mathcal{U}$ exists $U \in \mathcal{U}$ such that*

$$co(U \cap (K - K))) \subset V.$$

Definition 6 *Let X be linear topological space with fundamental system neighbourhoods zero $\mathcal{V}$, $\emptyset \neq D \subset X$ and $F, G : D \to \mathcal{P}(X)$. For point $y \in D$ is called, V-coincidence point ($V \in \mathcal{V}$) of mappings F and G if and only if $F(y) \cap (G(y) + V) \neq \emptyset$.*

Theorem 7 *Let X be complete metric space and closed set $K \in \mathcal{P}_{co}(X)$, $F : K \to \mathcal{P}_{co}(X)$ lower semicontinuous mapping and $G : K \to \mathcal{P}(X)$ quasiconvex mapping such that $F(x) \cap G(K) \neq \emptyset$, for all $x \in K$. If for each $V \in \mathcal{V}$ exists $U \in \mathcal{V}$ such that*

$$co(U \cap (G(K) - F(K))) \subset V,$$

and

$$\inf_{x\in K} \alpha[F^{+}((G(x) + V)^{c})] = 0,$$

then exists V-coincidence point mappings F and G for each $V \in \mathcal{V}$.

Proof. Let exists open and symmetrical neighbourhoods zero $V \in \mathcal{V}$ such that

$$F(x) \cap (G(x) + V) = \emptyset \text{ for all } x \in K.$$

Let U is open symmetrical neighbourhoods zero, such that

$$co(U \cap (G(K) - F(K))) \subset V.$$

Put $W = co(U \cap (G(K) - F(K)))$ and define mapping $T : K \to \mathcal{P}_{co}(K)$ with

$$T(x) = \{y \in K : F(y) \cap (G(x) + U) = \emptyset\}.$$

We have

$$K \backslash T(x) = \{y \in K : F(y) \cap (G(x) + U) \neq \emptyset\} = F^-(G(x) + U).$$

Since F lower semicontinuous, $T(x)$ is closed set for each $x \in K$.
We can prove that T is a KKM mapping, i. e. that for every $\{x_1, \ldots, x_n\} \subset C$

$$co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} T(x_i) \qquad (1)$$

If (1) does not hold, there exists $y = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i \geq 0$, $i = 1, \ldots, n$ $\sum_{i=1}^{n} \lambda_i = 1$ so that $y \notin \bigcup_{i=1}^{n} T(x_i)$. Then we have

$$F(y) \cap (G(x_i) + U) \neq \emptyset \text{ for } i = 1, \ldots, n,$$

and

$$F(y) \cap (G(x_i) + W) \neq \emptyset, \quad i = 1, \ldots, n.$$

Set $F(y) - W$ is convex and G quasiconvex mapping, therefore

$$F(y) \cap (G(y) + W) \neq \emptyset.$$

Hence,

$$F(y) \cap (G(y) + V) \neq \emptyset.$$

It is contradiction. Since

$$T(x) = \{y \in K : F(y) \subset (G(x) + V)^c\} = F^+((G(x) + V)^c)$$

then $\inf_{x \in K} \alpha(T(x)) = 0$. Using Theorem 2. and Theorem 4. we have

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$

Let $y_0 \in \bigcap_{x \in K} T(x)$. Then

$$F(y_0) \cap (G(x) + V) = \emptyset \quad x \in K \qquad (2)$$

Since $F(x) \cap G(K) \neq \emptyset$ for all $x \in K$, exists $x_0 \in K$ such that $F(y_0) \cap G(x_0) \neq \emptyset$, it is contadiction with (2).

Definition 8 *[4] $(E, ||\cdot||)$ is paranormed space if E is linear space on $\mathbb{R}$ and $||\cdot|| : E \to \mathbb{R}^+$ such that*

1. *$||x|| = 0 \Leftrightarrow x = 0$,*
2. *for each $x \in E$, $||-x|| = ||x||$,*
3. *for all $x, y \in E$, $||x+y|| \le ||x|| + ||y||$,*
4. *if $\lambda_n,\ \lambda \in \mathbb{R}$ and $x_n,\ x \in E$ such that*

$$|\lambda_n - \lambda| \to 0,\ ||x_n - x|| \to 0,$$

then

$$||\lambda_n x_n - \lambda x|| \to 0.$$

Let X be a paranormed space, with $\{V_\epsilon\}_{\epsilon>0}$ we denote

$$V_\epsilon = \{x \in X : ||x|| < \epsilon\}.$$

Corollary 9 *Let $(X, ||\cdot||)$ be complete paranormed space, K nonempty convex and closed subset of X and*

$$F : K \to \mathcal{P}_{co}(X) \text{ lower semicontinuous mapping,}$$

$$G : K \to \mathcal{P}(X) \text{ quasiconvex mapping}$$

such that exists $C(\widetilde{K})$, where $\widetilde{K} = G(K) \cup F(K)$ and

$$||\lambda(u-v)|| \le C(\widetilde{K})\lambda||u-v||, \quad \inf_{x\in K} \alpha[F^+((G(x)+V_\epsilon)^c)] = 0,$$

$$\text{for all } \lambda \in [0,1],\ u, v \in \widetilde{K},\ \epsilon > 0.$$

Then for each $\epsilon > 0$ exists V_ϵ-coincidence point mappings F and G.

Corollary 10 *[4] Let $(X, ||\cdot||)$ be complete paranormed space, K nonempty convex and closed subset of X, $F : X \to \mathcal{P}_{co}(X)$ lower semocontinuous such that $F(x) \cap K \neq \emptyset$ for all $x \in K$. If there exists $C(\widetilde{K})$, where $\widetilde{K} = K \cup F(K)$ such that*

$$||\lambda(u-v)|| \le C(\widetilde{K})\lambda||u-v|| \text{ for all } \lambda \in [0,1], u, v \in \widetilde{K}$$

and for all $\epsilon > 0$

$$\inf_{x\in K} \alpha[F^+((x+V_\epsilon)^C)] = 0$$

then F have V_ϵ-almost fixed point for each $\epsilon > 0$.

Theorem 11 *Let X be linear topological space, $\mathcal{V}$ family closed neighbourhood zero in X, K nonempty convex subset of X, $F : K \to \mathcal{P}_{co}(X)$ lower semicontinuous and $G : K \to \mathcal{P}(X)$ quasiconvex such that*

$$G(K) \cap F(x) \neq \emptyset \text{ for each } x \in K \text{ and for each } V \in \mathcal{V}$$

$$\text{exists finite } \{x_1, \dots, x_n\} \subset K \text{ such that } G(K) \subset \bigcup_{i=1}^{n} (G(x_i) + V).$$

If $G(K) \cup F(K)$ Zima's type then exists V-coincidence point mappings F and G for each $V \in \mathcal{V}$.

Let $V \in \mathcal{V}$, we shall prove that exists $x \in K$ such that

$$F(x) \cap (G(x) + V) \neq \emptyset.$$

Let $U \in \mathcal{V}$ such that $co(U \cap (G(K) - F(K))) \subset V$ and define mapping $T : K \to \mathcal{P}(K)$ with $T(x) = \{y \in K : F(y) \cap (G(x) + U) = \emptyset\}$.
F lower semicontinuous, therefore $T(x)$ is closed set for each $x \in K$. For choice U exists finite $\{x_1, \dots, x_n\} \subset K$ such that

$$G(K) \subset \bigcup_{i=1}^{n} (G(x_i) + U).$$

For each $y \in K$ we have

$$F(y) \cap G(K) \subset F(y) \cap (\bigcup_{i=1}^{n} (G(x_i) + U)) = \bigcup_{i=1}^{n} (F(y) \cap (G(x_i) + U)),$$

therefore $\bigcap_{i=1}^{n} T(x_i) = \emptyset$ since $F(y) \cap G(K) \neq \emptyset$. So, exists subset $\{z_1, \dots, z_m\}$ of K such that

$$co\{z_1, \dots, z_m\} \not\subseteq \bigcup_{i=1}^{m} T(z_i).$$

Therefore exist $\lambda_i \in [0, 1]$, $\sum_{i=1}^{m} \lambda_i = 1$ and $y = \sum_{i=1}^{m} \lambda_i z_i$, $y \notin T(z_i)$, for all $i = 1, \dots, m$, so $F(y) \cap (G(z_i) + U) \neq \emptyset$. Hence, exist w_i, $i = 1, \dots, m$ such that

$$w_i \in F(y) \text{ and } w_i \in G(z_i) + U, \; i = 1, \dots, m.$$

Therefore, $w_i \in F(y)$ and $w_i = v_i + u_i$ for some $v_i \in G(z_i)$, $u_i \in U$,

$$u_i = w_i - v_i \in (G(K) - F(K)) \cap U, \; i = 1, \dots, m.$$

$$u_i \in co(U \cap (G(K) - F(K))) \text{ and } u_i \in F(y) - G(z_i),$$
$$(F(y) - G(z_i)) \cap co(U \cap (G(K) - F(K))) \neq \emptyset, \; i = 1, \dots, m,$$

i. e.

$$(F(y) - co(U \cap (G(K) - F(K)))) \cap G(z_i) \neq \emptyset, \ \ i = 1, \ldots, m.$$

G is quasiconvex, therefore

$$(F(y) - co(U \cap (G(K) - F(K)))) \cap G(y) \neq \emptyset.$$

So, $F(y) \cap (G(y) + V) \neq \emptyset$.

Corollary 12 *[5] Let X be a linear topological space, $\mathcal{V}$ family open neighbouhood zero in X, K nonempty convex precompact subset of X and $F : K \to \mathcal{P}_{co}(X)$ lower semicontinuous mapping such that for each $x \in K$, $f(x) \cap K \neq \emptyset$. If $K \cup f(K)$ Zima's type then f have V-almost fixed point for each $V \in \mathcal{V}$.*

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Zoran D. Mitrović
University of Banja Luka
Faculty of Electrical Engineering
51000 Banja Luka, Patre 5
Bosnia and Herzegovina
zmitrovic@etfbl.net