BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 9(2019), 577-589
DOI: 10.7251/BIMVI1903577S

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

NEW INVESTIGATION OF ĆIRIĆ TYPE FUZZY SOFT CONTRACTIVE MAPPING IN FUZZY SOFT METRIC SPACES

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Abstract. This paper is view to introduce the notion of fuzzy soft metric space with some of its important properties. In this way we introduce and investigate Ciric type fuzzy soft contractive mappings to establish fixed point theorem of the mapping in the context of fuzzy soft metric spaces.

1. Introduction

In daily life, the problems in many fields concern with uncertain data and are not successfully modelled in classical mathematics. There are two types of mathematical tools to deal with uncertainties namely fuzzy set theory introduced by Zadeh [**16**] and the theory of soft sets initiated by Molodstov [**12**] which helps to solve problems in all areas. Maji et al. ([**10**], [**9**]) introduced several operations in soft sets, also Ahmad and Kharal [**1**] have coined fuzzy soft sets. Recently Beaula et al. [**3**] introduced the notion of fuzzy soft metric space as a genearlization of soft metric space.

The concept of fuzzy soft topology firstly introduced by Tanay and Kandemir [**14**]. They defined the concept of fuzzy soft topology as a topology over the given fuzzy soft set f_A. So a fuzzy soft topology in the sense of Tanay and Kandemir [**14**] is the collection τ of fuzzy soft subsets of f_A closed under arbitrary supremum and finite infimum. It also contains $\tilde{\phi}$ and $\tilde{E}$. But Roy and Samanta [**13**] redefined the concept of fuzzy soft topology. The most significant reason for such a change

2010 *Mathematics Subject Classification.* Primary 54A40, 54E50, 54E40; Secondary 03E72, 47H10.

Key words and phrases. Soft set, Fuzzy set, Fuzzy soft set, Fuzzy soft metric space, Ciric type fuzzy soft contractive mapping.

is to make sure that the DeMorgan Laws hold in the new definition of fuzzy soft topology. Here we recall the definition of fuzzy soft topology as introduced in [**13**].

In this paper, we examine some important properties of fuzzy soft metric spaces and prove some fixed point theorems of fuzzy soft contractive mappings. In this way, we generalize some fixed point results of soft metric spaces.

2. Preliminaries

Here, in this section we present some basic definitions of fuzzy soft set and fuzzy soft metric space.
Throughout our discussion, X refers to an initial universe, E the set of all parameters for X and $P(X)$ denotes the power set of X.

Definition 2.1. ([**16**]) A fuzzy set A in a non-empty set X is characterized by a membership function $\mu_A : X \to [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. Let I^X denotes the family of all fuzzy sets on X.

A member A in I^X is contained in a member B of I^X denoted $A \leqslant B$ if and only if $\mu_A(x) \leqslant \mu_B(x)$ for every $x \in X$ (see, [**16**]).

Let $A, B \in I^X$, we have the following fuzzy sets (see, [**16**]).

(1) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$. (Equality),
(2) $C = A \wedge B \in I^X$ by $\mu_C(x) = min\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$. (Intersction),
(3) $D = A \vee B \in I^X$ by $\mu_C(x) = max\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$.(Union),
(4) $E = A^c \in I^X$ by $\mu_E(x) = 1 - \mu_A(x)$ for all $x \in X$. (Complement).

Definition 2.2. ([**16**]) An empty fuzzy set denoted by $\bar{0}$ is a function which maps each $x \in X$ to 0. That is, $\bar{0}(x) = 0$ for all $x \in X$. A universal fuzzy set denoted by $\bar{1}$ is a function which maps each $x \in X$ to 1. That is, $\bar{1}(x) = 1$ for all $x \in X$.

Definition 2.3. ([**12**]) Let $A \subseteq E$. A pair (F, A) is called a soft set over X if F is a mapping $F : A \to P(X)$.

In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\epsilon \in A$, $F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A), or as the set of ϵ-approximate elements of the soft set.

Definition 2.4. ([**9**]) A pair (f, A), denoted by f_A, is called a fuzzy soft set over X, where f is a mapping given by $f : A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$ where

$$\mu^e_{f_A} = \begin{cases} \tilde{0}, & \text{if } e \notin A; \\ otherwise, & \text{if } e \in A. \end{cases}$$

$\widetilde{(X, E)}$ denotes the class of all fuzzy soft sets over (X, E) and is called a fuzzy soft universe (see, [**10**]).

Definition 2.5. ([**11**]) A fuzzy soft set F_A over X is said to be:

(a) NULL fuzzy soft set, denoted by $\tilde{\phi}$, if for all $e \in A, f_A(e) = \bar{0}$.
(b) absolute fuzzy soft set, denoted by $\tilde{E}$, if for all $e \in A, f_A(e) = \bar{1}$.

Definition 2.6. ([**13**]) The complement of a fuzzy soft set (f, A), denoted by $(f, A)^c$, is defined by $(f, A)^c = (f^c, A), f^c_A : E \to I^U$ is a mapping given by $\mu^e_{f^c_A} = \tilde{1} - \mu^e_{f_A}$, where $\tilde{1}(x) = 1$,for all$x \in X$. Clearly $(f^c_A)^c = f_A$.

Definition 2.7. ([**13**]) Let $f_A, g_B \tilde{\in} \widetilde{(X, E)}$. f_A is fuzzy soft subset of g_B, denoted by $f_A \tilde{\subseteq} g_B$, if $A \subseteq B$ and $\mu^e_{f_A} \leqslant \mu^e_{g_B}$for all$e \in A$, i.e. $\mu^e_{f_A}(x) \leqslant \mu^e_{g_B}(x)$for all$x \in X$ and $forall\ e \in A$.

Definition 2.8. ([**13**]) Let $f_A, g_B \tilde{\in} \widetilde{(X, E)}$. The union of f_A and g_B is also a fuzzy soft set h_C, where $C = A \cup B$ and for all $e \in C, h_C(e) = \mu^e_{h_c} = \mu^e_{f_A} \vee \mu^e_{g_B}$. Here we write $h_C = f_A \tilde{\cup} g_B$.

Definition 2.9. ([**13**]) Let $f_A, g_B \tilde{\in} \widetilde{(X, E)}$. The intersection of f_A and g_B is also a fuzzy soft set d_C, where $C = A \cap B$ and for all $e \in C, d_C(e) = \mu^e_{d_c} = \mu^e_{f_A} \wedge \mu^e_{g_B}$. Here we write $d_C = f_A \tilde{\cap} g_B$.

Definition 2.10. ([**8**]) The fuzzy soft set $f_A \tilde{\in} \widetilde{(X, E)}$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha(0 < \alpha \leqslant 1)$ and $\mu^e_{f_A}(y) = 0$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e.

Definition 2.11. ([**8**]) The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set (g, A), denoted by $x^e_\alpha \tilde{\in} (g, A)$, if for the element $e \in A$, $\alpha \leqslant \mu^e_{g_A}(x)$.

Definition 2.12. ([**3**]) Let f_A be fuzzy soft set over X. The two fuzzy soft points $f_{e_1}, f_{e_2} \in f_A$ are said to be equal if $\mu_{f_{e_1}}(x) = \mu_{f_{e_2}}(x)$ for all $x \in X$. Thus $f_{e_1} \neq f_{e_2}$ if and only $\mu_{f_{e_1}}(x) \neq \mu_{f_{e_2}}(x)$ for all $x \in X$.

Proposition 2.1 ([**3**]). *The union of any collection of fuzzy soft points can be considered as a fuzzy soft set and every fuzzy soft set can be expressed as the union of all fuzzy soft points.*
$f_A = \{\tilde{\cup}_{f_e \tilde{\in} f_A} f_e : e \in E\}$

Proposition 2.2 ([**3**]). *Let f_A, f_B be two fuzzy soft sets then $f_A \tilde{\subseteq} f_B$ if and only if $f_e \tilde{\in} f_A$ implies $f_e \tilde{\in} f_B$ and hence $f_A = f_B$ if and only if $f_e \tilde{\in} f_A$ and only if $f_e \tilde{\in} f_B$.*

Definition 2.13. ([**6**]) Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters, $A \subseteq E$. Then a mapping $f : A \to B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, it will be called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0, \tilde{1}(e) = 1$ for all $e \in E$ respectively.

The set of all soft real numbers is denoted by $\mathcal{R}(A)$ and the set of all non-negative soft real numbers by $\mathcal{R}(A)^*$.

Definition 2.14. ([**5**]) A (non negative) fuzzy soft real number is a fuzzy set on the set of all (non negative) soft real numbers $\mathcal{R}(A)$, that is, a mapping $\tilde{\tilde{\lambda}} : \mathcal{R}(A) \to [0,1]$, associating with each (non negative) soft real number $\tilde{t}$, its grade of membership $\tilde{\tilde{\lambda}}(\tilde{t})$ satisfying the following conditions:

(i) $\tilde{\tilde{\lambda}}$ is convex
that is, $\tilde{\tilde{\lambda}}(\tilde{t}) \geqslant \min\ (\tilde{\tilde{\lambda}}(\tilde{s}), \tilde{\tilde{\lambda}}(\tilde{r}))$ for $\tilde{s} \subseteq \tilde{t} \subseteq \tilde{r}$

(ii) $\tilde{\tilde{\lambda}}$ is normal
that is, there exists $\tilde{t_0} \in \mathcal{R}(A)^*$ such that $\tilde{\tilde{\lambda}}(\tilde{t_0}) = 1$.

(iii) $\tilde{\tilde{\lambda}}$ is upper semi continuous provided for all $\tilde{t} \in \mathcal{R}(A)$ and $\alpha \in [0,1]$
$\tilde{\tilde{\lambda}}(\tilde{t}) < \alpha$, there is a $\delta > 0$ such that $\| \tilde{s} - \tilde{t} \| \leqslant \delta$ implies that $\tilde{\tilde{\lambda}}(\tilde{s}) < \alpha$

Definition 2.15. ([**13**]) A fuzzy soft topology τ on X is a family of fuzzy soft sets over X satisfying the following properties

(i) $\tilde{\phi}, \tilde{E} \in \tau$

(ii) if $f_A, g_B \in \tau$, then $f_A \tilde{\cap} g_B \in \tau$,

(iii) if $f_{A_\alpha} \in \tau$ for all $\alpha \in \Delta$ an index set, then $\tilde{\bigcup}_{\alpha \in \Delta} f_{A_\alpha} \in \tau$.

Definition 2.16. ([**13**]) If τ is a fuzzy soft topology on X the triple (X, E, τ) is said to be a fuzzy soft topological space. Also each member of τ is called a fuzzy soft open set in (X, E, τ).

The complement of a fuzzy soft open set is a fuzzy soft closed set.

Definition 2.17. ([**14**]) Let (X, E, τ) be a fuzzy soft topological space. Let f_A be a fuzzy soft set over X. The fuzzy soft closure of f_A is defined as the intersection of all fuzzy soft closed sets which contained f_A and is denoted by $\bar{f_A}$ or $cl(f_A)$ we write

$$cl(f_A) = \tilde{\bigcap}\{g_B : g_B \text{ is fuzzy soft closed and } f_A \tilde{\subseteq} f_B\}.$$

Definition 2.18. ([**14**]) Let (X, E, τ) be a fuzzy soft topological space. Let f_A be a fuzzy soft set over X. The fuzzy soft interior of f_A denoted by f_A^o is the union of all fuzzy soft open subsets of f_A. Clearly, f_A^o is the largest fuzzy soft open set over X which contained in f_A.

Definition 2.19. ([**7**]) Let (X, E, τ) be a fuzzy soft topological space. Let f_A be a fuzzy soft set over X. The fuzzy soft boundary of f_A denoted by ∂f_A is defined as $\partial f_A = \bar{f_A} \tilde{\cap} \bar{f'_A}$.

Definition 2.20. ([**8**]) A fuzzy soft topological space $(X, E, \tau))$ is said to be a fuzzy soft normal space if for every pair of disjoint fuzzy soft closed sets h_A and k_A, $\exists$ disjoint fuzzy soft open sets g_{1_A}, g_{2_A} such that $h_A \tilde{\subseteq} g_{1_A}$ and $k_A \tilde{\subseteq} g_{2_A}$.

Definition 2.21. ([**2**]) Let $\widetilde{(X, E)}$ and $\widetilde{(Y, E')}$ be classes of fuzzy soft sets over X and Y with attributes (the set of all parameters)from E and E' respectively. Let $\varphi : X \longrightarrow Y$ and $\psi : E \longrightarrow E'$ be two mappings. Then $\varphi_\psi = (\varphi, \psi) : \widetilde{(X, E)} \longrightarrow \widetilde{(Y, E')}$ is called a fuzzy soft mapping from $\widetilde{(X, E)}$ to $\widetilde{(Y, E')}$.

If φ and ψ is injective then the fuzzy soft mapping $\varphi_\psi = (\varphi, \psi)$ is said to be injective.

If φ and ψ is surjective then the fuzzy soft mapping $\varphi_\psi = (\varphi, \psi)$ is said to be surjective.

If φ and ψ is constant then the fuzzy soft mapping $\varphi_\psi = (\varphi, \psi)$ is said to be constant.

Definition 2.22. ([**15**]) Let (X, E, τ_1) and (X, E, τ_2) be two fuzzy soft topological spaces.

(i) A fuzzy soft mapping $\varphi_\psi = (\varphi, \psi) : (\widetilde{X, E, \tau_1}) \longrightarrow (\widetilde{X, E, \tau_2})$ is called fuzzy soft continuous if $\varphi_{\psi^{-1}}(g_B) \in \tau_1$,for all$g_B \in \tau_2$.

(ii) A fuzzy soft mapping $\varphi_\psi = (\varphi, \psi) : (\widetilde{X, E, \tau_1}) \longrightarrow (\widetilde{X, E, \tau_2})$ is called fuzzy soft open if $\varphi_\psi \tilde{\in} \tau_2$,for all$f_A \in \tau_1$.

Let $A \subseteq E$ and The collection of all fuzzy soft points of a fuzzy soft set f_A over X be denoted by $FSC(f_A)$.

Let $\mathcal{R}(A)^*$ be the set of all non negative fuzzy soft real numbers. The fuzzy soft metric using fuzzy soft points is defined as follows:

Definition 2.23. ([**3**]) Let $A \subseteq E$ and $\tilde{E}$ be the absolute fuzzy soft set. A mapping $\tilde{d} : FSC(\tilde{E}) \times FSC(\tilde{E}) \to \mathcal{R}(A)^*$ is said to be a fuzzy soft metric on $\tilde{E}$ if $\tilde{d}$ satisfies the following conditions:
$(FSM_1) : \tilde{d}(f_{e_1}, f_{e_2}) \tilde{\geqslant} \tilde{0}$ for all $f_{e_1}, f_{e_2} \tilde{\in} \tilde{E}$,
$(FSM_2) : \tilde{d}(f_{e_1}, f_{e_2}) = \tilde{0}$ if and only if $f_{e_1} = f_{e_2}$ for al $f_{e_1}, f_{e_2} \tilde{\in} \tilde{E}$,
$(FSM_3) : \tilde{d}(f_{e_1}, f_{e_2}) = \tilde{d}(f_{e_2}, f_{e_1})$ for all $f_{e_1}, f_{e_2} \tilde{\in} \tilde{E}$,
$(FSM_4) : \tilde{d}(f_{e_1}, f_{e_3}) \tilde{\leqslant} \tilde{d}(f_{e_1}, f_{e_2}) + \tilde{d}(f_{e_2}, f_{e_3})$ for all $f_{e_1}, f_{e_2}, f_{e_3} \tilde{\in} \tilde{E})$.

The fuzzy soft set $\tilde{E}$ with the fuzzy soft metric $\tilde{d}$ is called the fuzzy soft metric space and is denoted by $(\tilde{E}, \tilde{d})$

Definition 2.24. ([**4**]) Let $(\tilde{E}, \tilde{d})$ be a fuzzy soft metric space and $\tilde{\tilde{t}}$ be a fuzzy soft real number and $\tilde{\epsilon} \in (0, 1)$. A fuzzy soft open ball centered at the fuzzy point $f_e \tilde{\in} \tilde{E}$ and radius $\tilde{\tilde{t}}$ is a collection of all fuzzy soft points g_e of $\tilde{E}$ such that $\tilde{d}(g_e, f_e) \tilde{<} \tilde{\tilde{t}}$. It is denoted by $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ where $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}) = \{g_e \tilde{\in} \tilde{E} | \tilde{d}(g_e, f_e) \tilde{<} \tilde{\tilde{t}}\}$ with $| \mu^a_{g_e}(x) - \mu^a_{f_e}(x) | < \tilde{\epsilon}$ for all $a \in E, x \in X$.

The fuzzy soft closed ball is denoted by $\tilde{\tilde{B}}[f_e, \tilde{\tilde{t}}, \tilde{\epsilon}] = \{g_e \tilde{\in} \tilde{E} | \tilde{d}(g_e, f_e) \tilde{\leqslant} \tilde{\tilde{t}}\}$ with $| \mu^a_{g_e}(x) - \mu^a_{f_e}(x) | \leqslant \tilde{\epsilon}$ for all $a \in E, x \in X$.

Definition 2.25. ([**4**]) A sequence $\{f_{e_n}\}$ in a fuzzy soft metric space $(\tilde{E}, \tilde{d})$ is said to converge to $f_{e'}$ if $\tilde{d}(f_{e_n}, f_{e'}) \to \tilde{0}$ as $n \to \infty$ for every $\tilde{\epsilon} \tilde{>} \tilde{0}$ there exists $\tilde{\delta} \tilde{>} \tilde{0}$ and a positive integer $N = N(\tilde{\epsilon})$ such that $\tilde{d}(f_{e_n}, f_{e'}) \tilde{<} \tilde{\delta}$ implies $| \mu^a_{f_{e_n}}(x) - \mu^a_{f_{e'}}(x) | < \tilde{\epsilon}$ whenever $n \geqslant N, a \in E$ and $x \in X$. It is usually denoted as lim $_{n \to \infty} f_{e_n} = f_{e'}$.

Definition 2.26. ([**4**]) A sequence $\{f_{e_n}\}$ in a fuzzy soft metric space $(\tilde{E}, \tilde{d})$ is said to be a Cauchy sequence if to every $\tilde{\epsilon} \tilde{>} \tilde{0}$ there exists $\tilde{\delta} \tilde{>} \tilde{0}$ and a positive integer $N = N(\tilde{\epsilon})$ such that $\tilde{d}(f_{e_n}, f_{e_m})\tilde{<}\tilde{\delta}$ implies $\mid \mu^a_{f_{e_n}}(x) - \mu^a_{f_{e_m}}(x) \mid < \tilde{\epsilon}$ for all $n, m \geqslant N, a \in E$ and $x \in X$ that is $\tilde{d}(f_{e_n}, f_{e_m}) \to \tilde{0}$ as $n, m \to \infty$.

Definition 2.27. ([**4**]) A fuzzy soft metric space $(\tilde{E}, \tilde{d})$ is said to be complete if every cauchy sequence in $\tilde{E}$ converges to some fuzzy soft point of $\tilde{E}$.

Definition 2.28. ([**4**]) A fuzzy soft set f_A in a fuzzy soft metric space $(\tilde{E}, \tilde{d})$ is said to be fuzzy soft open if for each fuzzy soft point f_e of f_A there exist a fuzzy soft open ball $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})\tilde{\subseteq} f_A$.

Lemma 2.1 ([**4**]). *Let $(\tilde{E}, \tilde{d})$ be a fuzzy soft metric space then the fuzzy soft open ball $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ is a fuzzy soft open set.*

Theorem 2.1 ([**4**]). *Given a fuzzy soft metric space $(\tilde{E}, \tilde{d})$. Let $\Im$ denote the set of all fuzzy soft open sets in $\tilde{E}$. Then $\Im$ has the following properties:*

(i) $\tilde{\phi}, \tilde{E} \in \Im$,
(ii) *if* $f_A, g_B \in \Im$ *then* $f_A \tilde{\cap} g_B \in \Im$,
(iii) *if* $f^{\alpha}_{A_\alpha} \in \Im$ *for all* $\alpha \in \Delta$ *an index set, then* $\tilde{\cup} f^{\alpha}_{A_\alpha} \in \Im$.

$\Im$ is called the topology determined by the fuzzy soft metric $\tilde{d}$.

3. Fuzzy Soft Topology Generated by Fuzzy Soft Metric

In this section, we study some important results of fuzzy soft metric spaces.

Let $\tilde{E}$ be the absolute fuzzy soft set over X and E be a parameter set and $\tilde{E}_e$ be a family of fuzzy soft points i. e. $\tilde{E}_e - \{f_e - (e, \tilde{1}) : \tilde{1}(x) - 1$, for all $x \in X, e \in E\}$. Then there exists a bijective mapping between the fuzzy soft set $\tilde{E}$ and the set X. If $e \neq \acute{e}0 \in E$, then $\tilde{E}_e \tilde{\cap} \tilde{E}_{\acute{e}} =$, and $FSC(\tilde{E}) = \tilde{\cup}_{e \in E} \tilde{E}_e$.

Let $(\tilde{E}, \tilde{d})$ be a fuzzy soft metric space. It is clear that $(\tilde{E}_e, \tilde{d}_e)$ is a fuzzy soft metric space for $e \in E$: Then by using the fuzzy soft metric $\tilde{d}_e$, we define a metric on X as $d_e(x, y) = \tilde{d}_e(f_e, g_e)$, for all $x, y \in X$ and $f_e, g_e \tilde{\in} \tilde{E}_e$.

Note that $e \neq \acute{e} \in E$, then d_e and $d_{\acute{e}}$ on X are generally different metrics.

Proposition 3.1. *Every fuzzy soft metric space is a family of parameterized metric spaces.*

Proof. It is obvious from above. □

The converse of Proposition 3.1 may not be true in general. This is shown by the following example.

Example 3.1. Let $E = R$ be a parameter set and (X, d) be a metric space. We define the function $\tilde{d} : FSC(\tilde{E}) \times FSC(\tilde{E}) \to R$ by

$$\tilde{d}(f_e, g_{\acute{e}}) = d(x, y)^{\tilde{1} + |\mu^a_{f_e}(x) - \mu^a_{g_{\acute{e}}}(y)|}, \text{ for all } f_e, g_{\acute{e}} \in FSC(\tilde{E}).$$

Then for all $e \in E, d_e$ is a metric on X. If $\tilde{d}(f_e, g_{\acute{e}}) = \tilde{0}$, then this does not always mean that $f_e = g_{\acute{e}}$, so $\tilde{d}$ is not a fuzzy soft metric on $\tilde{E}$.

PROPOSITION 3.2. *Let* $(\tilde{E}, \tilde{d})$ *be a fuzzy soft metric space and* $\tau_{\tilde{d}_e}$ *be a fuzzy soft topology generated by the fuzzy soft metric* $\tilde{d}$. *Then for every* $e \in E$, *the topology* $(\tau_{\tilde{d}})_e$ *on* X *is the topology* τ_{d_e} *generated by the metric* d_e *on* X.

PROOF. It is obvious. □

LEMMA 3.1. *Let* $(\tilde{E}, \tilde{d})$ *be a fuzzy soft metric space. Then the following expressions are true:*

(i) $f_e \tilde{\in} \overline{f_A} \Leftrightarrow \tilde{d}(f_e, f_A) = \tilde{0}$;
(ii) $f_e \tilde{\in} f_A^o \Leftrightarrow \tilde{d}(f_e, f_A^c) \tilde{>} \tilde{0}$;
(iii) $f_e \tilde{\in} \partial f_A \Leftrightarrow \tilde{d}(f_e, f_A) = \tilde{d}(f_e, f_A^c) = \tilde{0}$.

PROOF. It is clear. □

Note that if f_A is a fuzzy soft closed set in the fuzzy soft metric space $(\tilde{E}, \tilde{d})$ and $f_e \tilde{\notin} (f_A)$, then there exists a fuzzy soft open ball $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ such that $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}) \tilde{\cap} f_A = \tilde{\phi}$.

THEOREM 3.1. *Every fuzzy soft metric space is a fuzzy soft normal space.*

PROOF. Let h_A and k_B be two disjoint fuzzy soft closed sets in the fuzzy soft metric space $(\tilde{E}, \tilde{d})$. For every fuzzy soft points $f_e \tilde{\in} h_A$ and $g_{\acute{e}} \tilde{\in} k_B$, we choose fuzzy soft open balls $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ and $\tilde{\tilde{B}}(f_{\grave{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}})$ such that $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}) \tilde{\cap} k_B = \tilde{\phi}$ and $\tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}) \tilde{\cap} h_A = \tilde{\phi}$. Thus, we have $h_A \tilde{\subseteq} \widetilde{\bigcup} \tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}/3) = u_A$ and $k_B \tilde{\subseteq} \widetilde{\bigcup} \tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}/3) = v_B$. We want to show that $u_A \tilde{\cap} v_B = \tilde{\phi}$.

Assume that $u_A \tilde{\cap} v_B \neq \tilde{\phi}$. Then there exists a fuzzy soft point $w_{\acute{\acute{e}}}$ such that $w_{\acute{\acute{e}}} \tilde{\in} u_A \tilde{\cap} v_B$. Therefore, there exist fuzzy soft open balls $\tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}/3)$ and $\tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}/3)$ such that $w_{\acute{\acute{e}}} \tilde{\in} \tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon}/3)$ and $w_{\acute{\acute{e}}} \tilde{\in} \tilde{\tilde{B}}(g_{\acute{e}}, \tilde{\tilde{r}}, \tilde{\acute{\epsilon}}/3)$. Here, we have

$$\tilde{d}(f_e, w_{\acute{\acute{e}}}) \tilde{<} \epsilon/3 \text{ and } \tilde{d}(g_{\acute{e}}, w_{\acute{\acute{e}}}) \tilde{<} \acute{\epsilon}/3.$$

If we get max $\{\tilde{\epsilon}/3,\ \tilde{\acute{\epsilon}}/3\} = \tilde{\epsilon}/3$, then we have

$$\tilde{d}(f_e, g_{\acute{e}}) \tilde{\leqslant} \tilde{d}(f_e, w_{\acute{\acute{e}}}) + \tilde{d}(w_{\acute{\acute{e}}}, g_{\acute{e}}) \tilde{\leqslant} \epsilon/3 + \acute{\epsilon}/3 \tilde{\leqslant} \epsilon$$

and so $g_{\acute{e}} \tilde{\in} \tilde{\tilde{B}}(f_e, \tilde{\tilde{t}}, \tilde{\epsilon})$ and which contradicts with our assumption. Therefore, $u_A \tilde{\cap} v_B = \tilde{\phi}$. □

4. Ciric Type Fuzzy Soft Contractive Mapping

In this section we shall prove a fixed point theorem of Ciric type fuzzy contractive mapping.

DEFINITION 4.1. Let $(\tilde{E}, \tilde{d})$ and $(\tilde{E}', \tilde{\rho})$ be two fuzzy soft mappings. The mapping $\varphi_\psi = (\varphi, \psi) : (\tilde{E}, \tilde{d}) \to (\tilde{E}', \tilde{\rho})$ is a fuzzy soft mapping, if $\varphi : \tilde{E} \to \tilde{E}'$ and $\psi : E \to E'$ are two mappings.

PROPOSITION 4.1. *For each fuzzy soft point* $f_e \tilde{\in} FSC(\tilde{E}), \varphi_\psi(f_e)$ *is a fuzzy soft point in* $\tilde{E}'$.

Proof. Let $f_e\tilde{\in}FSC(\tilde{E})$ be a fuzzy soft point. Then

$$\varphi_\psi(f_e)(\acute{e})=\widetilde{\bigcup}_{e\in\psi^{-1}(\acute{e})}\varphi(f_e(e))=(\varphi(f_e))_{\psi(e)}.$$

□

Definition 4.2. Let $(\tilde{E},\tilde{d})$ and $(\tilde{E'},\tilde{\rho})$ be two fuzzy soft metric spaces and $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E'},\tilde{\rho})$ is a fuzzy soft continuous mapping at the fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$ if for every fuzzy soft open ball $\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\acute{\epsilon}})$ of $(\tilde{E'},\tilde{\rho})$, there exists a fuzzy soft open ball $\tilde{\tilde{B}}(f_e,\tilde{\tilde{t}},\tilde{\epsilon})$ of $(\tilde{d},\tilde{E})$ such that $\varphi(\tilde{\tilde{B}}(f_e,\tilde{\tilde{t}},\tilde{\epsilon}))\tilde{\subseteq}\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{r}},\tilde{\acute{\epsilon}})$.

If $\varphi_\psi(f_e)$ is a fuzzy soft continuous mapping at every fuzzy soft point f_e of $(\tilde{E},\tilde{d})$, then it is said to be fuzzy soft continuous mapping on $(\tilde{E},\tilde{d})$.

Now, this definition can be expressed using $\tilde{\varepsilon}-\tilde{\delta}$ as follows:

Definition 4.3. The mapping $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E'},\tilde{\rho})$ is said to be a fuzzy soft continuous mapping at the fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$ if for every $\tilde{\varepsilon}\tilde{>}\tilde{0}$ there exists a $\tilde{\delta}\tilde{>}\tilde{0}$ such that $\tilde{d}(f_e,g_{\acute{e}})\tilde{<}\tilde{\delta}$ implies that $\tilde{\rho}(\varphi_\psi(f_e),\varphi_\psi(g_{\acute{e}}))\tilde{<}\tilde{\varepsilon}$.

Theorem 4.1. *Let $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E'},\tilde{\rho})$ be a fuzzy soft mapping. Then the following conditions are equivalent:*

(1) *$\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E'},\tilde{\rho})$ is a fuzzy soft continuous mapping,*
(2) *For each fuzzy soft open set g_B in $(\tilde{E'},\tilde{\rho})$, $(\varphi_\psi)^{-1}(g_B)$ is a fuzzy soft open set in $(\tilde{E},\tilde{d})$,*
(3) *For each fuzzy soft closed set h_C in $(\tilde{E'},\tilde{\rho})$, $(\varphi_\psi)^{-1}(h_C)$ is a fuzzy soft closed set in $(\tilde{E},\tilde{d})$,*
(4) *For each fuzzy soft set f_A in $(\tilde{E},\tilde{d})$, $\varphi_\psi(\overline{f_A})\tilde{\subset}\overline{(\varphi_\psi(f_A))}$ is a fuzzy soft closed set in $(\tilde{E'},\tilde{\rho})$,*
(5) *For each fuzzy soft set g_B in $(\tilde{E'},\tilde{\rho})$, $\overline{((\varphi_\psi)^{-1}(g_B))}\tilde{\subset}(\varphi_\psi)^{-1}(\bar{g_B})$,*
(6) *For each fuzzy soft set g_B in $(\tilde{E'},\tilde{\rho})$, $(\varphi_\psi)^{-1}(g_B^o)\tilde{\subset}((\varphi_\psi)^{-1}(g_B))^o$.*

Proof. (1) ⇒ (2) Let φ_ψ be a fuzzy soft continuous mapping and g_B be a fuzzy soft open set in $(\tilde{E'},\tilde{\rho})$. Consider the fuzzy soft set $(\varphi_\psi)^{-1}(g_B)$. If $(\varphi_\psi)^{-1}(g_B)=\tilde{\phi}$, then the proof is completed. Let $(\varphi_\psi)^{-1}(g_B)\neq\tilde{\phi}$. In this case there exists at least one fuzzy soft point $f_e\tilde{\in}(\varphi_\psi)^{-1}(g_B)$. Then we have $\varphi_\psi(f_e)\tilde{\in}g_B$. Since g_B is a fuzzy soft open set, there exists a fuzzy soft open ball $\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\epsilon})$ such that $\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\epsilon})\tilde{\subset}\tilde{\subset}g_B$ holds. Also since φ_ψ is a fuzzy soft continuous mapping, there exists a fuzzy soft open ball $\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}})$ such that $\varphi_\psi(\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}}))\tilde{\subset}\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\epsilon})$. Thus, $\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}})\tilde{\subset}(\varphi_\psi)^{-1}(\varphi_\psi)(\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}}))\tilde{\subset}(\varphi_\psi)^{-1}(\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\epsilon})\tilde{\subset}(\varphi_\psi)^{-1}(g_B)$.

Consequently, $(\varphi_\psi)^{-1}(g_B)$ is a fuzzy soft open set.

(2) ⇒ (3) Let h_C be any fuzzy soft set in $(\tilde{E'},\tilde{\rho})$ Then h_C^c is a fuzzy soft open set. From (2), we have $(\varphi_\psi)^{-1}(h_C)^c$ is a fuzzy soft open set in $(\tilde{E},\tilde{d})$. Thus $(\varphi_\psi)^{-1}(h_C)$ is a fuzzy soft closed set.

(3) ⇒ (4) Let f_A be a fuzzy soft set in $(\tilde{E},\tilde{d})$. Since $f_A\tilde{\subset}(\varphi_\psi)^{-1}\varphi_\psi(f_A)$ and $\varphi_\psi(f_A)\tilde{\subset}\overline{(\varphi_\psi(f_A))}$,we have $f_A\tilde{\subset}(\varphi_\psi)^{-1}\varphi_\psi(f_A)\tilde{\subset}(\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))})$. By part (3),

since $(\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))})$ is a fuzzy soft closed set in $(\tilde{E},\tilde{d})$, $\bar{f_A}\tilde{\subset}(\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))})$. Thus $\varphi_\psi(\bar{f_A})\tilde{\subset}\varphi_\psi((\varphi_\psi)^{-1}(\overline{(\varphi_\psi(f_A))}))$ is obtained.

(4) $\Rightarrow$ (5) Let g_B be a fuzzy soft set in $(\tilde{E}',\tilde{\rho})$ and $(\varphi_\psi)^{-1}(g_B)=f_A$. By part (4), we have $\varphi_\psi(\bar{f_A})=\varphi_\psi(\overline{((\varphi_\psi)^{-1}(g_B))})\tilde{\subset}\overline{(\varphi_\psi(\varphi_\psi)^{-1}(g_B))}\tilde{\subset}\bar{g_B}$. Then

$$\overline{((\varphi_\psi)^{-1}(g_B))}=\bar{f_A}\tilde{\subset}(\varphi_\psi)^{-1}(\varphi_\psi(\bar{f_A}))\tilde{\subset}(\varphi_\psi)^{-1}(\bar{g_B}).$$

(5) $\Rightarrow$ (6) Let g_B be a fuzzy soft set in $(\tilde{E}',\tilde{\rho})$. Substituting g_B^c for condition in (5). Then $\overline{((\varphi_\psi)^{-1}(g_B^c))}\tilde{\subset}(\varphi_\psi)^{-1}(\bar{g_B^c})$. Since $g_B^o=(\bar{g_B^c})^c$, then we have

$$\begin{aligned}(\varphi_\psi)^{-1}(g_B^o)=(\varphi_\psi)^{-1}((\bar{g_B^c})^c)=((\varphi_\psi)^{-1}(\bar{g_B^c}))^c\tilde{\subset}(\overline{((\varphi_\psi)^{-1}(g_B^c))})^c\\=((\overline{(\varphi_\psi)^{-1}(g_B))^c})^c=((\varphi_\psi)^{-1}(g_B))^o.\end{aligned}$$

(6) $\Rightarrow$ (1) Let g_B be a fuzzy soft open set in $(\tilde{E}',\tilde{\rho})$. Then since

$$((\varphi_\psi)^{-1}(g_B))^o\tilde{\subset}(\varphi_\psi)^{-1}g_B=(\varphi_\psi)^{-1}(g_B^o)\tilde{\subset}((\varphi_\psi)^{-1}(g_B))^o,$$

$((\varphi_\psi)^{-1}(g_B))^o=(\varphi_\psi)^{-1}(g_B)$ is obtained. This implies that $(\varphi_\psi)^{-1}(g_B)$ is a fuzzy soft open set. □

Definition 4.4. The fuzzy soft mapping $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E}',\tilde{\rho})$ is said to be fuzzy soft sequentially continuous at the fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$ iff for every sequence of fuzzy soft points $\{f_{e_n}\}$ converging to the fuzzy soft point f_e in the fuzzy soft metric space $(\tilde{E},\tilde{d})$, the sequence $\varphi_\psi(\{f_{e_n}\})$ in $(\tilde{E}',\tilde{\rho})$ converges to a fuzzy soft point $\varphi_\psi(f_e)\tilde{\in}FSC(\tilde{E}')$.

Theorem 4.2. *Fuzzy soft continuity is equivalent to fuzzy soft sequential continuity in fuzzy soft metric spaces.*

Proof. Let $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E}',\tilde{\rho})$ be a fuzzy soft continuous mapping and $\{f_{e_n}\}$ be any sequence of fuzzy soft points converging to the fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$. Let $\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\epsilon})$ be a fuzzy soft open ball in $(\tilde{E}',\tilde{\rho})$. By fuzzy soft continuity of φ_ψ choose a fuzzy soft open ball $\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}})$ containing f_e such that $\varphi_\psi(\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}}))\tilde{\subseteq}\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\epsilon})$. Since $\{f_{e_n}\}$ converges to f_e there exists $n_0\in\mathbb{N}$ such that $\{f_{e_n}\}\tilde{\in}\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}})$ for all $n\geqslant n_0$. Therefore for all $n\geqslant n_0$ we have $\varphi_\psi(\{f_{e_n}\})\tilde{\in}\varphi_\psi(\tilde{\tilde{B}}(f_e,\tilde{\tilde{r}},\tilde{\acute{\epsilon}}))\tilde{\subseteq}\tilde{\tilde{B}}(\varphi_\psi(f_e),\tilde{\tilde{t}},\tilde{\epsilon})$, as required.

Conversely, assume for contradiction that $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E}',\tilde{\rho})$ is fuzzy soft sequential continuous but not fuzzy soft continuous mapping. Since φ_ψ is not fuzzy soft continuous at the fuzzy soft point f_e, there exists such that $\tilde{\varepsilon}\tilde{>}\tilde{0}$ for all $\tilde{\delta}\tilde{>}\tilde{0}$ there exists $g_{\acute{e}}\tilde{\in}FSC(\tilde{E})$ such that $\tilde{d}(f_e,g_{\acute{e}})\tilde{<}\tilde{\delta}$ and $\tilde{\rho}(\varphi_\psi(f_e),\varphi_\psi(g_{\acute{e}}))\tilde{>}\tilde{\epsilon}_0$. For $n\geqslant 1(n\in\mathbb{N})$, define $\tilde{\delta_n}=1/n$. For $n\geqslant 1$ we may choose $\{g_{\acute{e_n}}\}$ in $(\tilde{E},\tilde{d})$ such that $\tilde{d}(f_{e_n},g_{\acute{e}n})\tilde{<}\tilde{\delta_n}$ and $\tilde{\rho}(\varphi_\psi(f_e),\varphi_\psi(g_{\acute{e}}))\tilde{>}\tilde{\epsilon}_0$.

Therefore, by definition the sequence $\{g_{\acute{e_n}}\}(n\geqslant 1)$ converges to f_e. However, by definition the sequence $\{\varphi_\psi(g_{\acute{e_n}})\}(n\geqslant 1)$ does not converge to $\varphi_\psi(f_e)$. That is, φ_ψ is not fuzzy soft sequentially continuous at f_e. □

DEFINITION 4.5. Let $(\tilde{E},\tilde{d})$ be a fuzzy soft metric space. A function $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E},\tilde{d})$ is called a fuzzy soft contraction mapping if there exists a soft real number $\tilde{\alpha}$ with $\tilde{0}\tilde{\leqslant}\tilde{\alpha}\tilde{<}\tilde{1}$ such that for every fuzzy soft points $f_e,g_{\acute{e}}\tilde{\in}FSC(\acute{E})$ we have $\tilde{d}(\varphi_\psi(f_e),\varphi_\psi(g_{\acute{e}}))\tilde{\leqslant}\tilde{\alpha}\tilde{d}(f_e,g_{\acute{e}})$.

DEFINITION 4.6. Let $(\tilde{E},\tilde{d})$ be a fuzzy soft metric space. A function $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E},\tilde{d})$ is called a Ciric type fuzzy soft contractive mapping if there exists a soft real number $\tilde{\alpha}$ with $\tilde{0}\tilde{\leqslant}\tilde{\alpha}\tilde{<}\tilde{1}$ such that for every fuzzy soft points $f_e,g_{\acute{e}}\tilde{\in}FSC(\acute{E})$ we have

$$\tilde{d}(\varphi_\psi(f_e),\varphi_\psi(g_{\acute{e}}))\tilde{\leqslant}\tilde{\alpha}\cdot\max\Big\{\tilde{d}(f_e,g_{\acute{e}}),\tilde{d}(f_e,\varphi_\psi(f_e)),\tilde{d}(g_{\acute{e}},\varphi_\psi(g_{\acute{e}})),\frac{\tilde{d}(f_e,\varphi_\psi(g_{\acute{e}}))+\tilde{d}(g_{\acute{e}},\varphi_\psi(f_e))}{2}\Big\}$$

PROPOSITION 4.2. *Every fuzzy soft contraction mapping is a fuzzy soft continuous mapping.*

PROOF. Let $f_e\tilde{\in}FSC(\tilde{E})$ be any fuzzy soft point and $\tilde{\epsilon}\tilde{>}\tilde{0}$ be arbitrary. If we choose $\tilde{d}(f_e,g_{\acute{e}})\tilde{<}\tilde{\delta}\tilde{<}\tilde{\epsilon}$, then since φ_ψ is a fuzzy soft contraction mapping, we have $\tilde{d}(\varphi_\psi(f_e),\varphi_\psi(g_{\acute{e}}))\tilde{\leqslant}\tilde{d}(f_e,g_{\acute{e}})\tilde{<}\tilde{\alpha}\cdot\tilde{\delta}\tilde{<}\tilde{\epsilon}$ and so φ_ψ is a soft continuous mapping. □

THEOREM 4.3. *Let $(\tilde{E},\tilde{d})$ be a complete fuzzy soft metric space. If the mapping $\varphi_\psi:(\tilde{E},\tilde{d})\to(\tilde{E},\tilde{d})$ is a fuzzy soft contraction mapping on a complete fuzzy soft metric space, then there exists a unique fuzzy soft point $f_e\tilde{\in}FSC(\tilde{E})$ such that $\varphi_\psi(f_e)=f_e$.*

PROOF. Let f_e^0 be any fuzzy soft point in $FSC(\tilde{E})$. Set

$$f_{e_1}^1=\varphi_\psi(f_e^0)=(\varphi(f_e^0))_{\psi(e)},f_{e_2}^2=\varphi_\psi(f_{e_1}^1)=(\varphi^2(f_e^0))_{\psi^2(e)},...,$$
$$f_{e_{n+1}}^{n+1}=\varphi_\psi(f_{e_n}^n)=(\varphi^{n+1}(f_e^0))_{\psi^{n+1}(e)},....$$

We have

$$\begin{aligned}\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n)&=\tilde{d}(\varphi_\psi(f_{e_n}^n),\varphi_\psi(f_{e_{n-1}}^{n-1}))\\&\leqslant\tilde{\alpha}\cdot\max\Big\{\tilde{d}(f_{e_n}^n,f_{e_{n-1}}^{n-1}),\tilde{d}(\varphi_\psi(f_{e_{n-1}}^{n-1}),f_{e_{n-1}}^{n-1}),\tilde{d}(\varphi_\psi(f_{e_n}^n),f_{e_n}^n),\\&\qquad\frac{\tilde{d}(\varphi_\psi(f_{e_n}^n),f_{e_{n-1}}^{n-1})+\tilde{d}(\varphi_\psi(f_{e_{n-1}}^{n-1}),f_{e_n}^n)}{2}\Big\}\\&=\tilde{\alpha}\cdot\max\Big\{\tilde{d}(f_{e_n}^n,f_{e_{n-1}}^{n-1}),\tilde{d}(f_{e_n}^n,f_{e_{n-1}}^{n-1}),\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n),\\&\qquad\frac{\tilde{d}(\varphi_\psi(f_{e_n}^n),f_{e_{n-1}}^{n-1})+\tilde{d}(f_{e_n}^n,f_{e_n}^n)}{2}\Big\}\\&=\tilde{\alpha}\cdot\max\left\{\tilde{d}(f_{e_n}^n,f_{e_{n-1}}^{n-1}),\tilde{d}(f_{e_{n+1}}^{n+1},f_{e_n}^n),\frac{\tilde{d}(\varphi_\psi(f_{e_n}^n),f_{e_{n-1}}^{n-1})}{2}\right\}\end{aligned}$$

$$\tilde{\leqslant}\tilde{\tilde{\alpha}} \cdot \max\left\{\tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1}), \tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n), \frac{\tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n) + \tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1})}{2}\right\}$$

$$= \tilde{\tilde{\alpha}} \cdot \max\left\{\tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1}), \tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n)\right\}$$

If $\max\left\{\tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1}), \tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n)\right\} = \tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n)$, for some n

$$\tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n) \leqslant \tilde{\tilde{\alpha}} \cdot \tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n)$$

gives us a contradiction and hence we have

$$\begin{aligned}\tilde{d}(f_{e_{n+1}}^{n+1}, f_{e_n}^n) &\leqslant \tilde{\tilde{\alpha}} \cdot \tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1})\\ &\tilde{\leqslant}\tilde{\tilde{\alpha}}^2 \cdot \tilde{d}(f_{e_{n-1}}^{n-1}, f_{e_{n-2}}^{n-2})\\ &\dots \dots \dots \dots \dots \dots \dots \dots\\ &\tilde{\leqslant}\tilde{\tilde{\alpha}}^n \cdot \tilde{d}(f_{e_1}^1, f_{e_0}^0).\end{aligned}$$

So for $n > m$

$$\begin{aligned}\tilde{d}(f_{e_n}^n, f_{e_m}^m) &\tilde{\leqslant}\tilde{d}(f_{e_n}^n, f_{e_{n-1}}^{n-1}) + \tilde{d}(f_{e_{n-1}}^{n-1}, f_{e_{n-2}}^{n-2}) + \dots + \tilde{d}(f_{e_{m+1}}^{m+1}, f_{e_m}^m)\\ &\tilde{\leqslant}(\tilde{\tilde{\alpha}}^{n-1} + \tilde{\tilde{\alpha}}^{n-2} + \dots + \tilde{\tilde{\alpha}}^m) \cdot \tilde{d}(f_{e_1}^1, f_e^0)\\ &\tilde{\leqslant}\frac{\tilde{\tilde{\alpha}}^m}{\tilde{1} - \tilde{\tilde{\alpha}}} \cdot \tilde{d}(f_{e_1}^1, f_e^0)\end{aligned}$$

We get

$$\tilde{d}(f_{e_n}^n, f_{e_m}^m)\tilde{\leqslant}\frac{\tilde{\tilde{\alpha}}^m}{\tilde{1} - \tilde{\tilde{\alpha}}} \cdot \tilde{d}(f_{e_1}^1, f_e^0).$$

This implies $\tilde{d}(f_{e_n}^n, f_{e_m}^m) \to \tilde{0}$ as $(n, m \to \infty)$.

Hence $\{f_{e_n}^n\}$ is a fuzzy soft Cauchy sequence, by the completeness of $(\tilde{E}, \tilde{d})$, there is a fuzzy soft point $f_e^0 \tilde{\in} FSC(\tilde{E})$ such that $f_{e_n}^n \to f_e^0$ as $(n \to \infty)$.

Since

$$\begin{aligned}\tilde{d}(\varphi_\psi(f_e^0), f_e^0) &\tilde{\leqslant}\tilde{d}(\varphi_\psi(f_{e_n}^n), \varphi_\psi(f_e^0)) + \tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^0)\\ &\tilde{\leqslant}\tilde{\tilde{\alpha}} \cdot \max\{\, \tilde{d}(f_{e_n}^n, f_e^0), \tilde{d}(\varphi_\psi(f_e^0), f_e^0), \tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^n),\\ &\quad \left.\frac{\tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^0) + \tilde{d}(f_{e_n}^n, \varphi_\psi(f_e^0))}{2}\right\} + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^0)),\end{aligned}$$

we have

$$\begin{aligned}\tilde{d}(\varphi_\psi(f_e^0), f_e^0) \tilde{\leqslant}\tilde{\tilde{\alpha}} \cdot \max\{\, &\tilde{d}(f_{e_n}^n, f_e^0), \tilde{d}(\varphi_\psi(f_e^0), f_e^0), \tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^n),\\ &\left.\frac{\tilde{d}(\varphi_\psi(f_{e_n}^n), f_e^0) + \tilde{d}(f_{e_n}^n, \varphi_\psi(f_e^0))}{2}\right\} + \tilde{d}(f_{e_{n+1}}^{n+1}, f_e^0).\end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\tilde{d}(\varphi_\psi(f_e^0), f_e^0)\tilde{\leqslant}\tilde{\tilde{\alpha}} \cdot \max\left\{\tilde{d}(\varphi_\psi(f_e^0), f_e^0), \frac{\tilde{d}(\varphi_\psi(f_e^0), f_e^0)}{2}\right\} + \tilde{0}.$$

and $\tilde{d}(\varphi_\psi(f_e^0), f_e^0) \tilde{\leqslant} \tilde{\tilde{\alpha}} \cdot \tilde{d}(\varphi_\psi(f_e^0), f_e^0)$.

Hence, $\tilde{d}(\varphi_\psi(f_e^0), f_e^0) \to \tilde{0}$. This implies $\varphi_\psi(f_e^0) = f_e^0$. So the fuzzy soft point f_e^0 is a fixed fuzzy soft point of the mapping φ_ψ. Now, if $g_{\acute{e}}^0$ is another fixed fuzzy soft point of φ_ψ, then

$$\begin{aligned}\tilde{d}(f_e^0, g_{\acute{e}}^0) &= \tilde{d}(\varphi_\psi(f_e^0), \varphi_\psi(g_{\acute{e}}^0)) \\ &\tilde{\leqslant} \tilde{\tilde{\alpha}} \cdot \max\Big\{ \tilde{d}(f_e^0, g_{\acute{e}}^0), \tilde{d}(\varphi_\psi(f_e^0), f_e^0), \tilde{d}(\varphi_\psi(g_{\acute{e}}^0), g_{\acute{e}}^0), \\ &\quad \frac{\tilde{d}(f_e^0, \varphi_\psi(g_{\acute{e}}^0)) + \tilde{d}(\varphi_\psi(f_e^0), g_{\acute{e}}^0)}{2} \Big\} \\ &= \tilde{\tilde{\alpha}} \cdot \tilde{d}(f_e^0, g_{\acute{e}}^0).\end{aligned}$$

Hence, for $\tilde{\tilde{\alpha}} \tilde{\leqslant} \tilde{1}, \tilde{d}(f_e^0, g_{\acute{e}}^0) = \tilde{0} \Rightarrow f_e^0 = g_{\acute{e}}^0$. Therefore, the fixed fuzzy soft point of φ_ψ is unique. □

Conflict of Interests The authors declare that there is no conflict of interests.

Acknowledgements The authors are very grateful to the editor and the reviewers for their valuable suggestions.

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Receibed by editors 14.02.2018; Revised version 05.03.2019 and 05.05.2019; Available online 13.05.2019.

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