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# Kenmotsu manifolds with $\eta$-parallel Ricci tensor

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**Abstract**

We study a Kenmotsu manifold M with $\eta$-parallel Ricci tensor. We obtain a characterisation (Theorem 2.3) and some properties of the Ricci operator and the scalar curvature of M(cf.Prop. 2.2)

## Introduction.

In [2], K. Kenmotsu introduced a new class of almost contact metric manifold, later called, Kenmotsu manifolds. It is known that a Kenmotsu manifold M is not a compact manifold (because $div\xi = 2n$, $dimM = 2n$). Later N. Papaghiuc [5] and other geometers have studied and established important properties of Kenmotsu submanifolds. The purpose of this paper is to prove the important properties of the Ricci tensor, scalar curvature and the Ricci operator on a Kenmotsu manifold.

## 1 Preliminaries.

It is well-known that the structure tensors $(f, \xi, \eta, g)$ of an almost contact metric manifold M, satisfy

$$\begin{array}{llll} (a) & f^2X = -X + \eta(X)\xi, & (b) \quad f(\xi) = 0, & (c) \quad \eta(\xi) = 1, \\ (d) & \eta(X) = g(X, \eta), & (e) \quad g(fX, fY) = g(X, Y) - \eta(X)\eta(Y), & \end{array} \tag{1}$$

for any vector field $X, Y$ tangent to $M$. Throughout the paper, all manifolds and maps are differentiable of class $C^{\infty}$. We denote by $F(M)$ the algebra of the differentiable functions on $M$ and by $\Gamma(E)$ the $F(M)$-module of the sections of a vector bundle $E$ on $M$. [1]

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The Nijenhuis tensor field, denoted by $N_f$, with respect to the tensor $f$, is given by

$$N_f(X,Y) = [fX, fY] + f^2[X,Y] - f[X,Y] - f[X,fY], \quad \forall X,Y \in \Gamma(TM).$$

The almost contact metric manifold $M(f,\xi,\eta,g)$ is called normal if

$$N_f(X,Y) + 2d\eta(X,Y)\xi = 0, \quad \forall X,Y \in \Gamma(TM).$$

According to [2], we say an almost contact metric manifold $M$ is called an almost Kenmotsu manifold if

$$d\eta = 0 \text{ and } 3d\Phi(X,Y,Z) = 2(\eta(X)\Phi(Y,Z) + \eta(Y)\Phi(Z,X) +$$
$$+\eta(Z)\Phi(X,Y), \quad \forall X,Y \in \Gamma(TM),$$

where $\Phi$ is the fundamental 2-form, given by $\Phi(X,Y) = g(X,fY), X,Y \in \Gamma(TM)$. A Kenmotsu manifold is a normal almost Kenmotsu manifold. It is known [2],[5] that an almost contact metric manifold is a Kenmotsu manifld if and only if

$$(\nabla_X f)Y = g(fX,Y)\xi - \eta(Y)fX, \quad \forall X,Y \in \Gamma(TM), \tag{2}$$

where $\nabla$ is the Levi-Civita connection on $M$ with respect to the metric tensor g. By straightforward calculation, from (2) we deduce that

$$\nabla_X \xi = X - \eta(X)\xi, \quad \forall X,Y \in \Gamma(TM) \tag{3}$$

The curvatore tensor field is denoted by $K$ and is given by

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad \forall X,Y,Z \in \Gamma(TM).$$

Next we recall some properties of the curvature tensor field $K$ from [1],[2].

**Proposition 1.1** *Let $M$ be a Kenmotsu manifold with the curvature tensor $K$. Then we have*

$$\begin{aligned}
&(a) \quad K(X,Y)\xi = \eta(X)Y - \eta(Y)X,\\
&(b) \quad K(X,\xi)Y = g(X,Y)\xi - \eta(Y)X,\\
&(c) \quad K(X,Y)fZ = fK(X,Y)Z + g(Y,Z)fX - g(X,Z)fY +\\
&\qquad +g(X,fZ)Y - g(Y,fZ)X,\\
&(d) \quad K(fX,fY)Z = K(X,Y)Z + g(Y,Z)X - g(X,Z)Y -\\
&\qquad g(X,fZ)fY + g(Y,fZ)fX, \quad \forall X,Y,Z \in \Gamma(TM).
\end{aligned} \tag{4}$$

Next we consider an orthonormal field of frame $\{e_1,\cdots,e_{2n+1}\}$ with $e_{n+i} = fe_i, e_{2n+1} = \xi, i \in \{1,\cdots,n\}$. Then the Ricci tensor $S$ is given by

$$S(X,Y) = \sum_{i=1}^{2n+1} g(K(e_i,X)Y,e_i), \quad \forall X,Y \in \Gamma(TM) \tag{5}$$

and the scalar curvature, denoted by $r$ is given by $r = \sum_{i=1}^{2n+1} S(e_i,e_i)$. Also the Ricci operator, denoted by $Q$, is a tensor field of type $(1,1)$, given by $S(X,Y) = g(QX,Y) \quad \forall X,Y \in \Gamma(TM)$. If $S(X,Y) = ag(X,Y)$, with a constant, then M is Einstein manifold. A tensor field $T$ defined on $M$ is parallel if $\nabla T = 0$ and the length of the tensor field $T$ is denoted by $||T||$ and it is given by $||T||^2 = g(T,T)$.

# 2 Kenmotsu manifold with $\eta$-parallel Ricci tensor.

Let $M$ be a Kenmotsu manifold and $S$ be the Ricci tensor on $M$. Then using (4b) and (5) we obtain

**Proposition 2.1** *Let $M$ be a Kenmotsu manifold. The the Ricci tensor $S$ and the Ricci operator $Q$ verify*

$$\begin{array}{ll} (a) \quad S(X,\eta) = -2n\eta(X); \quad (b) \quad S(\xi,\xi) = -2n; \\ (c) \quad Q\xi = -2n\xi, \quad \forall X \in \Gamma(TM). \end{array} \tag{6}$$

**Theorem 2.1** *Let $M$ be a Kenmotsu manifold. Then the Ricci tensor $S$ verifies the next assertions*

$$\begin{array}{ll} (a) & S(X,Y) = \frac{1}{2}\sum_{i=1}^{2n+1} g(fK(X,fY)e_i,e_i) - \\ & -(2n-1)g(X,Y) - \eta(X)\eta(Y), \\ (b) & S(fX,fY) = S(X,Y) + 2n\eta(X)\eta(Y), \\ (c) & (\nabla_Z S)(X,Y) = (\nabla_X S)(Y,Z) + (\nabla_{fY}S)(fX,Z) + \\ & +\eta(X)S(fY,fZ) + 2(n-1)\eta(X)g(fY,fZ) + \\ & +2\eta(Z)g(fX,fY), \quad \forall X,Y,Z \in \Gamma(TM) \end{array} \tag{7}$$

*Proof.* Let $X, Y \in \Gamma(TM)$ and $e_1, \cdots, e_{2n+1}$ an orthonormal field of frame with $e_{n+i} = fe_i$, $e_{2n+1} = \xi$, $i \in \{1, \cdots, n\}$, and using (1), (4c), we get

$$\begin{array}{l} g(K(fe_i,X)Y,fe_i) = -g(K(fe_i,X)fe_i,Y) = \\ = -g(fK(fe_i,X)e_i,Y) - g(X,e_i)g(Y,f^2e_i) - g(fe_i,fe_i)g(X,Y) + \\ +g(X,fe_i)g(Y,fe_i) = g(K(fe_i,X)e_i,fY) - g(X,e_i)g(f^2Y,e_i) - \\ -g(fe_i,fe_i)g(X,Y) + g(fX,e_i)g(fY,e_i). \end{array} \tag{8}$$

By using the Bianchi identities we infer that

$$g(K(fe_i,X)e_i,fY) = g(K(fe_i,X)fY,e_i) + g(K(X,fY)e_i,fe_i). \tag{9}$$

Using again (4c) for the calculation of $g(K(fe_i,X)fY,e_i)$, then from (5), (8) and (9) we obtain assertion (a). Next from (7a), using (1a), (4b) and (6a) we infer that

$$\begin{array}{l} S(X,fY) = \frac{1}{2}\sum_{i=1}^{2n+1} g(fK(X,Y)e_i,e_i) - \\ -(2n-1)g(X,fY), \quad \forall X,Y \in \Gamma(TM) \end{array} \tag{10}$$

Now it is easy to see that $S(fX,Y) + S(X,fY) = 0$ and consequently the assertion (b) is proved. Next using the fact that $\nabla$ is a Levi-Civita connection, (2), (7a), and (10) we get

$$\begin{array}{l} (\nabla_{fY}S)(fX,Z) = -\frac{1}{2}\sum_{i=1}^{2n+1} g((\nabla_{fY}K)(X,Z)e_i,fe_i) - \\ -(2n-1)g((\nabla_{fY}f)(X,Z) - S((\nabla_{fY}f)X,Z) = \\ = -\frac{1}{2}\sum_{i=1}^{2n+1} g((\nabla_{fY}K)(X,Z)e_i,fe_i) - (2n-1)\eta(X)g(fY,fZ) - \\ -\eta(X)S(fY,fZ) - \eta(Z)g(fX,fY), \quad \forall X,Y,Z \in \Gamma(TM). \end{array} \tag{11}$$

In the same way we obtain

$$\begin{array}{l}(\nabla_Z S)(X,Y) = -\frac{1}{2}\sum_{i=1}^{2n+1}(g((\nabla_Z K)(X,fY)e_i, fe_i)+ \\ +g(K(X,(\nabla_Z f)Y)e_i, e_i)) - \eta(X)g(fY,fZ) - \eta(Y)g(fX,fZ) = \\ = -\frac{1}{2}\sum_{i=1}^{2n+1} g((\nabla_Z K)(X,fY)e_i, fe_i) - 2n\eta(Y)g(X,Z)- \\ -\eta(Y)S(X,Z) - \eta(X)g(fZ,fY), \quad \forall X,Y,Z \in \Gamma(TM)\end{array} \tag{12}$$

Finally, the assertion (7c), follows from (11), (12) and the Bianchi identity. Now we prove the following:

**Theorem 2.2** *Let $M$ be a Kenmotsu manifold. The Ricci tensor $S$ is parallel if and only if $M$ is Einstein manifold.*

**Proof.** Let $X, Y \in \Gamma(TM)$ and suppose that M is a Kenmotsu manifold with parallel Ricci tensor. Using (3), (6a), and the fact that $\nabla$ is a Levi-Civita connection, we infer that

$$\begin{array}{l}0 = (\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(X,\nabla_X \xi) = \\ = -2ng(Y, X - \eta(X)\xi) - S(Y, X - \eta(X)\xi) = \\ = -S(X,Y) - 2ng(X,Y), \quad \forall X,Y \in \Gamma(TM),\end{array}$$

and the assertion is proved. Conversely follow from the fact that $M$ is Einstein manifold and $\nabla$ is a metric connection.

Taking into account the previous result, we introduce a weaker condition for the Ricci tensor so called $\eta$-parallel Ricci tensor on a Kenmotsu manifold.

**Definition 2.1** *Let $M$ be a Kenmotsu manifold. We say that the Ricci tensor $S$ is $\eta$-parallel if*

$$(\nabla_X S)(fY, fZ) = 0, \quad \forall X,Y,Z \in \Gamma(TM) \tag{13}$$

Next we have

**Theorem 2.3** *Let $M$ be a Kenmotsu manifold. Then the next three assertions are equivalents*

(*i*) *The Ricci tensor $S$ is $\eta$-parallel*

(*ii*) $(\nabla_X S)(Y,Z) = -\eta(Y)S(X,Z) - \eta(Z)S(X,Y) -$

$$-2n\eta(Y)g(X,Z) - 2n\eta(Z)g(X,Y), \quad \forall X,Y,Z \in \Gamma(TM), \tag{14}$$

(*iii*) $\|\nabla Q\|^2 = 2\|Q\|^2 + 16n^3 + 8n^2 + 8nr$

**Proof.** The equivalence of the assertions (i) and (ii) it follows from (13) and (7b), by direct calculation. Next let $\{e_j, fe_i, \xi\}$ an orthonormal field of frame on $M$. Then using (3) and (6c), we get

$$(\nabla_{e_i}\xi) = \nabla_{e_i}Q\xi - Q\nabla_{e_i}\xi = -Qe_i - 2n\nabla_{e_i}\xi = -2ne_i - Qe_i, i = 1, \cdots, 2n+1,$$

and therefore the above result implies

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)\xi, (\nabla_{e_i} Q)\xi) = n^3 + 4nr + 4n^2 + \|Q\|^2. \quad (15)$$

On the other hand, using (2) and (6c), we obtain

$$\begin{aligned}(\nabla_{e_i} Q)fe_j &= \nabla_{e_i} Qfe_j - Q(\nabla_{e_i} f)e_j - Qf\nabla_{e_i} e_j = \\ &= -(\nabla_{e_i} f)Qe_j + f(\nabla_{e_i})e_j - Q(g(fe_i, e_j)\xi - \eta(e_j)fe_i = \\ &= g(fe_i, Qe_j)\xi + f(\nabla_{e_i} Q)e_j + 2ng(fe_i, e_j)\xi.\end{aligned}$$

From the above relation we obtain

$$\begin{aligned}g((\nabla_{e_i} Q)fe_j, (\nabla_{e_i} Q)fe_j) &= g(f(\nabla_{e_i} Q)e_j, f(\nabla_{e_i} Q)e_j) + \\ +(g(fe_i, Qe_j) + 2ng(fe_i, e_j))^2 &= g(f(\nabla_{e_i} Q)e_j, f(\nabla_{e_i} Q)e_j) + \\ +g^2(Qfe_i, e_j) + 4n^2 g^2(fe_i, e_j) &+ 4ng(fe_i, Qe_j)g(fe_i, e_j).\end{aligned} \quad (16)$$

Finally, using (15) and (16) we deduce that

$$\begin{aligned}&\|\nabla Q\|^2 = \textstyle\sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)e_j, (\nabla_{e_i} Q)e_j = \\ &= \textstyle\sum_{i=1}^{2n+1} \sum_{j=1}^{2n} g((\nabla_{e_i} Q)fe_j, (\nabla_{e_i} Q)fe_j) + \sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)\xi, (\nabla_{e_i} Q)\xi) = \\ &= \textstyle\sum_{i=1}^{2n+1} \sum_{j=1}^{2n} (g(f(\nabla_{e_i} Q)e_j, f(\nabla_{e_i} Q)e_j + \\ &+g(fQe_j, g(fQe_j, e_i)e_i) + 4n^2 g(fe_j, g(fe_j, e_i)e_i) + \\ &+4ng(fQe_j, g(fe_j, e_i)e_i)) + 8n^3 + 4nr + 4n^2 + \|Q\|^2 = \\ &= \textstyle\sum_{i=1}^{2n+1} \sum_{j=1}^{2n} g(f(\nabla_{e_i} Q)e_j, f(\nabla_{e_i} Q)e_j) + \|Q\|^2 + 16n^3 + 8n^2 + 8nr.\end{aligned}$$

The equivalence of the assertions (i) and (iii) follows from the above assertion because it is easy to see that the Ricci tensor $S$ is $\eta$-parallel if and only if $\sum_{i=1}^{2n+1} \sum_{j=1}^{2n} (g(f(\nabla_{e_i} Q)e_j, f(\nabla_{e_i} Q)e_j) = 0$.

**Proposition 2.2** *Let $M$ be a Kenmotsu manifold with $\eta$-parallel Ricci tensor $S$. Then*

*(a) the scalar curvature $r$ is constant,*

*(b) the length of the Ricci operator $Q$ is constant.*

**Proof.** Let $\{e_i, fe_i, \xi\}$, $i = 1, \cdots, n$ be an orthonormal field of frame on $M$ and using (6a), (14) we get that

$$\begin{aligned}&\nabla_X r = \textstyle\sum_{i=1}^{2n+1} (\nabla_X S)(e_i, e_i) = -4n \sum_{i=1}^{2n+1} \eta(e_i) g(X, e_i) - \\ &-2\textstyle\sum_{i=1}^{2n+1} \eta(e_i) S(X, e_i) = -4n\eta(X) - 2S(X, \xi) = 0, \quad \forall X \in \Gamma(TM),\end{aligned}$$

which proves the assertion (a). Next, because $S(X,Y) = g(QX,Y)$, using (6a), (6c) and (14) we deduce

$$\begin{aligned}&\nabla_X \|Q\|^2 = 2\textstyle\sum_{i=1}^{2n+1} g((\nabla_X Q)e_i, Qe_i) = 2\sum_{i=1}^{2n+1} (\nabla_X S)(e_i, Qe_i) = \\ &= -2\textstyle\sum_{i=1}^{2n+1} (\eta(e_i) S(Qe_i, X) + \eta(Qe_i) S(e_i, X) + \\ &+2n\eta(e_i) g(Qe_i, X) + 2n\eta(Qe_i) S(e_i, X)) = -4(S(X, Q\xi) + 2ng(X, Q\xi)) = 0\end{aligned}$$

and the assertion (b) is proved.

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