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SOME RESULTS ON ODD HARMONIOUS LABELING OF GRAPHS

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Abstract. A graph $G(p,q)$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. In this paper we prove that the m-mirror graph $M_m(G)$,m- shadow graph of C_{bn}, m-splitting graph of $K_{2,n}(r,s)$ and $\overline{W_(m,n)}$, $\langle C_n : K_{2,m} : C_r\rangle$, pyramid graph PY_n are odd harmonious graphs.

1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [**3**]. A graph $G = (V,E)$ with p vertices and q edges is called a (p,q) – graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [**1**]. Labeled graphs serves as useful mathematical models for many applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. Graham and Sloane [**2**] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \to Z_q$ such that the induced function $f^* : E(G) \to Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod q$ is a bijection and f is called harmonious labeling of G. The concept of odd harmonious labeling was due to Liang and Bai [**13**]. A graph G is said to be odd harmonious if there exists an injection $f : V(G) \to \{0,1,2,\cdots,2q-1\}$ such that the induced function $f^* : E(G) \to \{1,3,\cdots,2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. If $f(V(G)) = \{0,1,2,\cdots,q\}$ then f is called as strongly odd harmonious

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labeling and G is called as strongly odd harmonious graph. The odd harmoniousness of graph is useful for the solution of undetermined equations. Several results have been published on odd harmonious labeling. An interested reader can refer to [**4, 5, 6, 7, 8, 9, 10, 11, 12**].

We use the following definitions in the subsequent section.

Definition 1.1. The m-shadow graph $D_m(G)$ of a connected graph G is constructed by taking m- copies of G , say G_1,G_2,G_3,....,G_m , then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j ,$1 \leqslant j \leqslant m$.

Definition 1.2. The m-splitting graph $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v^1, v^2, ..., v^m$ such that $v^i, 1 \leqslant i \leqslant m$ is adjacent to every vertex that is adjacent to v in G.

Hence, the 2-shadow graph is the known shadow graph $D_2(G)$ and the 1-splitting graph is the known splitting graph.

Definition 1.3. Let G be any graph with p vertices and q edges. Let G_1, G_2, ..., G_m be m-copies of the graph G. The m-mirror graph $M_m(G)$ is defined as the disjoint union of $G_1, G_2, ..., G_m$ together with additional edges joining each vertex of G_i to its corresponding vertex in G_{i+1}, $1 \leqslant i \leqslant m-1$. Hence $M_m(G)$ has mp vertices and $mq + (m-1)p$ edges.

Definition 1.4. The wheel W_n is obtained by joining a new vertex with every vertices of n-cycle C_n by an edge.

Definition 1.5. Gear graph $\overline{W_n}$ is obtained from a wheel graph W_n by subdividing each edge of the n-cycle by a vertex. The vertex and the edge set of the gear graph $\overline{W_n}$ are $V(\overline{W_n}) = \{v_i/i \in [0, 2n]\}$, $E(\overline{W_n}) = \{v_0v_{2i-1}/i \in [1, n]\} \cup \{v_iv_{i-1}/i \in [2, 2n]\} \cup \{v_1v_{2n}\}$.

Definition 1.6. A pyramid graph PY_n is a graph with the vertex set

$$V(PY_n) = \{v_{11}, v_{21}, v_{22}, v_{31}, v_{32}, v_{33}, ..., v_{n1}, v_{n2}, ...v_{nn}\}$$

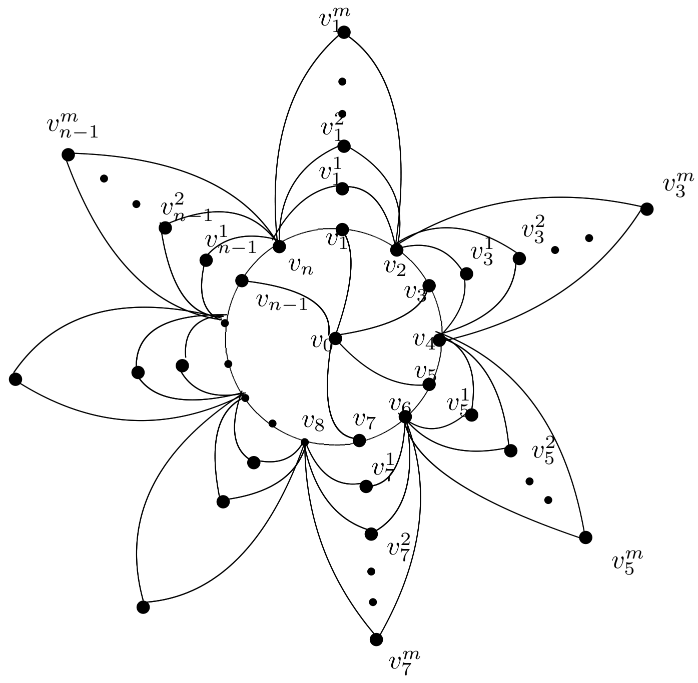
and the edge set $E(PY_n) = \{v_{ij}v_{(i+1)j}/1 \leqslant i \leqslant n-1 \text{ and } 1 \leqslant j \leqslant n-1\ \} \cup \{v_{ii}v_{(i+1)(i+1)}/1 \leqslant i \leqslant n-1\} \cup \{v_{ij}v_{(i+1)(j+1)}/2 \leqslant i \leqslant n-1 \text{ and } 1 \leqslant j \leqslant n-2\}$.

Definition 1.7. Let G be a graph on p vertices $v_1, v_2, ...v_p$ and $H_1, H_2, ...H_p$ be p graphs isomorphic to a graph H with n vertices. The corona graph $G \bigodot H$ is obtained by joining each vertex v_i of G with every vertices of H_i for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant n$. The comb graph is $P_n \bigodot K_1$.

Definition 1.8. ([**3**]) The graph $K_{2,n}(r, s)$ is obtained from $K_{2,n}, (n \geqslant 2)$ by adding r and s $(r, s \geqslant 1)$ pendent edges to the two vertices of degree n.

Definition 1.9. Consider the bipartite graph $K_{2,m}$ with the vertex set $V = V_1 \cup V_2$ where $V_1 = \{u, v\}$ and $V_2 = \{v_1, v_2, ..., v_m\}$. The graph $G = \langle C_n : K_{2,m} : C_r \rangle$ is obtained by identifying the vertex u of V_1 with a vertex of C_n and the other vertex v of V_1 with a vertex of C_r.

Definition 1.10. The graph $\overline{W_{m,n}}$ is obtained from the gear graph $\overline{W_n}$ as follows: Join the vertices v_i and v_{i+2} with the new vertices v_{i+1}^j for $1 \leqslant j \leqslant m$ and $2 \leqslant i \leqslant n-2$ and join v_n and v_2 with v_1^j for $1 \leqslant j \leqslant m$.



2. Main Results

In this section we prove that the m-mirror graph $M_m(G)$, m- shadow graph of C_{bn}, m-splitting graph of $K_{2,n}(r,s)$ and $\overline{W(m,n)}$, $\langle C_n : K_{2,m} : C_r \rangle$, pyramid graph PY_n are odd harmonious graphs.

Theorem 2.1. *If $G(p,q)$ is a strongly odd harmonious tree, then $M_m(G)$ is an odd harmonious graph.*

Proof. Let f be a strongly odd harmonious labeling of G. Then $f(V(G)) = \{0, 1, 2, ..., q\}$ and $f^*(E(G)) = \{1, 3, ...2q-1\}$. Define a new labeling $g : V(M_m(G)) \to \{0, 1, 2, ..., 2q-1\}$ as $g(u) = f(u) + (i-1)(2q+1)$ if $u \in G_i$, $1 \leqslant i \leqslant m$. The induced edge labels of the edges between the graphs G_i and G_{i+1} is

$= \{2f(u) + (i-1)(2q+1) + i(2q+1)/u \in G\}$ for $1 \leqslant i \leqslant m-1$.

$= \{2f(u) + (2q+1)(2i-1)/u \in G\}$ for $1 \leqslant i \leqslant m-1$.

$= \{(2q+1)(2i-1), 2+(2q+1)(2i-1), 4+(2q+1)(2i-1), ..., 2q+(2q+1)(2i-1)\}$ for $1 \leqslant i \leqslant m-1$.

Hence the induced edge labels of $M_m(G)$ is $= \{1, 3, ..., 2q - 1\} \cup \{2q + 1, 2q + 3, ..., 4q + 1\} \cup \{4q + 3, 4q + 5, ..., 6q + 1\} \cup \{6q + 3, 6q + 5, ...8q + 3\}....... \cup \{4mq - 6q + 2m - 3, ...4mq - 4q + 2m - 3\} \cup \{4mq - 4q + 2m - 1, ..., 4mq - 2q + 2m - 3\} = \{1, 3, 5, ..., 4mq - 2q + 2m - 3\}$. Hence $M_m(G)$ is odd harmonious. □

An odd harmonious labeling of $M_3(P_3 \ominus K_1)$ is shown in Figure 1.

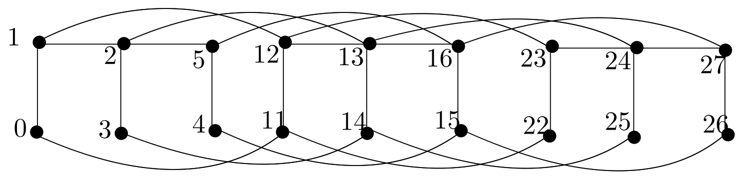


Figure 1: *An odd harmonious labeling of* $M_3(P_3 \ominus K_1)$

THEOREM 2.2. *The graph* $\overline{W_{m,n}}$ *for* $n \equiv 0(mod\ 4)$,$m \geqslant 1$ *is an odd harmonious graph.*

PROOF. Let $V(G) = \{v_0, v_1, \cdots, , v_n\} \cup \{v^j_{2i-1}/1 \leqslant i \leqslant \frac{n}{2}, 1 \leqslant j \leqslant m\}$. Join the vertices v_i and v_{i+2} with the vertices v^j_{i+1} for $1 \leqslant j \leqslant m$ and $2 \leqslant i \leqslant n-2$ and join v_n and v_2 with v^j_1 for $1 \leqslant j \leqslant m$. Let $G = \overline{W_{m,n}}$. Then $|V(G)| = n + \frac{mn}{2} + 1$ and $|E(G)| = n(m+1) + \frac{n}{2}$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(n(m+1) + \frac{n}{2}) - 1\}$ as follows:
$f(v_0) = 0$; $f(v_i) = i$, $1 \leqslant i \leqslant n$ and if i is odd;
$f(v_i) = (i-1) + n + 1$ and $1 \leqslant i \leqslant \frac{n}{2} - 2$ and if i is even;
$f(v_i) = (i-1) + n + 3$ and $\frac{n}{2} \leqslant i \leqslant n-2$ and if i is even;
$f(v_n) = n$; $f(v^j_{2i-1}) = 2nj + 2i - 1$, $1 \leqslant i \leqslant \frac{n}{2}$ and $1 \leqslant j \leqslant m$.
The induced edge labelings are
$f^*(v_0v_i) = i$, $1 \leqslant i \leqslant n$ and if i is odd; $f^*(v_nv_1) = n + 1$;
$f^*(v_iv_{i+1}) = 2i + n + 1$, $1 \leqslant i \leqslant \frac{n}{2} - 2$;
$f^*(v_iv_{i+1}) = 2i + n + 3$, $\frac{n}{2} - 1 \leqslant i \leqslant n - 2$;
$f^*(v_{n-1}v_n) = 2n - 1$;
$f^*(v^j_{i-1}v_i) = 2nj + 2i + n - 1$,$1 \leqslant i \leqslant \frac{n}{2} - 2$ and if i is even , $1 \leqslant j \leqslant m$;
$f^*(v^j_{i-1}v_i) = 2nj + 2i + n + 1$,$\frac{n}{2} \leqslant i \leqslant n - 2$ and if i is even , $1 \leqslant j \leqslant m$;
$f^*(v^j_{n-1}v_n) = 2nj + 2n - 1$;
$f^*(v_iv^j_{i+1}) = 2nj + 2i + n + 1$, $1 \leqslant i \leqslant \frac{n}{2} - 2$, if i is even and $1 \leqslant j \leqslant m$;

$f^*(v_i v_{i+1}^j) = 2nj + 2i + n + 3$, $\dfrac{n}{2} \leqslant i \leqslant n-2$, if i is even and $1 \leqslant j \leqslant m$;
$f^*(v_n v_1^j) = 2nj + n + 1$.
In view of the above defined labeling pattern, the graph $\overline{W_{m,n}}$ is an odd harmonious graph. □

An odd harmonious labeling of $\overline{W_{2,8}}$ is shown in Figure 2.

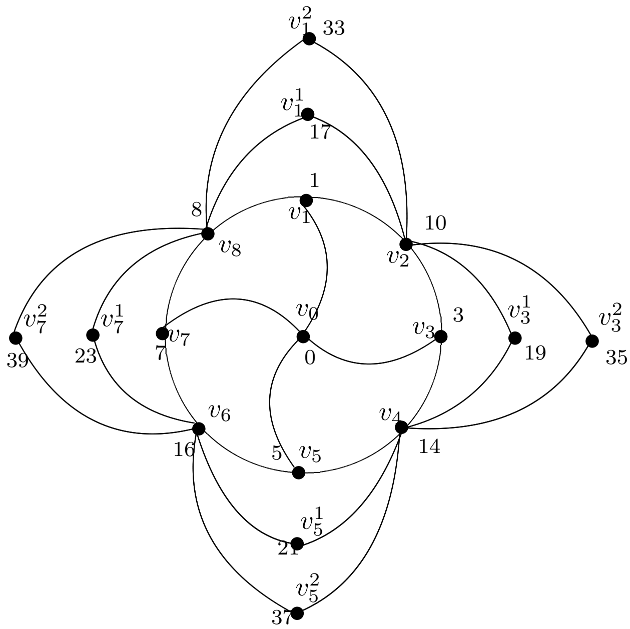


Figure 2: *An odd harmonious labeling of* $\overline{W_{2,8}}$.

Theorem 2.3. *The graph $D_m(C_{bn})$ is odd harmonious for all $m \geqslant 2$ and $n \geqslant 1$.*

Proof. Let $v_1^0, v_2^0, \cdots, v_n^0$ and $u_1^0, u_2^0, \cdots, u_n^0$ be the vertices of the graph C_{bn} and $v_1^j, v_2^j, \cdots, v_n^j$,$u_1^j, u_2^j, \cdots, u_n^j$ be the vertices of the jth copy of C_{bn} where $1 \leqslant j \leqslant m-1$. Let $G = D_m(C_{bn})$. Then $|V(G)| = 2mn$ and $|E(G)| = m^2(2n-1)$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2m^2(2n-1) - 1\}$ as follows:
$f(v_1^j) = 4mj$, $0 \leqslant j \leqslant m-1$;
$f(v_i^j) = (i-3)m^2 + 2m(2m+j-1)$, $i = 3, 5, \cdots, n(\text{odd})$ and $0 \leqslant j \leqslant m-1$;
$f(v_i^j) = (i-2)m^2 + 2m + 1 + 2j$, $i = 2, 4, \cdots, n(\text{even})$ and $0 \leqslant j \leqslant m-1$;
$f(u_1^j) = 1 + 2j$, $0 \leqslant j \leqslant m-1$;
$f(u_i^j) = 2m^2 n + (i-5)m^2 + 2m + 1 + 2j$, $i = 3, 5, \cdots, n(\text{odd})$ and $0 \leqslant j \leqslant m-1$;
$f(u_i^j) = 2m^2 n + (i-2)m^2 - 2m + 2mj$, $i = 2, 4, \cdots, n(\text{even})$ and $0 \leqslant j \leqslant m-1$.
The induced edge labels are
$f^*(v_1^j u_1^k) = 4mj + 1 + 2k$, $0 \leqslant j, k \leqslant m-1$;

$f^*(v_1^j v_2^k) = 4mj + 2m + 1 + 2k$, $0 \leqslant j, k \leqslant m-1$;
$f^*(v_2^j v_1^k) = 2m + 1 + 2j + 4mk$, $0 \leqslant j, k \leqslant m-1$;
$f^*(v_i^j v_{i+1}^k) = 2(i-2)m^2 + 2m + 1 + 2j + 2m(2m+k-1)$, $i = 2, 4, \cdots, (n-1)(\text{even})$, $0 \leqslant j, k \leqslant m-1$;
$f^*(v_i^j v_{i-1}^k) = (i-2)m^2 + 2m + 1 + 2j + (i-4)m^2 + 2m(2m+k-1)$ $i = 4, 6, \cdots, n(\text{even})$, $0 \leqslant j, k \leqslant m-1$;
$f^*(v_i^j v_{i+1}^k) = (i-3)m^2 + 2m(2m+j-1) + (i-1)m^2 + 2m + 1 + 2k$ $i = 3, 5, \cdots, (n-1)(\text{odd})$, $0 \leqslant j, k \leqslant m-1$;
$f^*(v_i^j v_{i-1}^k) = 2(i-3)m^2 + 2m + 1 + 2k + 2m(2m+j-1)$, $i = 3, 5, \cdots, n(\text{odd})$, $0 \leqslant j, k \leqslant m-1$;
$f^*(v_i^j u_i^k) = (i-3)m^2 + 2m(2m+j-1) + 2m^2 n + (i-5)m^2 + 2m + 1 + 2k$ $i = 3, 5, \cdots, n(\text{odd})$, $0 \leqslant j, k \leqslant m-1$;
$f^*(v_i^j u_i^k) = 2(i-2)m^2 + 2m + 1 + 2j + 2m^2 n - 2m + 2mk$, $i = 2, 4, \cdots, n(\text{even})$ $0 \leqslant j, k \leqslant m-1$.

In view of the above defined labeling pattern, the graph $D_m(C_{bn})$ is an odd harmonious graph. □

An odd harmonious labeling of $D_3(C_{b2})$ is shown in Figure 3.

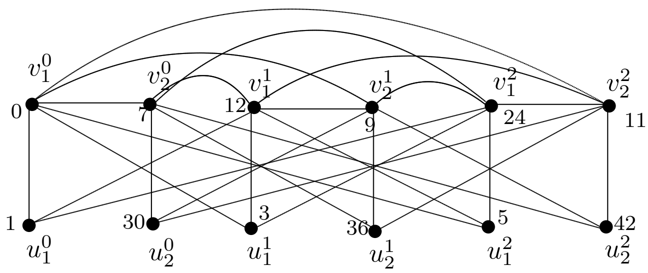


Figure 3: *An odd harmonious labeling of $D_3(C_{b2})$.*

THEOREM 2.4. *The graph $Spl_m[K_{2,n}(r,s)]$ is an odd harmonious graph for all $n, r, s \geqslant 1$.*

PROOF. Let $v_0, u_0, w_1, w_2, \cdots, w_r, v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_s$ be the vertices of the graph $K_{2,n}(r,s)$ and suppose $v_0^j, u_0^j, w_1^j, w_2^j, \cdots, w_r^j, v_1^j, v_2^j, \cdots, v_n^j, u_1^j, u_2^j, \cdots, u_s^j$, $1 \leqslant j \leqslant m$ be the vertices of the jth copy of $K_{2,n}(r,s)$. Let $G = Spl_m[K_{2,n}(r,s)]$. Then $|V(G)| = (m+1)(n+r+s+2)$ and $|E(G)| = (2m+1)(2n+r+s)$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(2m+1)(2n+r+s) - 1\}$ as follows:
$f(v_0) = 0$; $f(w_i) = 2i - 1$, $1 \leqslant i \leqslant r$; $f(v_i) = 2(i+r) - 1$, $1 \leqslant i \leqslant n$;
$f(u_0) = 2n$; $f(u_i) = 2(r+n+i) - 1$, $1 \leqslant i \leqslant s$;
$f(w_i^j) = (2j-1)(4n+2r+2s) + 2i - 1$, $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m$;
$f(v_0^j) = (2j-1)(4n+2r+2s) + 2r$, $1 \leqslant j \leqslant m$;

$f(v_i^j) = 6n + 6r + 2s + 2i - 1 + 4(j-1)(2n + r + s)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$;
$f(u_0^j) = 2(5n + 2r + s) + 4(j-1)(2n + r + s)$, $1 \leqslant j \leqslant m$;
$f(u_i^j) = 2(r + n + i) - 1 + 4j(2n + r + s)$, $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant m$.
The induced edge labels are
$f^*(v_0w_i) = 2i - 1$, $1 \leqslant i \leqslant r$;
$f^*(v_0w_i^j) = (2j-1)(4n + 2r + 2s) + 2i - 1$, $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m$;
$f^*(v_0^jw_i) = (2j-1)(4n + 2r + 2s) + 2r + 2i - 1$, $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m$;
$f^*(v_0v_i) = 2(i + r) - 1$, $1 \leqslant i \leqslant n$;
$f^*(v_0v_i^j) = 6n + 6r + 2s + 2i - 1 + 4(j-1)(2n + r + s)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$;
$f^*(v_iv_0^j) = 2(i + r) - 1 + (2j-1)(4n + 2r + 2s) + 2s$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$;
$f^*(v_iu_0^j) = 2(i + r) - 1 + 2(5n + 2r + s) + 4(j-1)(2n + r + s)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$;
$f^*(u_0v_i^j) = 8n + 6r + 2s + 2i - 1 + 4(j-1)(2n + r + s)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$;
$f^*(u_0u_i) = 2n + 2(r + n + i) - 1$, $1 \leqslant i \leqslant s$;
$f^*(u_0u_i^j) = 2n + 2(r + n + i) - 1 + 4j(2n + r + s)$, $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant m$;
$f^*(u_iu_0^j) = 2(r + n + i) - 1 + 2(5n + 2r + s) + 4(j-1)(2n + r + s)$, $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant m$.
In view of the above defined labeling pattern, the graph $Spl_m[K_{2,n}(r, s)]$ is odd harmonious. □

An odd harmonious labeling of $Spl[K_{2,3}(3, 2)]$ is shown in Figure 4.

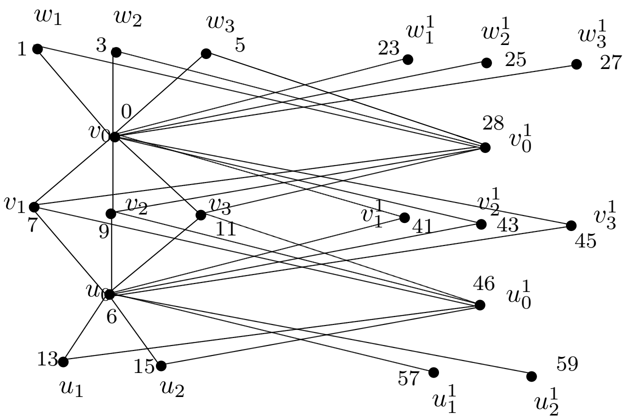


Figure 4: *An odd harmonious labeling of* $Spl[K_{2,3}(3, 2)]$.

THEOREM 2.5. *The pyramid graph PY_n is an odd harmonious graph for all $n \geqslant 2$.*

PROOF. The graph PY_n with $|V(G)| = n(n+1)$ and $|E(G)| = n(n-1)$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2n(n-1) - 1\}$ as follows: $f(v_{ij}) = (i-2)i + (2j-1)$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant i$. The induced edge labels are

$\{1, 3, 5, \cdots, 2n(n-1)-1\}$. In view of the above defined labeling pattern, the graph PY_n is an odd harmonious graph. $\square$

An odd harmonious labeling of PY_4 is shown in Figure 5.

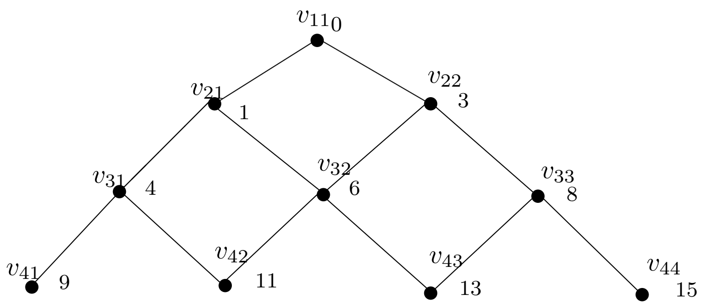


Figure 5: An odd harmonious labeling of PY_4.

THEOREM 2.6. *The graph $G = \langle C_n : K_{2,m} : C_r\rangle$ is an odd harmonious graph for $n, r \equiv 0(mod\ 4)$ and $m \geqslant 2$.*

PROOF. Let $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_r$ be the vertices of the cycles C_n and C_r respectively. Consider the bipartite graph $K_{2,m}$ with vertex set $V = V_1 \cup V_2$ where $V_1 = \{u, v\}$ and $V_2 = \{w_1, w_2, \cdots, w_m\}$. Then $G = \langle C_n : K_{2,m} : C_r\rangle$ is obtained by identifying the vertex u of V_1 with the vertex $v_{\frac{n}{2}}$ of C_n and the other vertex v of V_1 with the vertex u_1 of C_r. Then $|V(G)| = 2n + m$ and $|E(G)| = 2(m + n)$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 4(m + n) - 1\}$ as follows:

Case(i): When $n = r$

$f(v_i) = i - 1\ ,\ 1 \leqslant i \leqslant \frac{n}{2}$;

$f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even} \end{cases}, \frac{n}{2} + 1 \leqslant i \leqslant n;$

$f(w_i) = n + \frac{n}{2} + 2i,\ 1 \leqslant i \leqslant m;$

$f(u_i) = 2m + \frac{n}{2} + i - 2,\ i = 1, 3, 5, \cdots, \frac{n}{2} - 1;$

$f(u_i) = 2m + \frac{n}{2} + i,\ i = \frac{n}{2} + 1, \frac{n}{2} + 3, \cdots, n - 1;$

$f(u_i) = \frac{3n}{2} + 2m + i\ ,\ i = 2, 4, 6, \cdots, n.$

The induced edge labels are

$f^*(v_i v_{i+1}) = 2i - 1,\ 1 \leqslant i \leqslant \frac{n}{2} - 1;\ f^*(v_i v_{i+1}) = 2i + 1,\ \frac{n}{2} \leqslant i \leqslant n - 1;$

$f^*(v_{\frac{n}{2}} w_i) = 2(n + i) - 1,\ 1 \leqslant i \leqslant m;\ f^*(u_1 w_i) = 2(m + n + i) - 1,\ 1 \leqslant i \leqslant m;$

$f^*(u_iu_{i+1}) = 2(2m+n+i)-1,\ 1 \leqslant i \leqslant \frac{n}{2}-1;$
$f^*(u_{\frac{n}{2}}u_{\frac{n}{2}+1}) = 3n+4m+1;\ f^*(u_iu_{i+1}) = 2(2m+n+i)+1,\ \frac{n}{2}+1 \leqslant i \leqslant n-1.$

Case (ii): When $n \neq r$.
Without loss of generality assume that $n < r$.
$f(v_i) = i-1\ ,\ 1 \leqslant i \leqslant \frac{n}{2};\ f(v_i) = \begin{cases} i+1 & \text{if } i \text{ is odd,} \\ i-1 & \text{if } i \text{ is even} \end{cases},\frac{n}{2}+1 \leqslant i \leqslant n;$
$f(w_i) = n+\frac{n}{2}+2i,\ 1 \leqslant i \leqslant m;\ f(u_i) = 2m+\frac{n}{2}+i-2,\ i = 1,3,5,\cdots,\frac{r}{2}-1;$
$f(u_i) = 2m+\frac{n}{2}+i,\ i = \frac{r}{2}+1, \frac{r}{2}+3,\cdots,r-1;$
$f(u_i) = \frac{3n}{2}+2m+i\ ,\ i = 2,4,6,\cdots,r.$
The induced edge labels are
$f^*(v_iv_{i+1}) = 2i-1,\ 1 \leqslant i \leqslant \frac{n}{2}-1;\ f^*(v_iv_{i+1}) = 2i+1,\ \frac{n}{2} \leqslant i \leqslant n-1;$
$f^*(v_{\frac{n}{2}}w_i) = 2(n+i)-1,\ 1 \leqslant i \leqslant m;\ f^*(u_1w_i) = 2(m+n+i)-1,\ 1 \leqslant i \leqslant m;$
$f^*(u_iu_{i+1}) = 2(2m+n+i)-1,\ 1 \leqslant i \leqslant \frac{r}{2}-1;\ f^*(u_{\frac{n}{2}}u_{\frac{n}{2}+1}) = 2n+4m+r+1;$
$f^*(u_iu_{i+1}) = 2(2m+n+i)+1,\ \frac{r}{2}+1 \leqslant i \leqslant r-1.$
In view of the above defined labeling pattern, the graph $\langle C_n : K_{2,m} : C_r\rangle$ is odd harmonious. $\square$

An odd harmonious labeling of $\langle C_8 : K_{2,4} : C_8\rangle$ and $\langle C_8 : K_{2,4} : C_4\rangle$ are shown in Figures 6 and 7.

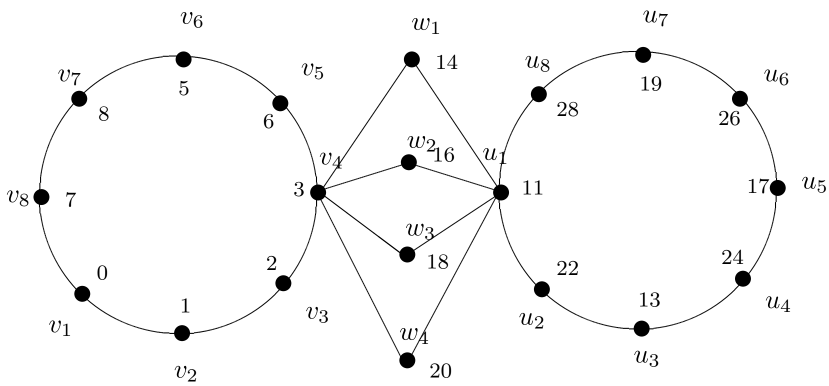


Figure 6: *An odd harmonious labeling of* $\langle C_8 : K_{2,4} : C_8\rangle$

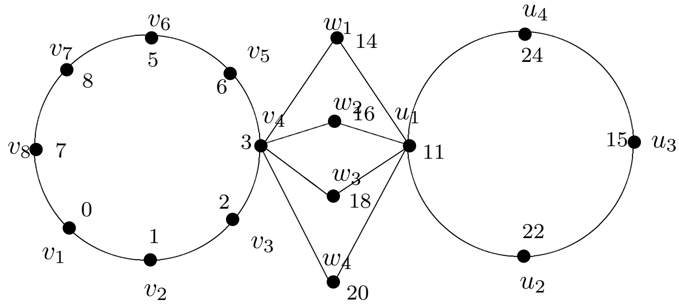


Figure 7: *An odd harmonious labeling of* $\langle C_8 : K_{2,4} : C_4 \rangle$

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