BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 9(2019), 553-565
DOI: 10.7251/BIMVI1903553C

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SOME PROPERTIES OF MULTIVALUED FUNCTIONS IN DIGITAL TOPOLOGY

Ismet Cinar and Ismet Karaca

Abstract. In this paper, we define the approximate fixed point property between digital multivalued functions. We also give the definition of universal digital multivalued functions. We introduce some properties of the smash product of digital multivalued functions. Finally, we give some results on morphological operators.

1. Introduction

Digital topology has dealt with developing image processing and computer graphics for several decades. The properties of digital objects are characterized with tools from topology by many researchers [**9, 14, 15, 17, 18, 19, 26**]. The notion of digital topology has been introduced by Rosenfeld [**28**] at the end of 1970s. He gives the concept of continuity of functions from a digital image to another digital image. Boxer [**2, 3**] expands the results of Rosenfeld by presenting digital versions of continuous functions, retractions and homotopies.

Digital continuous multivalued functions are introduced by Escribano et al. [**21**]. They state how the multivalued approach provides a better framework to define topological notions in a rather more realistic way than by using just single-valued digitally continuous functions. They define the notion of subdivision of a topological space. This notion is used to define continuity for multivalued functions. They show that the deletion of simple points can be completely characterized in terms of digitally continuous multivalued functions.

2010 *Mathematics Subject Classification.* Primary 54H99; Secondary 68R10.

Key words and phrases. Digital image, digital continuous multivalued function, digital smash product, Morphological operators, universal multivalued function, digital approximate fixed point.

Escribano et al. [**22**] show that the basic morphological operations of dilation and closing are continuous multivalued functions. Besides, they characterize thinning algorithms in terms of digitally continuous multivalued functions.

Boxer [**6**] studies uses of digitally continuous functions for enlarging and stretching digital images. Also, he indicates links between shy maps and digitally continuous multivalued functions.

Giraldo and Sastre [**23**] prove that the composition does not always preserve continuity among digitally continuous multivalued functions.

Boxer and Staecker [**7**] present connectivity preserving multivalued functions between digital images and demonstrate that these offer some advantages over continuous multivalued functions. One of these advantages is that the composition of connectivity preserving multivalued functions between digital images is connectivity preserving. Another advantage is that the concept of connectivity preservation of a function on a digital image can be defined without any restrictions on subsets of $\mathbb{Z}^n$. In addition, Tsaur and Smyth [**32**] describe weak and strong continuity for multivalued functions between digital images.

Boxer [**10**] studies properties of multivalued functions between digital images. He deals with properties of multivalued functions between digital images that are characterized by continuity, weak continuity, strong continuity and connectivity preservation.

In this paper, we construct the approximate fixed point property for digital multivalued functions. Moreover, we state the definition of a universal digital multivalued function. We develop some properties of the smash product of digital multivalued functions. Finally, we characterize some morphological operators such as dilation, erosion, opening, and closing.

2. Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the n-dimensional Euclidean space where $\mathbb{Z}$ is the set of integers. A (binary) digital image is a pair (X, κ), where $X \subset \mathbb{Z}^n$ for some positive integer n and κ represents certain adjacency relations in the study of digital images.

Let u be a positive integer, $1 \leqslant u \leqslant n$. Let $p, q \in \mathbb{Z}^n$, $p \neq q$. We say that p and q are *c_u-adjacent* [**4**] if

- there are at most l indices i for which $|p_i - q_i| = 1$, and
- for all indices j such that $|p_j - q_j| \neq 1$, we have $p_j = q_j$.

The notation c_u is sometimes also understood as the number of points $q \in \mathbb{Z}^n$ that are c_u-adjacent to given point $p \in \mathbb{Z}^n$. E.g.,

- in $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency;
- in $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency;
- in $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

More general adjacency relations appear in [**24**].

For two subsets $A, B \subset X$, we will say that A and B are *adjacent* when there exist points $a \in A$ and $b \in B$ such that these points are adjacent or equal.

Let $\kappa \in \{2, 4, 6, 8, 18, 26\}$. A *$\kappa$-neighbor* [**3**] of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ which is κ-adjacent to p. A digital image X is said to be *κ-connected* [**24**] if and only if for every pair of different points p and q in X, there is a sequence $\{p_0, p_1, ..., p_r\}$ of points of X such that $p = p_0$, $q = p_r$ and p_i and p_{i+1} are κ-adjacent where $i \in \{0, 1, ..., r-1\}$. Let $a, b \in \mathbb{Z}$ with $a < b$. A set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leqslant z \leqslant b\}$$

is called a *digital interval* [**2**].

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0 and κ_1-adjacency, respectively. Then a function $f : X \to Y$ is called *(κ_0, κ_1)-continuous* [**2, 29**] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is a digitally continuous.

A *digital simple closed curve* is a digital image $X = \{x_i\}_{i=0}^{m-1}$, with $m \geqslant 4$, such that the points of X are labeled circularly, i.e., x_i and x_j are adjacent if and only if $j = (i-1)\ (mod\ m)$ or $j = (i+1)\ (mod\ m)$.

Proposition 2.1 ([**3**]). *Let (X, κ), (Y, λ) and (Z, β) be digital images. If $f : X \to Y$ and $g : Y \to Z$ are, respectively, a (κ, λ)-continuous function and a (λ, β)-continuous function, then $g \circ f : X \to Z$ is a (κ, β)-continuous function.*

For the cartesian product of two digital images X_1 and X_2, *the normal product adjacency relation* [**1**] is defined as follows: Given points $x_i, y_i \in (X_i, \kappa_i)$, (x_0, y_0) and (x_1, y_1) are $k_*(\kappa_1, \kappa_2)$-adjacent in $X_1 \times X_2$ if and only if one of the following is satisfied:

- $x_0 = x_1$ and y_0 and y_1 are κ_1-adjacent; or
- x_0 and x_1 are κ_0-adjacent and $y_0 = y_1$; or
- x_0 and x_1 are κ_0-adjacent and y_0 and y_1 are κ_1-adjacent.

Proposition 2.2 ([**8**]). *Let (A, α), (B, β), (C, γ) and (D, δ) be digital images. $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (D, \delta)$. f and g are digitally continuous if and only if $f \times g : (A \times B, k_*(\alpha, \beta)) \to (C \times D, k_*(\gamma, \delta))$ defined by $(f \times g)(a, b) = (f(a), g(b))$ is digitally continuous.*

Definition 2.1. ([**21, 22**]) For any positive integer r, the r-subdivision of $\mathbb{Z}^n$ is

$$\mathbb{Z}^n_r = \{(z_1/r, z_2/r, ..., z_n/r) | (z_1, z_2, ..., z_n) \in \mathbb{Z}^n\}.$$

An adjacency relation κ on $\mathbb{Z}^n$ naturally induces an adjacency relation on $\mathbb{Z}^n_r$ as follows:

$(z_1/r, z_2/r, ..., z_n/r)$ and $(z_1'/r, z_2'/r, ..., z_n'/r)$ are κ-adjacent in $\mathbb{Z}^n_r$ if and only if $(z_1, z_2, ..., z_n)$ and $(z_1', z_2', ..., z_n')$ are κ-adjacent in $\mathbb{Z}^n$.

Given a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$ the *r-subdivision* of X is

$$S(X, r) = \{(x_1, x_2, ..., x_n) \in \mathbb{Z}^n_r | (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, ..., \lfloor x_n \rfloor) \in X\}.$$

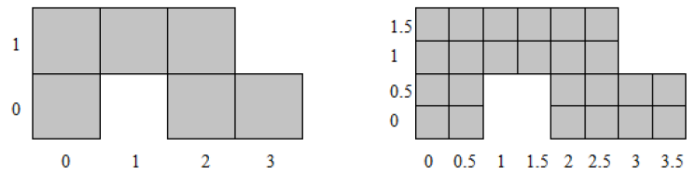

FIGURE 1. A digital image X and the corresponding 2-subdivision $S(X,2)$ [**6**]

Given a digital image $X \subset \mathbb{Z}^n$, let $x \in S(X,r)$ be represented as $x = (x_1, x_2, ..., x_n)$. Let $E_r : S(X,r) \to X$ be defined by $E_r(x) = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, ..., \lfloor x_n \rfloor)$.

REMARK 2.1. ([**21**]) Given a digital image $(X,\kappa) \subset (\mathbb{Z}^m, \kappa)$, any function $f : S(X,r) \to Y$ induces a multivalued function $F : X \multimap Y$ defined by

$$F(x) = \bigcup_{x' \in E_r^{-1}(x)} \{f(x')\}.$$

The function f is called by a *support function* for F.

DEFINITION 2.2. ([**22**]) Given a digital image $X \subset \mathbb{Z}^m$ and $Y \subset \mathbb{Z}^n$, a digital multivalued function $F : X \multimap Y$ is called a continuous if it is induced by a continuous single-valued function $f : S(X,r) \to Y$ for some integer $r > 0$.

DEFINITION 2.3. ([**7**]) Let (X,κ) be a digital image. Let $Y \subset X$. Then Y is a *multivalued retract* of X if there is a continuous multivalued function $F : X \multimap Y$ such that for all $y \in Y$, $F(y) = \{y\}$. The function F is called a multivalued retract.

PROPOSITION 2.3 ([**23**]). *Let $(X, \kappa = 3^m - 1) \subset \mathbb{Z}^m$, $(Y, \kappa') \subset \mathbb{Z}^n$, and $(Z, \kappa'') \subset \mathbb{Z}^p$ be digital images. If $F : X \multimap Y$ is a (κ, κ')-continuous multivalued function and $G : Y \multimap Z$ is a (κ', κ'')-continuous multivalued function, then $G \circ F : X \multimap Z$ is a (κ, κ'')-continuous multivalued function.*

DEFINITION 2.4. ([**27**]) Let (X,κ) and (Y,λ) be two digital images. A digital multivalued function $F : X \multimap Y$ is *connectivity preserving* if $F(A) \subset Y$ whenever $A \subset X$ is κ-connected.

PROPOSITION 2.4 ([**7**]). *For a digital image $(X,\kappa) \subset (\mathbb{Z}^n \kappa)$, if $F : X \multimap Y$ is a continuous digital multivalued function, then F is connectivity preserving.*

Let (W,κ) be a digital image such that $W = X \cup X'$, where $X \cap X' = \{x_0\}$ for some $x_0 \in W$. We say W is the *wedge* of X and X', written $W = X \vee X'$ or $(W,\kappa) = (X,\kappa) \vee (X',\kappa)$ [**31**].

Sphere-like digital images are defined as follows [**5**]:

$$S_n = [-1,1]_{\mathbb{Z}}^{n+1} \backslash \{0_{n+1}\} \subset \mathbb{Z}^{n+1},$$

where 0_n is the origin point of $\mathbb{Z}^n$. For instance,

$$S_0 = \{1, -1\}.$$

$$S_1 = \{c_0 = (1,0), c_1 = (1,1), c_2 = (0,1), c_3 = (-1,1), c_4 = (-1,0), c_5 = (-1,-1), c_6 = (0,-1), c_7 = (1,-1)\}$$ (See Figure 2).

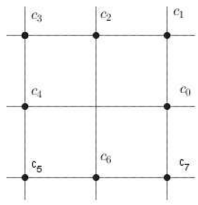

FIGURE 2. Digital 1-sphere S_1

Given two digital images X_1 and X_2, $X_1 \vee X_2$ is a subset of $X_1 \times X_2$. Since the normal product adjacency relation is defined for the cartesian product of two digital images, the smash product is constructed as follows:

DEFINITION 2.5. ([**13**]) Let (X, κ_1) and (Y, κ_2) be two digital images. The *digital smash product* $(X \wedge Y, k_*(\kappa_1, \kappa_2))$ is defined to be the quotient digital image $(X \times Y)/(X \vee Y)$, where $X \vee Y$ is a wedge union of X and Y.

EXAMPLE 2.1. ([**13**]) If we choose digital images $X = S_1$ and $Y = S_0$, then we get the following digital images in Figure 3.

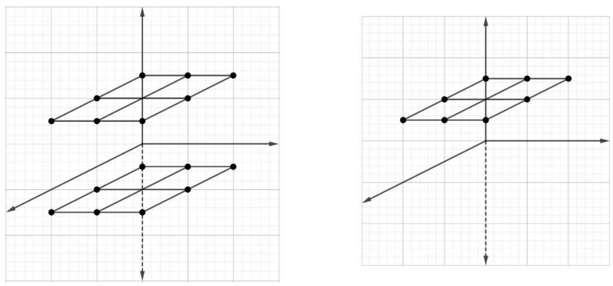

FIGURE 3. $S_1 \times S_0$ and $S_1 \wedge S_0$

Let $F : X \multimap Y$ be a multivalued function between digital images.
• F has *weak continuity* [**32**] if for each pair of adjacent points $x, y \in X$, $F(x)$ and $F(y)$ are adjacent subsets of Y.
• F has *strong continuity* [**32**] if for each pair of adjacent points $x, y \in X$, every point of $F(x)$ is adjacent or equal to some point of $F(y)$ and every point of $F(y)$ is adjacent or equal to some point of $F(x)$.

EXAMPLE 2.2. ([**7**]) Let $[0,1]_{\mathbb{Z}}$ and $[0,2]_{\mathbb{Z}}$ be two digital intervals. The multivalued function

$$\begin{aligned} G : [0,1]_{\mathbb{Z}} &\multimap [0,2]_{\mathbb{Z}} \\ 0 &\mapsto \{0,2\} \\ 1 &\mapsto \{1\} \end{aligned}$$

has both weak and strong continuity. In contrast, G is not a continuous multivalued function since G is not connectivity preserving.

EXAMPLE 2.3. ([**7**]) Let $G : [0,1]_{\mathbb{Z}} \multimap [0,2]_{\mathbb{Z}}$ be defined by $G(0) = \{0,1\}$, $G(1) = \{2\}$. Then G is continuous and has weak continuity but does not have strong continuity.

THEOREM 2.1 ([**10**]). *Let $f : (X,\kappa) \multimap (Y,\lambda)$ and $g : (Y,\lambda) \multimap (W,\mu)$ be multivalued functions between digital images.*

- *If f and g are both weakly continuous, then $g \circ f$ is weakly continuous.*
- *If f and g are both strongly continuous, then $g \circ f$ is strongly continuous.*

3. Universal Multivalued Functions

Boxer et al. [**12**] research universal functions on digital images and their relation to the AFPP. In this section, we introduce universal digital multivalued functions and prove some properties about these multivalued functions.

DEFINITION 3.1. Let (X,κ) and (Y,λ) be digital images. A (κ,λ)- continuous multivalued function $F : X \multimap Y$ is a *universal* for (X,Y) if given a (κ,λ)- continuous multivalued function $G : X \multimap Y$, there exists $x \in X$ such that $F(x) \leftrightarrow_\lambda G(x)$.

PROPOSITION 3.1. *Let X and Y be digital images. Suppose Y is finite. Then the multivalued function $F : X \multimap Y$ defined by $F(x) = Y$ for all $x \in X$ is universal.*

PROOF. This follows easily from Definition 3.1. □

THEOREM 3.1 ([**12**]). *Let (W,κ), (X,λ), and (Y,μ) be digital images. Let $f : W \to X$ be (κ,λ)-continuous and let $g : X \to Y$ be (λ,μ)-continuous. If $g \circ f$ is universal, then g is also universal.*

Theorem 3.1 does not hold for digital multivalued functions:

EXAMPLE 3.1. Let C_n be a digital simple closed curve of $n > 3$ points and let F be the multivalued self-map on C_n defined by $F(x) = C_n$ for all x in C_n. It is easy to see that F is continuous. Let G be the identity map on C_n. Then $G \circ F = F$ is easily seen to be universal, but G is not since we can take h to be a rotation of C_n by 2 points so that no x in C_n has $h(x)$ adjacent to $G(x) = x$.

THEOREM 3.2 ([**12**]). *$g : (U,\mu) \to (X,\kappa)$ and $h : (Y,\lambda) \to (V,\nu)$ are digital isomorphisms and $f : X \to Y$ is (κ,λ)-continuous, then the following are equivalent.*

(1) *f is a universal function for (X,Y).*
(2) *$f \circ g$ is universal.*
(3) *$h \circ f$ is universal.*

THEOREM 3.3. *Let (X,κ) and (Y,λ) be digital images and let U be a subset of X. If the restriction function $F|_U : (U,\kappa) \multimap (Y,\lambda)$ is a universal digital multivalued function for (U,Y), then $F : (X,\kappa) \multimap (Y,\lambda)$ is also a universal digital multivalued function for (X,Y).*

Proof. Let $H : X \multimap Y$ be (κ, λ)-continuous multivalued function. Since the restriction of F to U is a universal, there exists $u \in U \subset X$ such that

$$H(u) = H|_U(u) \leftrightarrow_\lambda F|_U(u) = F(u).$$

Therefore, F is a universal for (X, Y). $\square$

4. Approximate Fixed Points For Digital Multivalued Functions

Boxer et al. [**12**] introduce approximate fixed points and the Approximate Fixed Point Property (AFPP). They give examples of digital images that have, and that don't have, this property. In this section, we introduce approximate fixed points for digital multivalued functions.

Given a digital image (X, κ) and a (κ, κ)-continuous multivalued function $F : X \multimap X$, we say $p \in X$ is an *approximate fixed point* of F if

$$\text{either} \quad p \in F(p) \quad \text{or} \quad \{p\} \leftrightarrow F(p).$$

We say that a digital (X, κ) has the *approximate fixed point property* (AFPP) for multivalued functions if every (κ, κ)-continuous multivalued function $F : X \multimap X$ has an approximate fixed point.

Theorem 4.1 ([**29**]). *Let $I = \prod_{i=1}^{n} [a_i, b_i]_{\mathbb{Z}}$. Then (I, c_n) has the AFPP for single-valued functions.*

Theorem 4.2 ([**12**]). *Let A and B be digital images. Then $(A \vee B, \kappa)$ has the AFPP if and only if both (A, κ) and (B, κ) have the AFPP for single-valued functions.*

A digital multivalued function $F : A \multimap B$ is *injective* if $F(a) = F(b)$ implies $a = b$, *absolutely injective* if $a \neq b$ implies $F(A) \cap F(B) = \varnothing$, and *surjective* if every $b \in B$ belongs to some $F(a)$ for some $a \in A$. If F is both injective and surjective, it is said to be *bijective*. A bijective multivalued function $1_A : A \multimap A$ is said to be an *identity* if $a \in 1_A(a)$ for all $a \in A$.

Proposition 4.1. *Let (X, κ) be a digital image. Then (X, κ) has the AFPP for multivalued functions if and only if the identity multivalued function 1_X is a universal for (X, X).*

Proof. Let $F : X \multimap X$ be a multivalued function. Our hypothesis implies there exists $p \in X$ such that $\{p\} \leftrightarrow F(p)$. So 1_X is universal for (X, X). Conversely, suppose 1_X is a universal function for (X, X). Then given a continuous multivalued function $F : X \multimap X$ there exist $p \in X$ such that $\{p\} \in 1_X(p) \leftrightarrow F(p)$. Therefore, (X, κ) has the AFPP. $\square$

Theorem 4.3. *Let $(X_i, c_{n_i}) \subset \mathbb{Z}^{n_i}$ for $i = 0, 1, 2, ..., m$ and $s = \sum_{i=0}^{m} n_i$. Given the digital image $X = \prod_{i=0}^{m} X_i \subset \mathbb{Z}^s$, if (X, c_s) has the AFPP for multivalued functions, then each (X_i, c_{n_i}) has the AFPP for multivalued functions.*

Proof. Suppose that (X, c_s) has the AFPP for multivalued functions. Let $F_i : X_i \multimap X_i$ be (c_{n_i}, c_{n_i})-continuous multivalued function. Then the multivalued function $F : X \multimap X$ defined by

$$F(x_1, x_2, ..., x_m) = (F_1(x_1), F_2(x_2), ..., F_m(x_m))$$

is (c_s, c_s)-continuous. This claim follows from [**11**] (Proposition 3.1) and [**9**] (Proposition 3.4 and Theorem 11.4). From Proposition 4.1, 1_X is a universal for (X, X). Hence there is a point $x_* = (x_{1,*}, x_{2,*}, ..., x_{m,*}) \in X$ with $x_{i,*} \in X_i$ such that

$$\{x_*\} \leftrightarrow_{c_s} F(x_*).$$

So $x_{i,*} \leftrightarrow_{c_{n_i}} F_i(x_{i,*})$ for all $i \in \{0, 1, ..., m\}$. Since F_i was arbitrarily taken, it follow that (X_i, c_{n_i}) has the AFPP for multivalued functions. $\square$

Theorem 4.4. *Let $(X, \kappa = 3^n - 1) \subset \mathbb{Z}^n$ be a digital image, and let $Y \subset X$ be a (κ, κ)-multivalued retract of X. If (X, κ) has the AFPP, then (Y, κ) has the AFPP for multivalued functions.*

Proof. Let $R : X \multimap Y$ be a (κ, κ)-multivalued retraction and let $F : Y \multimap Y$ be a (κ, κ)-multivalued continuous function. Let $I : Y \multimap X$ be the inclusion function. By Proposition 2.3, $G = I \circ F \circ R : X \multimap X$ is a (κ, κ)-continuous multivalued function. Hence G has an approximate fixed point $x_0 \in X$. Let $x_1 \in G(x_0)$ such that $x_1 \leftrightarrow x_0$. Then

$$x_1 \in G(x_0) \leftrightarrow_\kappa G(x_1) = I \circ F \circ R(x_1) = I \circ F(x_1) = F(x_1).$$

As a result, x_1 is an approximate fixed point of F. $\square$

5. Some Results on Single Valued Digital Functions

In this section, we deal with some properties on single valued digital functions.

Theorem 5.1. *Let (X, κ) and (Y, λ) be digital images. If $f : (X, \kappa) \to (Y, \lambda)$ is a (κ, λ)-continuous function and if (A, κ) is a subset of (X, κ), then $f|_A : (A, \kappa) \to (Y, \lambda)$ is also a (κ, λ)-continuous function.*

Proof. The inclusion function $i : (A, \kappa) \to (X, \kappa)$ is a (κ, κ)-continuous. By Proposition 2.1, the composition of two digital continuous functions is also digital continuous. Thus, the composite function

$$f \circ i : (A, \kappa) \to (Y, \lambda)$$

is a (κ, λ)-continuous function. $\square$

Theorem 5.2. *Let (X, κ) and (Y, λ) be digital images. If (Z, λ) is a subset of (Y, λ) and $f : (X, \kappa) \to (Y, \lambda)$ is a digital continuous function such that $f(X)$ is a subset of digital image Z, then $f := f' : (X, \kappa) \to (Z, \lambda)$ is a digital continuous function.*

Proof. Since f is a digital continuous function, for every κ-connected subset A of X, $f(A)$ is a λ-connected subset of (Y, λ). From the hypothesis, we have the following:

$$f(A) \subset f(X) \subset Z \subset Y.$$

For every κ-connected digital image A in X, $f(A)$ is a λ-connected subset of (Z, λ). $\square$

6. Smash Product for Digital Multivalued Functions

Cinar et al. [**13**] construct the smash product for digital images. In this section, we give some properties of the smash product for digital multivalued functions.

THEOREM 6.1. *Let* (X, κ), $(X^{'}, \kappa)$, (Y, λ), *and* $(Y^{'}, \lambda)$ *be digital images. If* $f : S(X, r) \to Y$ *and* $g : S(X^{'}, r) \to Y^{'}$ *are continuous functions which induce digital continuous multivalued functions* $F : X \multimap Y$ *and* $G : X^{'} \multimap Y^{'}$, *respectively and* $f \times g(S(X, r) \wedge S(X^{'}, r)) \subset (Y \wedge Y^{'})$, *where* $S(X, r)$ *is an rth-subdivision of* X, *then the smash product of the multivalued functions*

$$F \wedge G : X \wedge X^{'} \multimap Y \wedge Y^{'}$$

is continuous.

PROOF. Let F and G be digital continuous multivalued functions. For $r \in \mathbb{N}$, there exist two digital continuous single-valued function

$$f : S(X, r) \to Y \quad \text{and} \quad g : S(X^{'}, r) \to Y^{'}$$

such that F and G are induced from f and g, respectively. By Proposition 2.2, we obtain that the function

$$f \times g : S(X, r) \times S(X^{'}, r) \to Y \times Y^{'}$$

is a digital continuous function. From Theorem 5.1, the function

$$(f \times g)|_{S(X,r) \wedge S(X^{'},r)} : S(X, r) \wedge S(X^{'}, r) \to Y \times Y^{'}$$

is also a digital continuous function. In addition,

$$f \times g(S(X, r) \wedge S(X^{'}, r)) \subset (Y \wedge Y^{'})$$

and by Theorem 5.2, we have

$$(f \times g)|_{S(X,r) \wedge S(X^{'},r)} : S(X, r) \wedge S(X^{'}, r) \to Y \wedge Y^{'}$$

is a digital continuous function. We conclude that the digital continuous single-valued function $(f \times g)|_{S(X,r) \wedge S(X^{'},r)}$ induces

$$F \wedge G : X \wedge X^{'} \multimap Y \wedge Y^{'}$$

a digital multivalued function. $\square$

COROLLARY 6.1. *Let* (X, κ), $(X^{'}, \kappa)$, (Y, λ), *and* $(Y^{'}, \lambda)$ *be digital images. If*

$$F \wedge G : X \wedge X^{'} \multimap Y \wedge Y^{'}$$

is a digital continuous multivalued function, then $F \wedge G$ *is a connectivity preserving function.*

PROOF. By the fact that $F \wedge G$ is a continuous multivalued function and by Proposition 2.4, the function

$$F \wedge G : X \wedge X^{'} \multimap Y \wedge Y^{'}$$

is a connectivity preserving function. $\square$

7. Morphological Operations in Digital Multivalued Functions

In this section, we consider the basic operations in mathematical morphology; dilation, erosion, closing, and opening operators (see [**30**]). These operators will denote, respectively, by D_κ, E_κ, C_κ, and O_κ. Escribano, Giraldo, and Sastre [**22**] show that these operators can be modeled as digitally continuous multivalued functions. In addition, Boxer and Steacker also show that these morphological operations can be modeled as digitally connectivity preserving multivalued functions.

Dilation [**30**] of a binary image can be regarded as a method of magnifying or swelling the image. A common method of performing a dilation of a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$ is to take the dilation

$$D_\kappa(X) = \bigcup_{x \in X} N_\kappa^*(x).$$

THEOREM 7.1. *Given a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, the digital multivalued function*

$$\begin{aligned} \tilde{D}_\kappa : X &\multimap D_\kappa(X) \\ x &\mapsto N_\kappa(x) \cup \{x\} \end{aligned}$$

is a strongly continuous multivalued function.

PROOF. Let x and y be two κ-adjacent points. Each element of $\tilde{D}_\kappa(y)$ is equal or κ-adjacent to $y \in \tilde{D}_\kappa(x)$. Similarly, because of $x \in \tilde{D}_\kappa(y)$, every element of $\tilde{D}_\kappa(x)$ is κ-adjacent to $x \in \tilde{D}_\kappa(y)$. So we get the required result. $\square$

EXAMPLE 7.1. Let $X = \{p = (0,0), q = (1,0)\} \subset \mathbb{Z}^2$ be a digital image. The multivalued function

$$\begin{aligned} \tilde{D}_4 : X &\multimap D_4(X) \\ (0,0) &\mapsto N_4((0,0)) \cup \{(0,0)\} \\ (1,0) &\mapsto N_4((1,0)) \cup \{(1,0)\} \end{aligned}$$

is strongly continuous.

There are nonequivalent definitions of the erosion operation in the literature. We will use the definition of [**22**]: the κ-erosion of $X \subset \mathbb{Z}^n$ is
$E_\kappa(X) = \mathbb{Z}^n \setminus D_\kappa(\mathbb{Z}^n \setminus X)$.

THEOREM 7.2. *For a digital image (X, κ) in $(\mathbb{Z}^n, \kappa)$, the multivalued function*

$$\begin{aligned} \bar{E}_\kappa : \mathbb{Z}^n \setminus X &\multimap \mathbb{Z}^n \\ x &\longmapsto N_\kappa(x) \cup \{x\} \end{aligned}$$

is a strongly continuous multivalued function, where $\bar{E}_\kappa$ is an erosion operator.

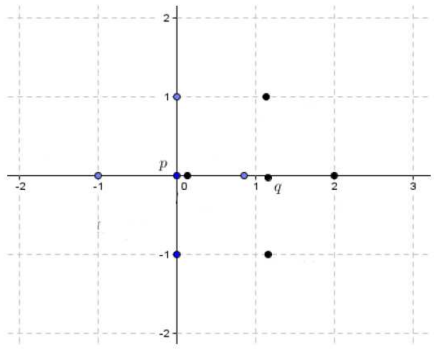

FIGURE 4. 4-adjacent points of p and q

PROOF. The assertion follows as in the proof of Theorem 7.1. $\square$

Like dilation, closing a digital image can be regarded as a way to swell the image. The closure operator C_κ is the result of a dilation followed by an erosion. Since we have defined an erosion on X as a dilation on $\mathbb{Z}^n \setminus X$, we cannot say that C_κ is a composition of a dilation and an erosion, since the corresponding composition $\bar{E}_\kappa \circ \tilde{D}_\kappa$ is not generally defined. The closure operator of X can be defined as

$$C_\kappa(X) = \mathbb{Z}^n \setminus \tilde{D}_\kappa(\mathbb{Z}^n \setminus \bigcup_{x \in X} N^*_\kappa(x)).$$

Given a digital image $(X, \kappa) \subset \mathbb{Z}^n$ and $x \in X$, the boundary of X in $\mathbb{Z}^n$ is defined as follows:

$$\partial_\kappa(X) = \{y \in X | N_\kappa(y) \setminus X \neq \varnothing\}.$$

THEOREM 7.3. *Given a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, the closure operator $\tilde{C}_\kappa$ is weakly continuous.*

PROOF. We define a digital multivalued function $\tilde{C}_\kappa : X \multimap C_\kappa(X)$ by

$$\tilde{C}_\kappa(x) = \begin{cases} \{x\}, & x \in X \setminus \partial_\kappa X \\ (\{x\} \cup N_\kappa(x)) \cap C_\kappa(X), & x \in \partial_\kappa X. \end{cases}$$

Suppose $x \leftrightarrow_\kappa x^{'}$ in X. We consider the following cases.

• $x, x^{'} \in X \setminus \partial_\kappa X$. $x \in \tilde{C}_\kappa(x)$ and $x^{'} \in \tilde{C}_\kappa(x^{'})$, hence $\tilde{C}_\kappa(x)$ and $\tilde{C}_\kappa(x^{'})$ are κ-adjacent sets.

• $x \in X \setminus \partial_\kappa X$, $x^{'} \in \partial_\kappa X$. Then x is an element of $\tilde{C}_\kappa(x^{'})$, $x \in \tilde{C}_\kappa(x) \cap \tilde{C}_\kappa(x^{'})$. Therefore, $\tilde{C}_\kappa(x)$ and $\tilde{C}_\kappa(x^{'})$ are κ-adjacent sets.

• $x \in \partial_\kappa X$, $x^{'} \in X \setminus \partial_\kappa X$. This is similiar to the previous case.

• $x, x^{'} \in \partial_\kappa X$. From the definition of $\tilde{C}_\kappa$, $x \in \tilde{C}_\kappa(x^{'})$. Consequently, $\tilde{C}_\kappa(x)$ and $\tilde{C}_\kappa(x^{'})$ are κ-adjacent sets.

Thus, $\tilde{C}_\kappa$ is weakly continuous. $\square$

As it happens in the case of the erosion, the opening operation (erosion composed with dilation) cannot be adequately modeled as a digitally continuous multivalued function on the set of black pixels. However, since the opening of a set agrees with the closing of its complement [**30**], the κ-opening operator can be modeled by a κ-continuous multivalued function on the set of white pixels. Thus, we define an opening operator for X as the closure operator on $\mathbb{Z}^n \setminus X$.

THEOREM 7.4. *Let* (X, κ) *be a digital image in* $(\mathbb{Z}^n, \kappa)$. *The* κ*-opening operation on* X *can be modeled as a weakly continuous multivalued function* $\bar{O}_\kappa : \mathbb{Z}^n \setminus X \multimap \mathbb{Z}^n$.

PROOF. The assertion follows from Theorem 7.3. □

8. Conclusion

This paper introduces some notions such as the approximate fixed point property for multivalued functions and universal digital multivalued functions. We give some properties of smash product of digital multivalued functions. Finally, we show that morphological operators such as dilation and erosion have strong continuity. Furthermore, opening and closing have weak continuity.

9. Acknowledgement

We would like to express our gratitude to the anonymous referees for their helpful suggestions and corrections. The first author is granted as fellowship by the Scientific and Technological Research Council of Turkey TUBITAK-2211-A. In addition, this work was partially supported by Research Fund of the Ege University (Project Number:FDK-2019-20333).

References

[1] C. Berge. *Graphs and Hypergraphs.* 2nd edition, North-Holland, Amsterdam, 1976. ISBN 10: 0720424534
[2] L. Boxer. Digitally continuous functions. *Pattern Recognition Letters*, **15**(8)(1994), 833–839.
[3] L. Boxer. A classical construction for the digital fundamental group. *J. Math. Imaging Vis.*, **10**(1)(1999), 51–62.
[4] L. Boxer. Homotopies properties of sphere-like digital images. *J. Math. Imaging Vis.*, **24**(2)(2006), 167–175.
[5] L. Boxer. Digital products, wedges and covering spaces. *J. Math. Imaging Vis.*, **25**(2)(2006), 159–171.
[6] L. Boxer. Remarks on digitally continuous multivalued functions. *J. Adv. Math.*, **9**(1)(2014), 1755–1762.
[7] L. Boxer and P. C. Staecker. Connectivity preserving multivalued functions in digital topology. *J. Math. Imaging Vis.*, **55**(3)(2016), 370–377.
[8] L. Boxer. Digital shy maps. *Appl. Gen. Topol.*, **18**(1)(2017), 143–152.
[9] L. Boxer. Generalized normal product adjacency in digital topology. *Appl. Gen. Topol.*, **18**(2)(2017), 401–427.
[10] L. Boxer. Multivalued functions in digital topology. *Note Mat.*, **37**(2)(2017), 61–76.
[11] L. Boxer and I. Karaca. Fundamental groups for digital products. *Adv. Appl. Math. Sci.*, **11**(4)(2012), 161–180.
[12] L. Boxer, O. Ege, I. Karaca, J. Lopez and J. Louwsma. Digital fixed points, approximate fixed points and universal functions. *Appl. Gen. Topol.*, **17**(2)(2016), 159–172.

[13] I. Cınar, O. Ege, and I. Karaca. The digital smash product. Preprint, 2019.
[14] I. Cınar and I. Karaca. Some new results on the connected sum of certain digital surfaces. Preprint, 2019.
[15] O. Ege and I. Karaca. Cohomology theory for digital images. *Rom. J. Inf. Sci. Tech.*, **16**(1)(2013), 10–28.
[16] O. Ege and I. Karaca. Fundamental properties of digital simplicial homology groups. *American Journal of Computer Technology and Application*, **1**(2)(2013), 25–41.
[17] O. Ege and I. Karaca. Some properties of digital H-spaces. *Turk. J. Elec. Eng. Comp. Sci.*, **24**(3)(2016), 1930–1941.
[18] O. Ege and I. Karaca. Digital fibrations. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, **87**(1)(2017), 109–114.
[19] O. Ege and I. Karaca. Graph topology on finite digital images. *Utilitas Mathematica*, **109**(2018), 211–218.
[20] O. Ege and I. Karaca. Digital cohomology groups of certain minimal surfaces. *Journal of Linear and Topological Algebra*, **7**(4)(2018), 293–305.
[21] C. Escribano, A. Giraldo and M. A. Sastre. Digitally continuous multivalued functions. In: *Proceedings of the 14th IAPR international conference on Discrete geometry for computer imagery* (pp. 81-92). Lecture Notes in Computer Science, Springer, Berlin, 2008. ISBN:3-540-79125-6 978-3-540-79125-6
[22] C. Escribano, A. Giraldo and M. A. Sastre. Digitally continuous multivalued functions, morphological operations and thinning algorithms. *J. Math. Imaging Vis.*, **42**(2)(2012), 76–91.
[23] A. Giraldo and M. Sastre. On the Composition of Digitally Continuous Multivalued Functions. *J. Math. Imaging Vis.*, **53**(2)(2015), 196–209.
[24] G. T. Herman. Oriented surfaces in digital spaces. *CVGIP: Graphical Models and Image Processing*, **55**(5)(1993), 381–396.
[25] I. Karaca and O. Ege. Some results on simplicial homology groups of 2D digital images. *International Journal of Information and Computer Science*, **1**(8)(2012), 198–203.
[26] I. Karaca and I. Cınar. The cohomology structure of digital Khalimsky spaces. *Rom. J. Math. Comp. Sci.*, **8**(2)(2018), 110–128.
[27] V. Kovalevsky. A new concept for digital geometry. In: Y. L. O, A. Toet, D. Foster, H. J. A. M. Heijmans and P. Meer (Eds.). *Shape in Picture* (pp. 37-51). NATO ASI Series (Series F: Computer and Systems Sciences) , vol 126. Springer, Berlin, Heidelberg Shape in Picture. Springer, New York, 1994. ISBN 978-3-642-08188-0
[28] A. Rosenfeld. Digital topology. *Amer. Math. Monthly*, **86**(8)(1979), 621–630.
[29] A. Rosenfeld. 'Continuous' functions in digital pictures. *Pattern Recognition Letters*, **4**(3)(1986), 177–184.
[30] P. Soille. Morphological operators. In: B. Jahne, H. Haussecker and P. Geissler (Eds.) *Handbook of computer vision and applications. Vol. 2: Signal Processing and Pattern Recognition* (pp. 627–682). San Diego: Academic Press, 1999. ISBN 0-12-379772-1.
[31] E. H. Spanier. *Algebraic Topology.* Springer-Verlag, New York, 1966. ISBN 978-0-387-94426-5
[32] R. Tsaur anbd M. Smyth. "Continuous" multifunctions in discrete spaces with applications to fixed point theory. In: G. Bertrand, A. Imiya and R. Klette (Eds.). *Digital and Image Geometry: Advanced Lectures* (pp. 75-88). Lecture Notes in Computer Science, Springer, New York, 2001. ISBN: 3-540-43079-2

Received by editors 16.03.2019; Revised version 16.04.2019; Available online 06.05.2019.

Department of Mathematics, Ege University, Izmir, Turkey
E-mail address: `ismet___cinar@hotmail.com`

Department of Mathematics, Ege University, Izmir, Turkey
E-mail address: `ismet.karaca@ege.edu.tr`