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PERIPHERAL WIENER INDEX OF GRAPH OPERATIONS

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Abstract. Peripheral Wiener index of a graph is the sum of the distance of the peripheral vertices of a graph. In this paper the peripheral Wiener index of graph operations is investigated.

1. Introduction

Throughout this paper we consider only simple connected graphs. For a graph G, the sets $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. The distance between two vertices of a graph G is denoted by $d_G(u,v)$ (or simply $d(u,v)$, if there is no ambiguity), which is defined as the length of a shortest path between vertex u and vertex v in G.

A graph invariant is a real number related to a graph G which is invariant under graph isomorphism. In chemistry, graph invariants are known as topological indices. Topological indices have many applications as tools for modelling chemical and other properties of molecules [**6, 4, 12**]. The Wiener index is one of the first and most studied topological indices, both from theoretical point of view and applications in chemistry. The Wiener index of a graph G, denoted by $W(G)$, was introduced in 1947 by chemist Harold Wiener [**14**] as the sum of distances between all vertices of G:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v).$$

The peripheral Wiener index is also defined in [**11**] as the sum of the distances of unordered pair of peripheral vertices of a graph. That is

$$PW(G) = \sum_{\{u,v\}\subseteq PV(G)} d(u,v)$$

where, $PV(G)$ is the set of peripheral vertices of graph G.

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Computing topological indices of graph operations has been the object of some papers. For instance, Yeh and Gutman in [**13**] computed the Wiener index in the case of graphs that are obtained by means of certain binary operations (such as product, join, and composition) on pairs of graphs. In this paper the peripheral Wiener index of some graph operations is investigated. Throughout this paper our notations are taken mainly from [**2, 3**].

The Cartesian product $G\Box H$ of graphs G and H has the vertex set $V(G\Box H) = V(G) \times V(H)$ and $(a,x)(b,y)$ is an edge of $G\Box H$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $x = y$ [**7**].

Cartesian product of graphs has interesting properties.

Proposition 1.1 ([**9**]). *The Cartesian product of two graphs is connected if and only if both factors are connected.*

Another property of Cartesian product is:

Corollary 1.1 ([**9**]). *Let (u,v) and (x,y) be arbitrary vertices of the Cartesian product $G\Box H$. Then*

$$d_{G\Box H}((u,v),(x,y)) = d_G(u,x) + d_H(v,y). \tag{1.1}$$

Moreover, if Q is the shortest path between (u,v) and (x,y), then the projection p_1Q is a shortest path in G from u to x and the projection p_2Q is a shortest path in H from v to y.

By restricting the distance of vertices to the diameters of each factor graphs, and number of peripheral vertices in each vertices sets in equation (1.1) of Corollary 1.1, we have the following corollary.

Corollary 1.2. *If diameters of G and H are d_1 and d_2, respectively, then the diameter $diam(G\Box H) = d_1 + d_2$. Moreover, If G and H has peripheral vertices of k_1 and k_2, respectively, then $G\Box H$ has $k_1 \times k_2$ number of peripheral vertices.*

Another graph operation is the join of two graphs. The join $G_1 + G_2$ of two graphs G_1 and G_2 is a graph formed by a vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{u_1u_2 | u_1 \in V(G_1), u_2 \in V(G_2)\}$ [**8**].

Lemma 1.1.

$$|V(G_1 + G_2)| = |V(G_1)| + |V(G_2)|.$$
$$|E(G_1 + G_2)| = |E(G_1)| + |E(G_2)| + |V(G_1)| \times |V(G_2)|.$$

Another well-known graph operation is the composition of two graphs. In [**7**] the composition $G_1[G_2]$ of graphs G_1 and G_2 is defined as $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and two vertices $u = (u_1,u_2)$, $v = (v_1,v_2)$ of $G_1[G_2]$ are adjacent whenever $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$ or $[u_1v_1 \in E(G_1)]$.

Lemma 1.2.

$$|V(G_1[G_2])| = |V(G_1)| \times |V(G_2)|.$$
$$|E(G_1[G_2])| = |V(G_1)||E(G_2)| + |V(G_2)|^2|E(G_1)|.$$

In [**7**], the strong product $G \boxtimes H$ of G and H is defined as follows:

$$V(G \boxtimes H) = V(G) \times V(H).$$

$$E(G \boxtimes H) = \{(u,v)(x,y) | u = x, \quad xy \in E(H), \quad or \quad ux \in E(G), \quad v = y, \quad or \quad ux \in E(G), \quad vy \in E(H)\}.$$

PROPOSITION 1.2 ([**7**]). *If (g,h) and (g',h') are vertices of a strong product $G \boxtimes H$, then*

$$d_{G\boxtimes H}((g,h),(g',h')) = max\{d_G(g,g'), d_H(h,h')\}. \tag{1.2}$$

COROLLARY 1.3 ([**7**]). *(Distance Formula) If $G = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$ and $x, y \in V(G)$, then*

$$d_G(x,y) = \max_{1\leqslant i\leqslant k}\{d_{G_i}(p_i(x), p_i(y))\}$$

where, p_i is the i^{th} projection.

COROLLARY 1.4 ([**7**]). *A strong product of graphs is connected if and only if every one of its factors is connected.*

Corona product of graphs is also one of the well-studied graph operations. The corona $G_1 \circ G_2$ of graphs G_1 and G_2 is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and by joining each vertex of the i^{th} copy of G_2 to the i^{th} vertex of $G_1, i = 1, 2, \ldots, |V(G_1)|$ [**5**].

2. Main results

THEOREM 2.1. *Let G_1 and G_2, be graphs with radius $r_1 \geqslant 2$, and $r_2 \geqslant 2$, respectively. Then*

$$PW(G_1 + G_2) = 2\binom{n_1}{2} - m_1 + 2\binom{n_2}{2} - m_2 + n_1 n_2$$

where, n_i are number of peripheral vertices and m_i are number of edges of G_i, $i = 1, 2$, respectively.

PROOF. Note that, since $r_i \geqslant 2$, $i = 1, 2$ and the diameter of $G_1 + G_2$ is 2, the graph is a self-centered graph. Consequently, the peripheral vertex set of $G_1 + G_2$ consists of all the vertices $V(G_1) \cup V(G_2)$ and the peripheral Wiener index is the same as Wiener index of $G_1 + G_2$. Therefore, by Theorem 2 of [**13**], the assertion holds. □

THEOREM 2.2. *Let G_1 be a graph with k_1 number of peripheral vertices. For any graph G_2 of order n_2 and size m_2, the peripheral Wiener index of the composition $G_1[G_2]$ is*

$$PW(G_1[G_2]) = n_2^2 PW(G_1) + k_1 \left[2\binom{n_2}{2} - m_2\right].$$

PROOF. Let G_1 be a graph of order n_1, with k_1 peripheral vertices. Let G_2 be any graph of order n_2 and size m_2. Observe that the composition $G_1[G_2]$ can be obtained by taking n_1 copies of G_2 and by joining all the vertices of the i^{th} and the

j^{th} copy of G_2 if and only if the i^{th} and the j^{th} vertices of G_1 are adjacent. Hence the vertices in each copy of G_2 of $G_1[G_2]$ are either adjacent (if they were adjacent in G_2) or are at a distance two (if they were non-adjacent in G_2). Moreover, the number of peripheral vertices of $G_1[G_2]$ is k_1n_2. A pair of vertices of $G_1[G_2]$ belonging to different copies of G_2 has the same distance as the corresponding two vertices of G_1. That is, the diameter of $G_1[G_2]$ is equal to the diameter of G_1. Next we calculate the sum of the distances between all pairs of peripheral vertices by considering the following two cases: Case 1, within one copy of G_2 and case 2, within different copies of G_2.

Case 1: If u, v are peripheral vertices within one copy of G_2, then

$$\sum d_{G_1[G_2]}(u,v) = 2\left[\binom{n_2}{2} - m_2\right] + m_2 = 2\binom{n_2}{2} - m_2.$$

Since we have k_1 number of copies of G_2 in the peripheral vertex set of $G_1[G_2]$, this sum should be multiplied k_1 times.

Case 2: If u, v are taken from different copies of G_2 in the peripheral vertices set of $G_1[G_2]$,

$$\begin{aligned}\sum_{G_1[G_2]} d(u,v) &= n_2^2 \sum_{\{u,v\}\in PV(G_1)} d_{G_1}(u,v)\\ &= n_2^2 PW(G_1).\end{aligned}$$

Thus,

$$PW\left(G_1[G_2]\right) = \sum_{\{u,v\}\subseteq PV(G_1[G_2])} d_{G_1[G_2]}(u,v) = n_2^2 PW(G_1) + k_1\left[2\binom{n_2}{2} - m_2\right].$$

□

Theorem 2.3. *The peripheral Wiener index of strong product $G \boxtimes H$ of graphs G and H is*

$$PW(G \boxtimes H) = n_2^2 PW(G) + k_1 W(H)$$

where, n_1 and n_2 are the orders of G and H, respectively.

Proof. Let G and H be two graphs with k_1 and k_2 peripheral vertices, respectively. Without loss of generality, assume $diam(G) \geqslant diam(H)$. By definition of $\boxtimes$, the peripheral vertices of $G \boxtimes H$ is $k_1 \times k_2$. By equation (1.2), the diameter of $G \boxtimes H$ is $max\{diam(G), diam(H)\}$. Thus, if we denote $u = (u_i, u_j)$ and $v = (v_k, v_r)$ arbitrary vertices from peripheral vertex set of $G \boxtimes H$, by simple counting, we have

$$\begin{aligned}PW(G \boxtimes H) &= \sum_{\{u.v\}\subseteq PV(G\boxtimes H)} d_{G\boxtimes H}(u,v)\\ &= n_2^2 \sum_{\{u_i,v_k\}\subseteq PV(G)} d_G(u_i,v_k) + k_1 \sum_{(u_j,v_r)\in V(H)} d_H(u_j,v_r)\\ &= n_2^2 PW(G) + k_1 W(H).\end{aligned}$$

□

THEOREM 2.4. *Let the diameter of a graph G_1, $diam(G_1) = D \geqslant 2$. Let G_1 has k_1 number of peripheral vertices and the order and size of G_2 are n_2 and m_2, respectively. The peripheral Wiener index of the corona product of $G_1 \circ G_2$ is given by*

$$PW(G_1 \circ G_2) = k_1 \left[2\binom{n_2}{2} - m_2 \right] + (D+2)k_1 n_2.$$

PROOF. Let G_1 be a graph of order n_1, and $diam(G_1) = D \geqslant 2$. Since the corona product $G_1 \circ G_2$ is obtained by taking one copy of G_1 and n_1 copies of G_2 and joining the i^{th} vertex of G_1 to each vertex of the i^{th} copy of G_2, the vertices in each copy of G_2 in $G_1 \circ G_2$ are either adjacent (if they were adjacent in G_2) or are at a distance two (if they were non-adjacent in G_2). Moreover the number of peripheral vertices in $G_1 \circ G_2 = k_1 \times n_2$. The diameter of $G_1 \circ G_2 = D + 2$. Considering the two cases similar to the proof of Theorem 2.2, we arrive at:

$$PW(G_1 \circ G_2) = k_1 \left[m_2 + 2\left[\binom{n_2}{2} - m_2 \right] \right] + (D+2)k_1 n_2.$$

$\square$

The peripheral Wiener index of Cartesian product of G and H is calculated in [**11**]. Here we give an alternative proof.

THEOREM 2.5. *The peripheral Wiener index of the Cartesian product $G \square H$ of two graphs G and H is*

$$PW(G \square H) = k_2^2 PW(G) + k_1^2 PW(H),$$

where, k_1 and k_2 are the orders of $PV(G)$ and $PV(H)$, respectively.

PROOF. Let $V_1 \subseteq V(G)$ and $V_2 \subseteq V(H)$. Then for $V = V_1 \times V_2$, it follows that

$$\begin{aligned} d(V,V) &= d(V_1 \times V_2, V_1 \times V_2) \\ &= \sum_{(u_1,u_2)} \sum_{(v_1,v_2)} d((u_1,u_2),(v_1,v_2)) \\ &= \sum_{(u_1,u_2)} \sum_{(v_1,v_2)} (d(u_1,v_1) + d(u_2,v_2)) \qquad \textit{by Corollary } 1.1 \\ &= |V_2|^2 \sum_{u_1} \sum_{v_1} d(u_1,v_1) + |V_1|^2 \sum_{u_2} \sum_{v_2} d(u_2,v_2) \\ &= |V_2|^2 d(V_1,V_1) + |V_1|^2 d(V_2,V_2). \end{aligned}$$

By letting $V = V_1 \times V_2 = PV(G) \times PV(H)$, we obtain the result of the theorem. $\square$

G. Indulala and R. Balakrishnan defined the Indu-Bala product $G_1 \blacktriangledown G_2$ of graphs G_1 and G_2 in [**10**] as a graph obtained from two disjoint copies of the join $G_1 + G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2. Observe that:

(1) $|V(G_1 \blacktriangledown G_2)| = 2(n_1 + n_2)$, where $n_i = V(G_i)$, $i = 1, 2$.
(2) $|E(G_1 \blacktriangledown G_2)| = 2(m_1 + m_2 + n_1 n_2) + n_2$, where, $n_i = |V(G_i)|$, $m_i = |E(G_i)|$, $i = 1, 2$.

THEOREM 2.6. *Let G_1 be a graph of order n_1 and size m_1. Then for any graph G_2; the peripheral Wiener index of the Indu-Bala product $G_1 \blacktriangledown G_2$ of G_1 and G_2 is*

$$PW(G_1 \blacktriangledown G_2) = 3n_1^2 + 2n_1(n_1 - 1) - 2m_1.$$

PROOF. Since $G_1 \blacktriangledown G_2$ can be obtained from two disjoint copies of the join $G_1 + G_2$ of G_1 and G_2 by joining the corrosponding vertices in the two copies of G_2, the maximum distance of vertices occurs between vertices of different copies of G_1. Moreover, the maximum distance is 3. The set of the peripheral vertices of $G_1 \blacktriangledown G_2$ contains only the vertices of the two copies of $V(G_1)$. Hence, if we denote the peripheral vertex set of $G_1 \blacktriangledown G_2$ by $PV(G_1 \blacktriangledown G_2) = V_1(G_1) \cup V_2(G_1)$, where $V_1(G_1)$ and $V_2(G_1)$ are the two copies of $V(G_1)$, then

$$\begin{aligned} PW(G_1 \blacktriangledown G_2) &= \sum_{\{u,v\} \subseteq PV(G_1 \blacktriangledown G_2)} d(u,v) \\ &= \sum_{\{u_i,v_i\} \subseteq V_i(G_1)} d(u_i,v_i) + \sum_{\{u_i,v_j\} \subseteq V_i(G_1) \cup V_j(G_1)} d(u_i,v_j), i \neq j, \quad i,j = 1,2 \\ &= \left(2\left[\binom{n_1}{2} - m_1\right] + m_1\right) 2 + 3n_1^2 \\ &= 2n_1(n_1 - 1) - 2m_1 + 3n_1^2. \end{aligned}$$

□

A. Arivalagan and et al [**1**] defined two graph operations . The first one is defined as follows:

Let G be a graph. If we put two similar graphs G side by side, and any vertex of the first graph G is connected by edges with those vertices which are adjacent to the corresponding vertex of the second graph G and the resultant graph is denoted by $K_2 \bullet G$, then we have $V(K_2 \bullet G) = 2V(G)$ and $E(K_2 \bullet G) = 4E(G)$. This means that $K_2 \bullet G$ is the graph of K_2 and G with the vertex set $V(K_2 \bullet G) = V(K_2) \times V(G)$ and $(u_i, v_j)(u_k, v_r)$ is an edge of $K_2 \bullet G$ whenever $u_i = u_k$ and $v_j v_r \in E(G)$ or $u_i \neq u_k$ and $v_j v_r \in E(G)$.

The second graph operation is defined as a graph formed by putting two copies of graph G side by side, and any vertex of the first copy of G is connected by edges with those vertices which are nonadjacent to the corresponding vertex (including the corresponding vertex itself) of the second copy of G and the resultant graph is denoted by $K_2 \star G$, then we have $|V(K_2 \star G)| = 2|V(G)|$ and $|E(K_2 \star G)| = |V(G)|^2$. Moreover, $K_2 \star G$ is the graph of K_2 and G with the vertex set $V(K_2 \star G) = V(K_2) \times V(G)$ and $(u_i, v_j)(u_k, v_r)$ is an edge of $K_2 \star G$ whenever $u_i = u_k$ and $v_j v_r \in E(G)$ or $u_i \neq u_k$ and $v_j v_r \in E(G)$.

By simple calculation, we have the following two results.

THEOREM 2.7. *Let G be a graph with k_1 peripheral vertices and diameter, $D \geqslant 3$. Then,*

$$PW(K_2 \bullet G) = 4PW(G) + 2k_1$$

THEOREM 2.8. *Let G be a graph of order n and size m. Then*

$$PW(K_2 \star G) = 3n^2 - 2n + m.$$

References

[1] A. Arivalagan, K. Pattabiraman and V. S. A. Subramanian. The Wiener related indices of some graph operations. *Int. J. Math. Research*, **6**(2)(2014), 121–134.
[2] F. Buckley abd F. Harary. *Distance in Graphs.* Addison-Wesley, Red-wood 1990.
[3] G. Chartrand, L. Lesniak and P. Zhang. *Graphs and Digraphs, (6^{th} ed.),* CRC Press, Boka Raton 2016.
[4] M.V. Diudea, I. Gutman and L. Jantschi. *Molecular Topology.* Huntington, New York 2001.
[5] G. H. Fath-Tabar, A. Hamzeh and S. Hossein-Zadeh. GA_2 index of some graph operations. *Filomat*, **24**(1)(2010), 21–28.
[6] S. Gupta, M. Singh and A. K. Madan. Application of graph theory: Relationship of eccentric connectivity index and Wiener's index with anti-inflammatory activity. *J. Math. Anal. Appl.*, **266**(2)(2002), 259–268.
[7] R. Hammack, W. Imrich and S. Klavžar. *Handbook of Product Graphs (2^{nd} Ed.),* CRC Press, Boka Raton 2011.
[8] F. Harary. *Graph Theory.* Narosa Publishing House, New Delhi 1969.
[9] W. Imrich and S. Klavžar. *Product Graphs Structure and Recognition.* John Wiley and Sons Inc., New York 2000.
[10] G. Indulala and R. Balakrishnan. Distance spectrum of InduBala product of graphs. *AKCE International Journal of Graphs and Combinatorics*, **13**(3)(2016), 230–234.
[11] K. Narayankar and S. B. Lokesh. Peripheral Wiener index of a graph. *Communications in Combinatorics and Optimization*, **2**(1)(2017), 43–56.
[12] N. Trinajstić. *Chemical Graph Theory.* CRC Press, Boca Raton FL, 1992.
[13] Y-N. Yeh and I. Gutman. On the sum of all distances in composite graphs. *Discrete Math.*, **135**(1-3)(1994), 359-365.
[14] H. Wiener. Structural determination of the paraffin boiling points. *J. Am. Chem. Soc.*, **69**(1)(1947), 17–20.

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