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NEW GENERALIZED CLASSES OF IDEAL NANOTOPLOGICAL SPACES

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Abstract. In this paper, we introduce a new class of subsets called ξ-nI-open subsets and $\mathcal{Q}$-nI-closed subsets, where ξ-nI-open subsets are weaker than α-nI-open subsets and $\mathcal{Q}$-nI-closed subsets are stronger than β-nI-open subsets. Also a new class of subsets called semi*-nI-closed subsets is introduced which are equivalent to t-nI-sets.

1. Introduction

An ideal I [**11**] on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

(1) $A \in I$ and $B \subset A$ imply $B \in I$ and
(2) $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X. If $\wp(X)$ is the family of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [**1**]. The closure operator defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ [**10**] is a Kuratowski closure operator which generates a topology $\tau^\star(I, \tau)$ called the $\star$-topology finer than τ. The topological space together with an ideal on X is called an ideal topological space or an ideal space denoted by (X, τ, I). We will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$.

The notion of an ideal nanotopological space was introduced by Parimala et al. [**2, 4**].

Recently, Rajasekaran and Nethaji [**7**] introduced the notions of α-nI-open sets and β-nI-open sets and studied in detail.

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In this paper, we introduce a new class of subsets called ξ-nI-open subsets and $\mathcal{Q}$-nI-closed subsets, where ξ-nI-open subsets are weaker than α-nI-open subsets and $\mathcal{Q}$-nI-closed subsets are stronger than β-nI-open subsets. Also a new class of subsets called semi*-nI-closed subsets is introduced which equivalent to t-nI-set. Also decompositions of a semi-nI-open subset, an α-nI-open subset and decompositions of a $\mathcal{Q}$-nI-closed subset have been obtained.

2. Preliminaries

Definition 2.1. ([**5**]) Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

(1) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.

(2) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.

(3) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2. ([**9**]) Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:

(1) U and $\phi \in \tau_R(X)$,

(2) The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,

(3) The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nanotopology with respect to X and $(U, \tau_R(X))$ is called the nanotopological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed. In the rest of the paper, we denote a nanotopological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset A of U are denoted by n-$int(A)$ and n-$cl(A)$, respectively.

A nanotopological space $(U, \mathcal{N})$ with an ideal I on U is called [**2**] an ideal nanotopological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, denotes [**2**] the family of nano open sets containing x.

In future an ideal nanotopological space $(U, \mathcal{N}, I)$ will be simply called a space.

Definition 2.3. ([**2**]) Let $(U, \mathcal{N}, I)$ be a space with an ideal I on U. Let $(.)^\star_n$ be a set operator from $\wp(U)$ to $\wp(U)$ ($\wp(U)$ is the set of all subsets of U). For a subset $A \subseteq U$, $A^\star_n(I, \mathcal{N}) = \{x \in U : G_n \cap A \notin I$, for every $G_n \in G_n(x)\}$ is called the nano local function (briefly, n-local function) of A with respect to I and $\mathcal{N}$. We will simply write $A^\star_n$ for $A^\star_n(I, \mathcal{N})$.

THEOREM 2.1 ([**2**]). *Let $(U,\mathcal{N},I)$ be a space and A and B be subsets of U. Then*

(1) $A \subseteq B \Rightarrow A^\star_n \subseteq B^\star_n$,
(2) $A^\star_n = n\text{-}cl(A^\star_n) \subseteq n\text{-}cl(A)$ *($A^\star_n$ is a n-closed subset of n-cl(A))*,
(3) $(A^\star_n)^\star_n \subseteq A^\star_n$,
(4) $(A \cup B)^\star_n = A^\star_n \cup B^\star_n$,
(5) $V \in \mathcal{N} \Rightarrow V \cap A^\star_n = V \cap (V \cap A)^\star_n \subseteq (V \cap A)^\star_n$,
(6) $J \in I \Rightarrow (A \cup J)^\star_n = A^\star_n = (A - J)^\star_n$.

THEOREM 2.2 ([**2**]). *Let $(U,\mathcal{N},I)$ be a space with an ideal I and A be a subset of U. If $A \subseteq A^\star_n$, then $A^\star_n = n\text{-}cl(A^\star_n) = n\text{-}cl(A)$.*

DEFINITION 2.4. ([**2**]) Let $(U,\mathcal{N},I)$ be a space. The set operator $n\text{-}cl^\star$ called a nano $\star$-closure is defined by $n\text{-}cl^\star(A) = A \cup A^\star_n$ for $A \subseteq U$. It can be easily observed that $n\text{-}cl^\star(A) \subseteq n\text{-}cl(A)$.

THEOREM 2.3 ([**4**]). *In a space $(U,\mathcal{N},I)$, if A and B are subsets of U, then the following results are true for the set operator $n\text{-}cl^\star$.*

(1) $A \subseteq n\text{-}cl^\star(A)$,
(2) $n\text{-}cl^\star(\phi) = \phi$ *and* $n\text{-}cl^\star(U) = U$,
(3) *If* $A \subset B$, *then* $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(B)$,
(4) $n\text{-}cl^\star(A) \cup n\text{-}cl^\star(B) = n\text{-}cl^\star(A \cup B)$,
(5) $n\text{-}cl^\star(n\text{-}cl^\star(A)) = n\text{-}cl^\star(A)$.

DEFINITION 2.5. ([**8**])

(1) A subset A of U in a space $(U,\mathcal{N},I)$ is called nano dense (briefly *n*-dense) if $n\text{-}cl(A) = U$.

(2) A space $(U,\mathcal{N},I)$ is called nano submaximal (briefly *n*-submaximal) if each *n*-dense subset of U is *n*-open.

DEFINITION 2.6. ([**3**]) A subset A of a space $(U,\mathcal{N},I)$ is $n\star$-dense in itself (resp. $n\star$-perfect and $n\star$-closed) if $A \subseteq A^\star_n$ (resp. $A = A^\star_n$, $A^\star_n \subseteq A$).

DEFINITION 2.7. ([**9**]) A subset A of a space $(U,\mathcal{N})$ is called nano semi-open if $A \subseteq n\text{-}cl(n\text{-}int(A))$. The complement of a nano semi-open is said to be nano semi-closed.

DEFINITION 2.8. ([**7**]) A subset A of a space $(U,\mathcal{N},I)$ is said to be

(1) nano α-*I*-open (briefly α-*nI*-open) if $A \subseteq n\text{-}int(n\text{-}cl^\star(n\text{-}int(A)))$,
(2) nano semi-*I*-open (briefly semi-*nI*-open) if $A \subseteq n\text{-}cl^\star(n\text{-}int(A))$,
(3) nano pre-*I*-open (briefly *pre-nI*-open) if $A \subseteq n\text{-}int(n\text{-}cl^\star(A))$,
(4) nano b-*I*-open (briefly *b-nI*-open) if $A \subseteq n\text{-}int(n\text{-}cl^\star(A)) \cup n\text{-}cl^\star(n\text{-}int(A))$,
(5) nano β-*I*-open (briefly β-*nI*-open) if $A \subseteq n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A)))$.

The complements of the above mentioned sets are called their respective closed sets.

DEFINITION 2.9. ([**6**]) A subset A of a space $(U,\mathcal{N},I)$ is called a nano t-*I*-set (briefly t-*nI*-set) if $n\text{-}int(A) = n\text{-}int(n\text{-}cl^\star(A))$.

Remark 2.1. [**7**] For any subset of a space $(U, \mathcal{N}, I)$, we have the following Diagram.

n-open	$\longrightarrow$	**α-nI-open**	$\longrightarrow$	**pre-nI-open**
		$\downarrow$		$\downarrow$
		semi-nI-open	$\longrightarrow$	**b-nI-open**
				$\downarrow$
				β-nI-open

In this Diagram, none of the implications is reversible.

Theorem 2.4 ([**7**]). *A subset A of a space $(U, \mathcal{N}, I)$ is α-nI-open $\iff$ A is semi-nI-open and pre-nI-open.*

3. On nano semi*-I-open sets and nano ξ-I-closed sets

Definition 3.1. A subset A of a space $(U, \mathcal{N}, I)$ is called nano semi*-I-open (briefly semi*-nI-open) if $A \subseteq n\text{-}cl(n\text{-}int^\star(A))$.

The complement of a semi*-nI-open set is said to be semi*-nI-closed.

Proposition 3.1. *In a space $(U, \mathcal{N}, I)$, each nano semi-open set is semi*-nI-open.*

Proof. If A is nano semi-open, then $A \subseteq n\text{-}cl(n\text{-}int(A)) \subseteq n\text{-}cl(n\text{-}int^\star(A))$. Therefore A is semi*-nI-open. □

Remark 3.1. The converse of Proposition 3.1 is not true as illustrated in the following Example.

Example 3.1. Let $U = \{p, q, r, s\}$ with $U/R = \{\{p\}, \{s\}, \{q, r\}\}$ and $X = \{p, r\}$. Then $\mathcal{N} = \{\phi, \{p\}, \{q, r\}, \{p, q, r\}, U\}$. If the ideal $I = \{\phi, \{r\}\}$, then $\{q\}$ is semi*-nI-open but not nano semi open in the space $(U, \mathcal{N}, I)$. $n\text{-}int^\star(\{q\}) = \{q\}$ and $n\text{-}cl(n\text{-}int^\star(\{q\})) = \{q, r, s\}$. Thus $\{q\} \subseteq n\text{-}cl(n\text{-}int^\star(\{q\}))$ and hence $\{q\}$ is semi*-nI-open. But $\{q\}$ is not nano semi open for $n\text{-}int(\{q\}) = \phi$.

Proposition 3.2. *A subset A of a space $(U, \mathcal{N}, I)$ is semi*-nI-open if and only if $n\text{-}cl(A) = n\text{-}cl(n\text{-}int^\star(A))$.*

Proof. If A is semi*-nI-open set, then $A \subseteq n\text{-}cl(n\text{-}int^\star(A))$ and $n\text{-}cl(A) \subseteq n\text{-}cl(n\text{-}int^\star(A))$. But $n\text{-}cl(n\text{-}int^\star(A)) \subseteq n\text{-}cl(A)$. Hence $n\text{-}cl(A) = n\text{-}cl(n\text{-}int^\star(A))$.

Conversely, $A \subseteq n\text{-}cl(A) = n\text{-}cl(n\text{-}int^\star(A))$ by assumption. Therefore A is semi*-nI-open. □

Definition 3.2. A subset A of U in the space $(U, \mathcal{N}, I)$ is said to be nano ξ-I-closed (briefly ξ-nI-closed) if $n\text{-}int^\star(n\text{-}cl(A)) \subseteq n\text{-}cl(n\text{-}int^\star(A))$.

The complement of a ξ-nI-closed set is said to be ξ-nI-open.

It is easily seen that a subset A of U in a space $(U, \mathcal{N}, I)$ is ξ-nI-open if $n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}cl^\star(n\text{-}int(A))$.

Proposition 3.3. *For a subset A of U in the space $(U, \mathcal{N}, I)$, the following properties are true:*

(1) *A is α-nI-open $\Rightarrow$ A is ξ-nI-open.*

(2) *A is a t-nI-set $\Rightarrow$ A is ξ-nI-open.*
(3) *A is n-open $\Rightarrow$ A is ξ-nI-open.*

Proof. (1) Since A is α-nI-open,
$A \subseteq n\text{-}int(n\text{-}cl^\star(n\text{-}int(A))) \subseteq n\text{-}cl^\star(n\text{-}int(A))$.
So $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(n\text{-}int(A))$ and $n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(n\text{-}int(A))$. Therefore A is ξ-nI-open.

(2) Since A is a t-nI-set, $n\text{-}int(n\text{-}cl^\star(A)) = n\text{-}int(A) \subseteq A$. Then
$n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}int(A) \subseteq n\text{-}cl^\star(n\text{-}int(A))$.
Therefore A is ξ-nI-open.

(3) A is n-open $\Rightarrow$ A is α-nI-open $\Rightarrow$ A is ξ-nI-open by Remark 2.1 and by(1) of Proposition 3.3. □

Remark 3.2. The converses in Proposition 3.3 are not true in general as seen from the following Example.

Example 3.2. In Example 3.1,

(1) $A = \{q, r, s\}$ is ξ-nI-open but not α-nI-open.
$n\text{-}int(n\text{-}cl^\star(A)) = \{q, r\} \subseteq \{q, r, s\} = n\text{-}cl^\star(n\text{-}int(A))$
which verifies that A is ξ-nI-open. But
$n\text{-}int(n\text{-}cl^\star(n\text{-}int(A))) = \{q, r\}$ and $A \nsubseteq \{q, r\}$.
Hence A is not α-nI-open.

(2) $B = \{p, q, r\}$ is ξ-nI-open but not a t-nI-set. B is n-open and so ξ-nI-open by(3) of Proposition 3.3. But $n\text{-}int(B) = B$ whereas $n\text{-}int(n\text{-}cl^\star(B)) = U$. Hence B is not a t-nI-set.

Proposition 3.4. *A subset A of a space $(U, \mathcal{N}, I)$ is β-nI-closed if and only if $n\text{-}int^\star(n\text{-}cl(n\text{-}int^\star(A))) = n\text{-}int^\star(A)$.*

Proof. Since A is β-nI-closed, $n\text{-}int^\star(n\text{-}cl(n\text{-}int^\star(A))) \subseteq A$ and hence
$n\text{-}int^\star(n\text{-}cl(n\text{-}int^\star(A))) \subseteq n\text{-}int^\star(A)$.
Also $n\text{-}int^\star(A) \subseteq n\text{-}cl(n\text{-}int^\star(A))$ and hence $n\text{-}int^\star(A) \subseteq n\text{-}int^\star(n\text{-}cl(n\text{-}int^\star(A)))$. Thus $n\text{-}int^\star(n\text{-}cl(n\text{-}int^\star(A))) = n\text{-}int^\star(A)$.

Conversely, let the condition be true. Then
$n\text{-}int^\star(n\text{-}cl(n\text{-}int^\star(A))) = n\text{-}int^\star(A) \subseteq A$.
Therefore A is β-nI-closed. □

Theorem 3.1. *For a subset A of U in a space $(U, \mathcal{N}, I)$, the following properties are equivalent.*

(1) *A is semi-nI-open.*
(2) *A is β-nI-open and ξ-nI-open.*

Proof. (1) $\Rightarrow$ (2): Let A be semi-nI-open. Then by the Remark 2.1, A is β-nI-open. Also A is semi-nI-open implies $A \subseteq n\text{-}cl^\star(n\text{-}int(A))$ and so $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(n\text{-}int(A))$. But $n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(n\text{-}int(A))$ which proves that A is ξ-nI-open.

(2) $\Rightarrow$ (1): Since A is ξ-nI-open, $n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}cl^\star(n\text{-}int(A))$ which implies $n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A))) \subseteq n\text{-}cl^\star(n\text{-}int(A))$. But A is β-nI-open implies $A \subseteq n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A))) \subseteq n\text{-}cl^\star(n\text{-}int(A))$. This proves that A is semi-nI-open. □

REMARK 3.3. In a space, the family of β-nI-open sets and the the family of ξ-nI-open sets are independent as illustrated in the following Example.

EXAMPLE 3.3. In Example 3.1,

(1) $A = \{p, r\}$ is ξ-nI-open for

$n\text{-}int(n\text{-}cl^\star(A)) = \{p\} \subseteq \{p, s\} = n\text{-}cl^\star(n\text{-}int(A))$.

But A is not β-nI-open for $A = \{p, r\} \nsubseteq \{p, s\} = n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A)))$.

(2) $B = \{q\}$ is β-nI-open for $B = \{q\} \subseteq \{q, r, s\} = n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(B)))$. But B is not ξ-nI-open for $n\text{-}int(B) = \phi$.

THEOREM 3.2. *Let $(U, \mathcal{N}, I)$ be a space. Then a subset of U is α-nI-open if and only if it is both ξ-nI-open and pre-nI-open.*

PROOF. Assuming that A is α-nI-open we prove that A is ξ-nI-open and pre-nI-open. Since A is α-nI-open, $A \subseteq n\text{-}int(n\text{-}cl^\star(n\text{-}int(A)))$. It implies that $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(n\text{-}int(A))$ and

$n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}int(n\text{-}cl^\star(n\text{-}int(A))) \subseteq n\text{-}cl^\star(n\text{-}int(A))$.

Hence, A is ξ-nI-open. Also, since A is α-nI-open, A is pre-nI-open by Remark 2.1.

Conversely assuming that A is both ξ-nI-open and pre-nI-open, we prove that A is α-nI-open. Since A is ξ-nI-open, $n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}cl^\star(n\text{-}int(A))$ and hence $n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}int(n\text{-}cl^\star(n\text{-}int(A)))$. Since A is pre-nI-open, $A \subseteq n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}int(n\text{-}cl^\star(n\text{-}int(A)))$ which proves that A is a α-nI-open set. □

REMARK 3.4. In a space the family of ξ-nI-open sets and the family of pre-nI-open sets are independent as illustrated in the following Example.

EXAMPLE 3.4. In Example 3.1,

(1) $A = \{r\}$ is ξ-nI-open since $n\text{-}int(n\text{-}cl^\star(A)) = \phi$. But A is not pre-nI-open for $A = \{r\} \nsubseteq \phi = n\text{-}int(n\text{-}cl^\star(A))$.

(2) $B = \{q\}$ is pre-nI-open since $\{q\} \subseteq \{q, r\} = n\text{-}int(n\text{-}cl^\star(B))$. But B is not ξ-nI-open for $n\text{-}int(B) = \phi$.

PROPOSITION 3.5. *Let A and B be subsets of U of a space $(U, \mathcal{N}, I)$. If $A \subseteq B \subseteq n\text{-}cl^\star(A)$ and A is ξ-nI-open in U, then B is ξ-nI-open in U.*

PROOF. Suppose that $A \subseteq B \subseteq n\text{-}cl^\star(A)$ and A is ξ-nI-open in U. Then $n\text{-}int(n\text{-}cl^\star(A)) \subseteq n\text{-}cl^\star(n\text{-}int(A)) \subseteq n\text{-}cl^\star(n\text{-}int(B))$. Since

$B \subseteq n\text{-}cl^\star(A)$, $n\text{-}cl^\star(B) \subseteq n\text{-}cl^\star(n\text{-}cl^\star(A)) = n\text{-}cl^\star(A)$

and $n\text{-}int(n\text{-}cl^\star(B)) \subseteq n\text{-}int(n\text{-}cl^\star(A))$. Therefore

$n\text{-}int(n\text{-}cl^\star(B)) \subseteq n\text{-}cl^\star(n\text{-}int(B))$.

This shows that B is a ξ-nI-open set. □

DEFINITION 3.3. A subset A of U in a space $(U, \mathcal{N}, I)$ is said to be nano $\star$-dense (briefly $n\star$-dense) if $n\text{-}cl^\star(A) = U$.

COROLLARY 3.1. *Let $(U, \mathcal{N}, I)$ be a space. If $A \subseteq U$ is ξ-nI-open and $n\star$-dense in $(U, \mathcal{N}, I)$, then each subset of X containing A is ξ-nI-open.*

PROOF. Let B be any subset of U such that $A \subseteq B$. Since A is $n\star$-dense, $n\text{-}cl^\star(A) = U$. Hence $A \subseteq B \subseteq U = n\text{-}cl^\star(A)$. Also A is $\xi\text{-}nI$-open implies B is $\xi\text{-}nI$-open by Proposition 3.5. □

PROPOSITION 3.6. *In a space $(U, \mathcal{N}, I)$, each $n\star$-closed subset is a $t\text{-}nI$-set.*

PROOF. Let A be $n\star$-closed. Then $A = n\text{-}cl^\star(A)$ and $n\text{-}int(n\text{-}cl^\star(A)) = n\text{-}int(A)$ which proves that A is a $t\text{-}nI$-set. □

REMARK 3.5. The converse of Proposition 3.6 is not true as shown in the following Example.

EXAMPLE 3.5. In Example 3.1, $A = \{p\}$ is a t-nI-set for
$n\text{-}int(A) = n\text{-}int(n\text{-}cl^\star(A)) = \{p\}$.
But A is not $n\star$-closed for $n\text{-}cl^\star(A) = \{p, s\} \neq \{p\} = A$.

THEOREM 3.3. *A subset A of a space $(U, \mathcal{N}, I)$ is semi*-nI-closed if and only if A is a $t\text{-}nI$-set.*

PROOF. A is semi*-nI-closed in $U \Leftrightarrow U - A$ is semi*-nI-open $\Leftrightarrow n\text{-}cl(U - A) = n\text{-}cl(n\text{-}int^\star(U - A))$ by Proposition 3.2 $\Leftrightarrow U - n\text{-}int(A) = U - n\text{-}int(n\text{-}cl^\star(A)) \Leftrightarrow n\text{-}int(A) = n\text{-}int(n\text{-}cl^\star(A)) \Leftrightarrow$ A is a $t\text{-}nI$-set. □

PROPOSITION 3.7. *If A and B are $t\text{-}nI$-sets of a space $(U, \mathcal{N}, I)$, then $A \cap B$ is a $t\text{-}nI$-set.*

PROOF. Let A and B be $t\text{-}nI$-sets. Then $n\text{-}int(A \cap B) \subseteq n\text{-}int(n\text{-}cl^\star(A \cap B)) \subseteq n\text{-}int(n\text{-}cl^\star(A) \cap n\text{-}cl^\star(B)) = n\text{-}int(n\text{-}cl^\star(A)) \cap n\text{-}int(n\text{-}cl^\star(B)) = n\text{-}int(A) \cap n\text{-}int(B) = n\text{-}int(A \cap B)$. Thus $n\text{-}int(A \cap B) = n\text{-}int(n\text{-}cl^\star(A \cap B))$ and hence $A \cap B$ is a $t\text{-}nI$-set. □

4. On $\mathcal{Q}$-nI-closed sets

DEFINITION 4.1. A subset A of a space $(U, \mathcal{N}, I)$ is called nano $\mathcal{Q}$-I-closed (briefly $\mathcal{Q}$-nI-closed) if $A = n\text{-}cl^\star(n\text{-}int(A))$.

The complement of a $\mathcal{Q}$-nI-closed set is said to be $\mathcal{Q}$-nI-open.

EXAMPLE 4.1. In Example 3.1, $A = \{q, r, s\}$ is $\mathcal{Q}$-nI-closed for
$n\text{-}cl^\star(n\text{-}int(A)) = n\text{-}cl^\star(\{q, r\}) = \{q, r, s\} = A$.

THEOREM 4.1. *For a subset $A(\neq \phi)$ of a space $(U, \mathcal{N}, I)$, the following properties are equivalent.*

(1) A is $\mathcal{Q}$-nI-closed.

(2) There exists a non-empty n-open set G such that $G \subseteq A = n\text{-}cl^\star(G)$.

(3) There exists a non-empty n-open set G such that $A = G \cup (n\text{-}cl^\star(G) - G)$.

PROOF. (1) $\Rightarrow$ (2): Suppose $A(\neq \phi)$ is $\mathcal{Q}$-nI-closed set. Then
$A = n\text{-}cl^\star(n\text{-}int(A))$.
Let $G = n\text{-}int(A)$. If $G = \phi$ then $A = n\text{-}cl^\star(\phi) = \phi$ which is a contradiction for $A \neq \phi$. Thus $G \neq \phi$ and G is the required non empty n-open set such that $G \subseteq A = n\text{-}cl^\star(G)$.

(2) ⇒ (3): Since $A = n\text{-}cl^\star(G) = G \cup (n\text{-}cl^\star(G) - G)$ where G is a nonempty n-open set, (3) follows.

(3) ⇒ (1): $A = G \cup (n\text{-}cl^\star(G) - G)$ implies that

$A = n\text{-}cl^\star(G) = n\text{-}cl^\star(n\text{-}int(G)) \subseteq n\text{-}cl^\star(n\text{-}int(A))$,

since G is n-open and $G \subseteq A$. Also $n\text{-}cl^\star(n\text{-}int(A)) \subseteq n\text{-}cl^\star(A) = n\text{-}cl^\star(G) = A$. Therefore $A = n\text{-}cl^\star(n\text{-}int(A))$ which implies that A is $\mathcal{Q}$-nI-closed. □

THEOREM 4.2. *For each β-nI-open subset A of U in a space $(U, \mathcal{N}, I)$, $n\text{-}cl^\star(A)$ is $\mathcal{Q}$-nI-closed.*

PROOF. Suppose A is β-nI-open. Then $A \subseteq n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A)))$ and so $n\text{-}cl^\star(A) \subseteq n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A))) \subseteq n\text{-}cl^\star(A)$ which implies that

$n\text{-}cl^\star(A) = n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A)))$.

Therefore $n\text{-}cl^\star(A)$ is $\mathcal{Q}$-nI-closed. □

THEOREM 4.3. *For a subset A of a space $(U, \mathcal{N}, I)$, the following properties are equivalent.*

(1) *A is $\mathcal{Q}$-nI-closed.*
(2) *A is semi-nI-open and $n\star$-closed.*
(3) *A is β-nI-open and $n\star$-closed.*

PROOF. (1) ⇒ (2): If A is $\mathcal{Q}$-nI-closed, then $A = n\text{-}cl^\star(n\text{-}int(A))$. Since $A \subseteq n\text{-}cl^\star(n\text{-}int(A))$, A is semi-nI-open. Also, $A \subseteq n\text{-}cl^\star(n\text{-}int(A))$ implies $n\text{-}cl^\star(A) \subseteq A$ and so $A = n\text{-}cl^\star(A)$ and thus A is $n\star$-closed.

(2) ⇒ (3): It follows from the fact that each semi-nI-open set is β-nI-open by Remark 2.1.

(3) ⇒ (1): Suppose A is β-nI-open and $n\star$-closed. Then

$A \subseteq n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(A)))$

and $A = n\text{-}cl^\star(A)$. Now $n\text{-}cl^\star(n\text{-}int(A)) \subseteq n\text{-}cl^\star(A) = A$. Also,

$A \subseteq n\text{-}cl^\star(n\text{-}int(A))$.

Therefore $A = n\text{-}cl^\star(n\text{-}int(A))$ which implies that A is $\mathcal{Q}$-nI-closed. □

REMARK 4.1. In a space $(U, \mathcal{N}, I)$,

(1) the family of semi-nI-open sets and the family of $n\star$-closed sets are independent.
(2) the family of β-nI-open sets and the family of $n\star$-closed sets are independent.

EXAMPLE 4.2. In Example 3.1,

(1) $A = \{p\}$ is n-open and hence β-nI-open by Remark 2.1. But $n\text{-}cl^\star(A) = \{p, s\} \neq \{p\} = A$ and thus A is not $n\star$-closed.
(2) $B = \{r\}$ is $n\star$-closed for $B_n^\star = \phi$. But B is not β-nI-open for $B = \{r\} \nsubseteq \phi = n\text{-}cl^\star(n\text{-}int(n\text{-}cl^\star(B)))$.
(3) $A = \{p\}$ is n-open and hence semi-nI-open by Remark 2.1. But A is not $n\star$-closed by(1).
(4) $B = \{r\}$ is $n\star$-closed by(2). But B is not β-nI-open by(2) and hence by Remark 2.1 B is not semi-nI-open.

(1) and (2) of Example 4.2 verify (2) of Remark 4.1.
(3) and (4) of Example 4.2 verify (1) of Remark 4.1.

Definition 4.2. A space $(U, \mathcal{N}, I)$ is called nano I-submaximal (briefly nI-submaximal) if each $n\star$-dense subset of U is n-open.

Proposition 4.1. *Each n-submaximal space is nI-submaximal for any ideal I.*

Proof. Let $(U, \mathcal{N})$ be n-submaximal and A be a $n\star$-dense subset in a space $(U, \mathcal{N}, I)$, for any ideal I. Then $n\text{-}cl^\star(A) = U$. But $U = n\text{-}cl^\star(A) \subseteq n\text{-}cl(A)$ implies $n\text{-}cl(A) = U$. Thus A is n-dense in U and $(U, \mathcal{N})$ is submaximal implies A is n-open in U. This shows that $(U, \mathcal{N}, I)$ is nI-submaximal. □

Remark 4.2. The converse of Proposition 4.1 is not true as shown in the following Example.

Example 4.3. Let $U = \{e_1, e_2, e_3\}$ with $U/R = \{\{e_1\}, \{e_2, e_3\}\}$ and $X = \{e_2, e_3\}$. Then $\mathcal{N} = \{\phi, \{e_2, e_3\}, U\}$. Let the ideal be $I = \wp(U)$.

In the space $(U, \mathcal{N}, I)$ if A is any $n\star$-dense subset, then $n\text{-}cl^\star(A) = U$ which implies $A \cup A_n^\star = U$. Since $I = \wp(U)$, $A_n^\star = \phi$ and so $A = U$ which is n-open. Thus each $n\star$-dense subset of U is n-open which means that $(U, \mathcal{N}, I)$ is nI-submaximal. On the other hand $B = \{e_1, e_2\}$ is n-dense in U for $n\text{-}cl(B) = U$ whereas B is not n-open. Hence $(U, \mathcal{N})$ is not n-submaximal.

Definition 4.3. A subset A of U in a nanotopological space $(U, \mathcal{N})$ is called nano-codense (briefly n-codense) if $U - A$ is n-dense.

Definition 4.4. A subset A of U in a space $(U, \mathcal{N}, I)$ is called nano $\star$-codense (briefly $n\star$-codense) if $U - A$ is $n\star$-dense.

Proposition 4.2. *Each $n\star$-codense subset in a space $(U, \mathcal{N}, I)$ is n-codense $(U, \mathcal{N})$.*

Proof. Let A be $n\star$-codense in $(U, \mathcal{N}, I)$. Then $U - A$ is $\star$-dense in U. Therefore $n\text{-}cl^\star(U - A) = U$ and $U = n\text{-}cl^\star(U - A) \subseteq n\text{-}cl(U - A)$ which implies $n\text{-}cl(U - A) = U$. Thus $U - A$ is n-dense in U and so A is n-codense in U. □

Remark 4.3. The converse of Proposition 4.2 is not true as illustrated in the following Example.

Example 4.4. In Example 4.3, $C = \{e_3\}$ is n-codense in U for $U - C = \{e_1, e_2\}$ is n-dense in U. But $C = \{e_3\}$ is not $n\star$-codense for $U - C = \{e_1, e_2\}$ is not $n\star$-dense in U with $n\text{-}cl^\star(U - C) = n\text{-}cl^\star(\{e_1, e_2\}) = \{e_1, e_2\} \neq U$.

Theorem 4.4. *For a space $(U, \mathcal{N}, I)$, the following are equivalent.*

(1) *U is nI-submaximal,*
(2) *each $n\star$-codense subset of U is n-closed.*

Proof. U is nI-submaximal $\Leftrightarrow$ each $n\star$-dense subset of U is n-open $\Leftrightarrow$ each $n\star$-codense subset of U is n-closed since a subset A is $n\star$-codense in U if and only if $U - A$ is $n\star$-dense in U. □

5. Decompositions

(1) From Theorem 3.1 and Remark 3.3, in a space $(U, \mathcal{N}, I)$ we get a decomposition of a semi-nI-open subset into a β-nI-open subset and a ξ-nI-open subset.

(2) From Theorem 3.2 and Remark 3.4, in a space $(U, \mathcal{N}, I)$ we obtain a decomposition of an α-nI-open subset into a pre-nI-open subset and a ξ-nI-open subset.

(3) By Theorem 4.3 and (2) of Remark 4.1, we obtain a decomposition of a $\mathcal{Q}$-nI-closed subset of a space $(U, \mathcal{N}, I)$ into a β-nI-open subset and a $n\star$-closed subset.

(4) By Theorem 4.3 and (1) of Remark 4.1, we get a decomposition of a $\mathcal{Q}$-nI-closed subset of a space $(U, \mathcal{N}, I)$ into a semi-nI-open subset and a $n\star$-closed subset.

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