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YANG-BAXTER EQUATIONS IN MTL-ALGEBRAS

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Abstract. In this work, we handle set-theoratical solutions of Yang-Baxter equation problem in fundamental algebraic structures. The aim of this paper is to construct new set-theoretical solutions for the Yang–Baxter equation by using MTL-algebra.

1. Introduction

The Yang-Baxter equation was primarily given in theoretical physics [**12**] and in statistical mechanics ([**1**], [**2**], [**11**]). It has been brought about many applications not only these areas but also quantum groups, quantum computing, knot theory, etc. [**8**]. Recent advances show to us in other areas like as C^* algebras, Hopf algebras, conformal field theory, etc. clarify the importance of the equation. Many authors have taken advantage of the axioms of these algebraic structures to sort out this equation. We want to observe the Yang-Baxter equation in associated with MTL-algebras.

Oner and Katican have constructed a new set-theoretical solutions to the Yang–Baxter equation using BL-algebras [**15**], and Wasjberg Algebras [**14**].

All sorts of fuzzy logical algebras have been widely identified and analysed, for example, MV-algebras [**3**] , BL-algebras [**5**] and NM-algebras [**4**]. Between these algebras, MTL-algebras are the most important since the others are special conditions of MTL-algebras. Therefore, MTL-algebras play a crucial role in studying fuzzy logics and their relevant structures.

In this paper, we examine the Yang-Baxter equation in relation with MTL-algebra. We give fundamental definitions and theorems of MTL-algebras and give some solutions of the set-theoretical Yang-Baxter equation in MTL-algebras.

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2. Preliminaries

The following fundamental notions are taken from [**16**].

Definition 2.1. An algebraic structure $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ of type (2, 2, 2, 2, 0, 0) is called an MTL-algebra if it satisfies the following conditions:
(a) $\mathcal{L}$ is a bounded lattice,
(b) $(L, \odot, 1)$ is a commutative monoid,
(c) $s_1 \odot s_2 \leqslant s_3$ if and only if $s_1 \leqslant s_2 \to s_3$,
(d) $(s_1 \to s_2) \vee (s_2 \to s_1) = 1$, for any $s_1, s_2, s_3 \in L$.

For any $s \in L$ and a natural number n, we define

$\neg s = s \to 0$, $\neg\neg s = \neg(\neg s)$, $s^0 = 1$ and $s^n = s^{n-1} \odot s$ for all $n \geqslant 1$.

Let $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ be an MTL-algebra, unless otherwise is stated.

Proposition 2.1. *In every MTL-algebra* $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ *, the following hold for all* $s \in L$*:*
(1) $(s \to 0) \to 0 = s$,
(2) $s \to s = 1$, $1 \to s = s$, $s \to 1 = 1$,
(3) $\neg s = \neg\neg\neg s$,
(4) $1 \odot s = s$, $s \odot 0 = 0$,
(5) $\neg 0 = 1$, $\neg 1 = 0$.

Definition 2.2. Let $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ be an MTL-algebra. Then $\mathcal{L}$ is called
(a) a Gödel algebra if $s_1 \odot s_2 = s_1 \wedge s_2 = s_1 \odot (s_1 \to s_2)$ for any $s_1, s_2 \in L$,
(b) an IMTL-algebra if $\neg\neg s = s$ for any $s \in L$.

3. Solutions to the Yang-Baxter Equation in MTL-Algebras

In this paper, we present some results in connection with the (set-theoretical) Yang- Baxter equation in MTL-algebras.

Let V be a vector space over a field F, which is algebraically closed and of characteristic zero.

Definition 3.1. ([**9**]) A linear automorphism φ of $V \otimes V$ is a solution of the Yang-Baxter equation, if the following equality holds in the automorphism group of $V \otimes V \otimes V$:

$$(\varphi \otimes id_V) \circ (id_V \otimes \varphi) \circ (\varphi \otimes id_V) = (id_V \otimes \varphi) \circ (\varphi \otimes id_V) \circ (id_V \otimes \varphi). \tag{3.1}$$

In the following definitions φ^{nm} means φ acting on the n-th and m-th component.

Definition 3.2. ([**10**]) φ is a solution of the Yang-Baxter equation if

$$\varphi^{12} \circ \varphi^{23} \circ \varphi^{12} = \varphi^{23} \circ \varphi^{12} \circ \varphi^{23}, \tag{3.2}$$

Definition 3.3. ([**9**]) φ is a solution of the quantum Yang-Baxter equation if

$$\varphi^{12} \circ \varphi^{13} \circ \varphi^{23} = \varphi^{23} \circ \varphi^{13} \circ \varphi^{12}. \tag{3.3}$$

Let T be the twist map $T : V \otimes V \to V \otimes V$ defined by $T(u \otimes v) = v \otimes u$. Then, φ satisfies (3.2) if and only if $\varphi \circ T$ satisfies (3.3) if and only if $T \circ \varphi$ satisfies (3.3).

A connection between the set-theoretical Yang-Baxter equation and MTL-algebras is constituted by the following definition.

Definition 3.4. ([**10**]) Let X be a set and $\varphi : X^2 \to X^2, \varphi(p,q) = (p', q')$ be a map. The map φ is a solution for the set-theoretical Yang-Baxter equation if it satisfies (3.2), which is also equivalent to (3.3), where

$$\begin{aligned}
\varphi^{12} : X^3 \to X^3, &\quad \varphi^{12}(s_1, s_2, s_3) = (s_1', s_2', s_3),\\
\varphi^{23} : X^3 \to X^3, &\quad \varphi^{23}(s_1, s_2, s_3) = (s_1, s_2', s_3'),\\
\varphi^{13} : X^3 \to X^3, &\quad \varphi^{13}(s_1, s_2, s_3) = (s_1', s_2, s_3').
\end{aligned}$$

Now we construct solutions to the set theoretical Yang-Baxter equation by using MTL-algebras.

Lemma 3.1. *Let $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ be an MTL-algebra. Then the following are a solution of the set-theoretical Yang-Baxter equation:*
(a) $\varphi(s_1, s_2) = (s_1 \odot 1, s_2 \odot 1)$,
(b) $\varphi(s_1, s_2) = ((1 \odot (s_1 \to 0)) \to 0, (1 \odot (s_2 \to 0)) \to 0)$,
(c) $\varphi(s_1, s_2) = (((s_1 \to 0) \odot (s_2 \to 0)) \to 0, 0)$,
(d) $\varphi(s_1, s_2) = (s_2 \odot s_1, 1)$,
(e) $\varphi(s_1, s_2) = (s_1 \odot s_2, 1)$.

Proof. The proofs of (a) and (b) are clear.
(c) We define

$$\begin{aligned}
\varphi^{12}(s_1, s_2, s_3) &= (((s_1 \to 0) \odot (s_2 \to 0)) \to 0, 0, s_3),\\
\varphi^{23}(s_1, s_2, s_3) &= (s_1, ((s_2 \to 0) \odot (s_3 \to 0)) \to 0, 0).
\end{aligned}$$

For all $(s_1, s_2, s_3) \in L^3$, using Proposition 2.1 (1), (2) and (4) we have

$$\begin{aligned}
(\varphi^{12} \circ \varphi^{23} \circ \varphi^{12})(s_1, s_2, s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1, s_2, s_3)))\\
&= \varphi^{12}(\varphi^{23}(((s_1 \to 0) \odot (s_2 \to 0)) \to 0, 0, s_3))\\
&= \varphi^{12}(((s_1 \to 0) \odot (s_2 \to 0)) \to 0, ((0 \to 0) \odot (s_3 \to\\
&\quad 0)) \to 0, 0)\\
&= \varphi^{12}(((s_1 \to 0) \odot (s_2 \to 0)) \to 0, s_3, 0)\\
&= ((((((s_1 \to 0) \odot (s_2 \to 0)) \to 0) \to 0) \odot (s_3 \to 0))\\
&\quad \to 0, 0, 0)\\
&= ((((s_1 \to 0) \odot (s_2 \to 0)) \odot (s_3 \to 0)) \to 0, 0, 0)
\end{aligned}$$

and

$$\begin{aligned}(\varphi^{23}\circ\varphi^{12}\circ\varphi^{23})(s_1,s_2,s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1,s_2,s_3)))\\ &= \varphi^{23}(\varphi^{12}(s_1,((s_2\to 0)\odot(s_3\to 0))\to 0,0))\\ &= \varphi^{23}(((s_1\to 0)\odot((((s_2\to 0)\odot(s_3\to 0))\to 0)\to\\ &\quad 0))\to 0,0,0)\\ &= (((s_1\to 0)\odot((s_2\to 0)\odot(s_3\to 0)))\to 0,((0\to 0)\\ &\quad \odot(0\to 0))\to 0,0)\\ &= (((s_1\to 0)\odot((s_2\to 0)\odot(s_3\to 0)))\to 0,0,0).\end{aligned}$$

Since $(L,\odot,1)$ is a commutative monoid, it is a solution.

(d) φ^{12} and φ^{23} are defined by

$$\begin{aligned}\varphi^{12}(s_1,s_2,s_3) &= (s_2\odot s_1,1,s_3),\\ \varphi^{23}(s_1,s_2,s_3) &= (s_1,s_3\odot s_2,1).\end{aligned}$$

For all $(s_1,s_2,s_3)\in L^3$, by using Proposition 2.1 (4), we get

$$\begin{aligned}(\varphi^{12}\circ\varphi^{23}\circ\varphi^{12})(s_1,s_2,s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1,s_2,s_3)))\\ &= \varphi^{12}(\varphi^{23}(s_2\odot s_1,1,s_3))\\ &= \varphi^{12}(s_2\odot s_1,s_3\odot 1,1)\\ &= \varphi^{12}(s_2\odot s_1,s_3,1)\\ &= (s_3\odot(s_2\odot s_1),1,1)\end{aligned}$$

and

$$\begin{aligned}(\varphi^{23}\circ\varphi^{12}\circ\varphi^{23})(s_1,s_2,s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1,s_2,s_3)))\\ &= \varphi^{23}(\varphi^{12}(s_1,s_3\odot s_2,1))\\ &= \varphi^{23}((s_3\odot s_2)\odot s_1,1,1)\\ &= ((s_3\odot s_2)\odot s_1,1,1).\end{aligned}$$

Since $(L,\odot,1)$ is a commutative monoid, it is a solution, as required.

(e) The proof is similar to (d). □

THEOREM 3.1. *Let* $(L,\wedge,\vee,\odot,\to,0,1)$ *be an MTL-algebra. If the MTL-algebra is a Gödel algebra, then* $\varphi(s_1,s_2)=(s_2\odot s_1,s_1)$ *is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. We define

$$\begin{aligned}\varphi^{12}(s_1,s_2,s_3) &= (s_2\odot s_1,s_1,s_3),\\ \varphi^{23}(s_1,s_2,s_3) &= (s_1,s_3\odot s_2,s_2).\end{aligned}$$

By using Definition 2.1 (b) and Definition 2.2 (a), for all $(s_1, s_2, s_3) \in L^3$, we obtain

$$\begin{aligned}(\varphi^{12} \circ \varphi^{23} \circ \varphi^{12})(s_1, s_2, s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1, s_2, s_3))) \\ &= \varphi^{12}(\varphi^{23}(s_2 \odot s_1, s_1, s_3) \\ &= \varphi^{12}(s_2 \odot s_1, s_3 \odot s_1, s_1) \\ &= ((s_3 \odot s_1) \odot (s_2 \odot s_1), s_2 \odot s_1, s_1) \\ &= ((s_3 \odot s_2) \odot (s_1 \odot s_1), s_2 \odot s_1, s_1) \\ &= ((s_3 \odot s_2) \odot s_1, s_2 \odot s_1, s_1)\end{aligned}$$

and

$$\begin{aligned}(\varphi^{23} \circ \varphi^{12} \circ \varphi^{23})(s_1, s_2, s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1, s_2, s_3))) \\ &= \varphi^{23}(\varphi^{12}(s_1, s_3 \odot s_2, s_2)) \\ &= \varphi^{23}((s_3 \odot s_2) \odot s_1, s_1, s_2) \\ &= ((s_3 \odot s_2) \odot s_1, s_2 \odot s_1, s_1).\end{aligned}$$

Since the MTL- algebra is a Gödel algebra, it is a solution. □

Corollary 3.1. *Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be an MTL-algebra. The following are solutions to the set-theoretical Yang-Baxter equation in MTL-algebras since the $(L, \odot, 1)$ is a commutative monoid:*

(a) $\varphi(s_1, s_2) = (s_1 \odot s_2, s_1)$,
(b) $\varphi(s_1, s_2) = (s_2 \odot s_1, s_2)$,
(c) $\varphi(s_1, s_2) = (s_1 \odot s_2, s_2)$.

Lemma 3.2. *Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be an IMTL-algebra. Thus $\varphi(s_1, s_2) = (\neg s_2, \neg s_1)$ is a solution of the set-theoretical Yang-Baxter equation.*

Proof. We define φ^{12} and φ^{23} as follows:

$$\begin{aligned}\varphi^{12}(s_1, s_2, s_3) &= (\neg s_2, \neg s_1, s_3), \\ \varphi^{23}(s_1, s_2, s_3) &= (s_1, \neg s_3, \neg s_2).\end{aligned}$$

For all $(s_1, s_2, s_3) \in L^3$ we have

$$\begin{aligned}(\varphi^{12} \circ \varphi^{23} \circ \varphi^{12})(s_1, s_2, s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1, s_2, s_3))) \\ &= \varphi^{12}(\varphi^{23}(\neg s_2, \neg s_1, s_3) \\ &= \varphi^{12}(\neg s_2, \neg s_3, \neg\neg s_1) \\ &= (\neg\neg s_3, \neg\neg s_2, \neg\neg s_1) \\ &= (s_3, s_2, s_1)\end{aligned}$$

and

$$\begin{aligned}(\varphi^{23} \circ \varphi^{12} \circ \varphi^{23})(s_1, s_2, s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1, s_2, s_3))) \\ &= \varphi^{23}(\varphi^{12}(s_1, \neg s_3, \neg s_2) \\ &= \varphi^{23}(\neg\neg s_3, \neg s_1, \neg s_2) \\ &= (\neg\neg s_3, \neg\neg s_2, \neg\neg s_1) \\ &= (s_3, s_2, s_1).\end{aligned}$$

□

LEMMA 3.3. *Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be an MTL-algebra. Then the following are solutions of the set-theoretical Yang-Baxter equation:*
(a) $\varphi(s_1, s_2) = (\neg\neg s_1, s_2)$,
(b) $\varphi(s_1, s_2) = (\neg\neg s_1, s_1)$.

PROOF. (a) We define

$$\begin{aligned} \varphi^{12}(s_1, s_2, s_3) &= (\neg\neg s_1, s_2, s_3), \\ \varphi^{23}(s_1, s_2, s_3) &= (s_1, \neg\neg s_2, s_3). \end{aligned}$$

By using Proposition 2.1 (3), for all $(s_1, s_2, s_3) \in L^3$, we obtain

$$\begin{aligned} (\varphi^{12} \circ \varphi^{23} \circ \varphi^{12})((s_1, s_2, s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1, s_2, s_3))) \\ &= \varphi^{12}(\varphi^{23}(\neg\neg s_1, s_2, s_3) \\ &= \varphi^{12}(\neg\neg s_1, \neg\neg s_2, s_3) \\ &= (\neg\neg\neg\neg s_1, \neg\neg s_2, s_3) \\ &= (\neg\neg s_1, \neg\neg\neg\neg s_2, s_3) \end{aligned}$$

and

$$\begin{aligned} (\varphi^{23} \circ \varphi^{12} \circ \varphi^{23})(s_1, s_2, s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1, s_2, s_3))) \\ &= \varphi^{23}(\varphi^{12}(s_1, \neg\neg s_2, s_3) \\ &= \varphi^{23}(\neg\neg s_1, \neg\neg s_2, s_3) \\ &= (\neg\neg s_1, \neg\neg\neg\neg s_2, s_3). \end{aligned}$$

Therefore, it is a solution.
(b) It is similar to (a). □

LEMMA 3.4. *Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be an MTL-algebra. Then $\varphi(s_1, s_2) = ((s_2 \rightarrow 0) \odot 1, (s_1 \rightarrow 0) \odot 1)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. The proof is obtained from Proposition 2.1 (1) and (3). □

EXAMPLE 3.1. Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be an MTL-algebra, where $L = \{0, a, 1\}$. We define $\rightarrow$ and $\odot$ by the following Cayley tables: Then $\varphi(s_1, s_2) = ((s_2 \rightarrow 0) \odot s_1, 0)$ is a solution of the set-theoretical Yang-Baxter equation in MTL-algebra.

$\rightarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

$\odot$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

PROOF. φ^{12} and φ^{23} are defined as follows:

$$\begin{aligned} \varphi^{12}(s_1, s_2, s_3) &= ((s_2 \rightarrow 0) \odot s_1, 0, s_3), \\ \varphi^{23}(s_1, s_2, s_3) &= (s_1, ((s_3 \rightarrow 0) \odot s_2, 0). \end{aligned}$$

By using Proposition 2.1 (2) and (4), we get

$$\begin{aligned}(\varphi^{12}\circ\varphi^{23}\circ\varphi^{12})(s_1,s_2,s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1,s_2,s_3)))\\ &= \varphi^{12}(\varphi^{23}((s_2\to 0)\odot s_1,0,s_3))\\ &= \varphi^{12}((s_2\to 0)\odot s_1,(s_3\to 0)\odot 0,0)\\ &= \varphi^{12}((s_2\to 0)\odot s_1,0,0)\\ &= ((0\to 0)\odot((s_2\to 0)\odot s_1),0,0)\\ &= ((s_2\to 0)\odot s_1,0,0) \qquad (5)\end{aligned}$$

and

$$\begin{aligned}(\varphi^{23}\circ\varphi^{12}\circ\varphi^{23})(s_1,s_2,s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1,s_2,s_3)))\\ &= \varphi^{23}(\varphi^{12}(s_1,(s_3\to 0)\odot s_2,0))\\ &= \varphi^{23}((((s_3\to 0)\odot s_2)\to 0)\odot s_1,0,0)\\ &= ((((s_3\to 0)\odot s_2)\to 0)\odot s_1,(0\to 0)\odot 0,0)\\ &= ((((s_3\to 0)\odot s_2)\to 0)\odot s_1,0,0). \qquad (6)\end{aligned}$$

Since the equation (5) is equal to equation (6) for all $(s_1,s_2,s_3)\in L$, we get $\varphi(s_1,s_2)=((s_2\to 0)\odot s_1,0)$ is a solution in this MTL-algebra, but it is not in MTL-algebras. □

Similarly, $\varphi(s_1,s_2)=(0\to(s_2\odot s_1),s_1)$ and $\varphi(s_1,s_2)=((s_1\odot(s_2\to 0))\to 0,s_1)$ are also solutions for MTL-algebra in Example 3.1 whereas they are not solutions in MTL-algebras.

Example 3.2. Notice that the map $\varphi(s_1,s_2)=(s_1\to s_2,s_1)$ is a solution to the set-theoretical Yang-Baxter equation in Boolean algebras [**7**] while it is not a solution in Wajsberg algebra [**14**] and also in MTL-algebras. Since

$$\begin{aligned}(\varphi^{12}\circ\varphi^{23}\circ\varphi^{12})(s_1,s_2,s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1,s_2,s_3)))\\ &= \varphi^{12}(\varphi^{23}(s_1\to s_2,s_1,s_3))\\ &= \varphi^{12}(s_1\to s_2,s_1\to s_3,s_1)\\ &= ((s_1\to s_2)\to(s_1\to s_3),s_1\to s_2,s_1)\end{aligned}$$

and

$$\begin{aligned}(\varphi^{23}\circ\varphi^{12}\circ\varphi^{23})(s_1,s_2,s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1,s_2,s_3)))\\ &= \varphi^{23}(\varphi^{12}(s_1,s_2\to s_3,s_2)\\ &= \varphi^{23}(s_1\to(s_2\to s_3),s_1,s_2)\\ &= (s_1\to(s_2\to s_3),s_1\to s_2,s_1)\end{aligned}$$

We get $\varphi^{12}\circ\varphi^{23}\circ\varphi^{12}(s_1,s_2,s_3)\neq\varphi^{23}\circ\varphi^{12}\circ\varphi^{23}(s_1,s_2,s_3)$. If we define $s_1\to(s_2\to s_3)=(s_1\to s_2)\to(s_1\to s_3)$, then $\varphi(s_1,s_2)=(s_1\to s_2,s_1)$ is a solution in MTL-algebras.

Proposition 3.1 ([**6**]). *MTL-algebras are distributive lattices.*

LEMMA 3.5. *Let $(L,\wedge,\vee,\odot,\rightarrow,0,1)$ be an MTL-algebra. Then $\varphi(s_1,s_2)=(s_1\wedge s_2,s_1\vee s_2)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. It is easy to get the proof by using Proposition 3.1. □

EXAMPLE 3.3. $\varphi(s_1,s_2)=(s_1\wedge s_2,s_1\vee s_2)$ is a solution of the set-theoretical Yang-Baxter equation in MV-algebras [**13**] and MTL-algebras from Proposition 3.1 and Lemma 3.5 while it is usually not a solution of the set-theoretical Yang-Baxter equation in Wasjberg-algebras [**14**].

THEOREM 3.2. *Let $(L,\wedge,\vee,\odot,\rightarrow,0,1)$ be an MTL-algebra. If the MTL-algebra is an IMTL-algebra, then $\varphi(s_1,s_2)=(\neg(\neg s_1\odot\neg s_2),0)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. Let φ^{12} and φ^{23} be defined as follows:

$$\begin{aligned}\varphi^{12}(s_1,s_2,s_3) &= (\neg(\neg s_1\odot\neg s_2),0,s_3),\\ \varphi^{23}(s_1,s_2,s_3) &= (s_1,\neg(\neg s_2\odot\neg s_3),0).\end{aligned}$$

By using Definition 2.2 (b) and Proposition 2.1 (4) and (5), for all $(s_1,s_2,s_3)\in L^3$ we get

$$\begin{aligned}(\varphi^{12}\circ\varphi^{23}\circ\varphi^{12})(s_1,s_2,s_3) &= \varphi^{12}(\varphi^{23}(\varphi^{12}(s_1,s_2,s_3)))\\ &= \varphi^{12}(\varphi^{23}(\neg(\neg s_1\odot\neg s_2),0,s_3))\\ &= \varphi^{12}(\neg(\neg s_1\odot\neg s_2),\neg(\neg 0\odot\neg s_3),0)\\ &= \varphi^{12}(\neg(\neg s_1\odot\neg s_2),s_3,0)\\ &= (\neg(\neg\neg(\neg s_1\odot\neg s_2)\odot\neg s_3),0,0))\\ &= (\neg((\neg s_1\odot\neg s_2)\odot\neg s_3),0,0)\end{aligned}$$

and

$$\begin{aligned}(\varphi^{23}\circ\varphi^{12}\circ\varphi^{23})(s_1,s_2,s_3) &= \varphi^{23}(\varphi^{12}(\varphi^{23}(s_1,s_2,s_3)))\\ &= \varphi^{23}(\varphi^{12}(s_1,\neg(\neg s_2\odot\neg s_3),0))\\ &= \varphi^{23}(\neg(\neg s_1\odot\neg\neg(\neg s_2\odot\neg s_3),0,0)))\\ &= \varphi^{23}(\neg(\neg s_1\odot(\neg s_2\odot\neg s_3),0,0))\\ &= (\neg(\neg s_1\odot(\neg s_2\odot\neg s_3),\neg(\neg 0\odot\neg 0),0)\\ &= (\neg(\neg s_1\odot(\neg s_2\odot\neg s_3),0,0).\end{aligned}$$

Since $(L,\odot,1)$ is a commutative monoid,

$$(\varphi^{12}\circ\varphi^{23}\circ\varphi^{12})(s_1,s_2,s_3)=(\varphi^{23}\circ\varphi^{12}\circ\varphi^{23})(s_1,s_2,s_3) \text{ for all } (s_1,s_2,s_3)\in L^3.$$

Therefore, if the MTL- algebra is an IMTL-algebra then $\varphi(s_1,s_2)=(\neg(\neg s_1\odot\neg s_2),0)$ is a solution in MTL-algebras. □

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