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STABILITY OF POSITIVE ALMOST PERIODIC SOLUTIONS FOR A FISHING MODEL WITH MULTIPLE TIME VARYING VARIABLE DELAYS ON TIME SCALES

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Abstract. In this paper we consider fishing model with a multiple time varying variable delays. Under appropriate conditions, we establish a criterion for the existence of positive almost periodic solutions by applying the contraction mapping principle. We also investigate global exponential stability of the positive almost periodic solution of the system.

1. Introduction

In the study of population dynamics of fisheries, the following differential equation

$$\frac{u'}{u} = f(t,u) - g(t,u) - h(t) \tag{1.1}$$

is widely used [**1**, **6**–**8**], where $u(t)$ denotes the population biomass, $f(t,u)$ denotes the per capita fecundity rate, $g(t,u)$ denotes the per capita mortality rate, and $h(t)$ is per-capita harvesting rate of the species.

Taking account of the delay and the varying environments, Berezansky and Idels [**2**] proposed the following time-lag model based on (1.1),

$$u'(t) = u(t)\Big[\frac{a(t)}{1+\big(\frac{u(t-\tau(t))}{K(t)}\big)^{\gamma}} - b(t)\Big], \tag{1.2}$$

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where $r, b, K, \tau : \mathbb{R} \to (0, +\infty)$ are almost periodic functions and parameter $\gamma > 0$ and studied the local and global stability of the periodic solutions of equation (1.2). Later, some more results on the stability and existence of periodic solutions for (1.2) were established in Wang [**15**]. In [**16**], Zhang, Gong and Shao developed a new criterion to obtain a condition for the global exponential stability of the positive almost periodic solutions of (1.2) and unveiled exponential convergent rate.

Many authors believe that the discrete time model governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Consequently, the studies of dynamic systems governed by difference equations have received great attention from researchers [**4**, **9**–**11**, **13**, **14**].

In [**18**], Zhang, Li and Ye considered the discrete fishing model with feedback control,

$$u(n+1) = u(n) \exp\Big[\frac{a(t)}{1 + \big(\frac{u(n)}{K(n)}\big)^{\gamma}} - b(n) - c(n)v(n)\Big]$$
$$\Delta v(n) = -\alpha(n)v(n) + \beta(n)u(n).$$

They derived sufficient conditions for the persistence of the system, and also investigated the existence and uniformly asymptotical stability of an almost periodic solution of the system.

The study of dynamical systems on time scales is now an active area of research. This study reveals that the existence of positive periodic solutions of population models, it is not worthwhile to establish results for differential equations and again for difference equations separately. One can unify such problems in the frame of dynamic equations on time scales.

Motivated by the aforementioned facts, in this paper, we consider the generalized fishing model with N time varying variable delays on time scales,

$$u^{\Delta}(t) = -b(t)u(t) + \sum_{r=1}^{N} \frac{a_r(t)}{1 + \Big(\frac{u(t-\tau_r(t))}{K(t)}\Big)^{\gamma_r}}, \; t \in \mathbb{T}, \tag{1.3}$$

where $\mathbb{T}$ is a time scale, $b, a_r, K, \tau_r : \mathbb{T} \to (0, +\infty)$ are almost periodic functions and parameter $\gamma_r > 0$. Under appropriate conditions, we establish a criterion for the existence of positive almost periodic solutions of (1.3) by virtue of the contraction mapping principle and then we investigate global exponential stability of the positive almost periodic solution of (1.3).

For convenience, we introduce few notations and assumptions:

(H1) For any bounded function $f(t)$, we denote $f^U = \sup\limits_{t\in\mathbb{T}} f(t), f^L = \inf\limits_{t\in\mathbb{T}} f(t)$.

(H2) We assume that the bounded almost periodic functions $b(t)$, $K(s)$, $a_r(t)$, $\tau_r(t)$ satisfy

$$0 < b^L \leqslant b(t) \leqslant b^U,\ 0 < K^L \leqslant K(t) \leqslant K^U,\ 0 < a_r^L \leqslant a_r(t) \leqslant a_r^U,$$
$$0 < \tau_r^L \leqslant \tau_r(t) \leqslant \tau_r^U$$

for $r = 1, 2, 3, \cdots, N$ and $-b(t) \in \mathcal{R}^+$ where $\mathcal{R}^+$ is the set of all positively regressive functions and rd-continuous functions.

(H3) Due to biological realistic of the model (1.3), positive solutions are only meaningful. So, we restrict our attention to positive solutions of equation (1.3).

(H4) The initial functions associated with equation (1.3) is given by

$$u(t;\varphi) = \varphi(t) \text{ for } t \in [-\tau^*, 0]_{\mathbb{T}},\ \tau^* = \max_{1 \leqslant r \leqslant N}\{\tau_r^U\}$$

where $\varphi(\cdot)$ denotes a real-valued bounded rd-continuous function defined on $[-\tau^*, 0]_{\mathbb{T}}$.

The rest of the paper is organized as follows. In Section 2, we present some notations, definitions and lemmas which are useful to establish our main results. Sufficient conditions for the existence of unique positive almost periodic solution of system (1.3) are established in Section 3 and in Section 4 we discussed global exponential stability of unique positive almost periodic solution of the system (1.3). Finally in Section 5, an example is given to illustrate our results.

2. Preliminaries

In this section, we introduce some definitions and state some preliminary results which are useful in the sequel.

Definition 2.1. ([**3**]) A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$,

$$\sigma(t) = \inf\{r \in \mathbb{T} : r > t\}, \quad \rho(t) = \sup\{r \in \mathbb{T} : r < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has left-scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise let $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2. ([**3**]) By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Similarly other intervals can be defined.

Definition 2.3. ([**3**]) A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.4. ([**3**]) A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$ If p is regressive function, then the generalized exponential function e_p is defined by

$$e_p(t, s) = \exp\Big\{\int_s^t \xi_{\mu(x)}(p(x))\Delta x\Big\}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{Log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Definition 2.5. ([**3**]) A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$; $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive provided $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^k$ The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathcal{R})$ and the set of all positively regressive functions and rd-continuous functions will be denoted by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathcal{R})$.

Lemma 2.1 ([**3**]). *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions; then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;

(ii) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;

(iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$;

(iv) $(e_p(t, s))^\Delta = p(t)e_p(t, s)$.

Lemma 2.2 ([**3**]). *Suppose that $p \in \mathcal{R}^+$, then*

(i) $e_p(t, s) > 0$ *for all* $t, s \in \mathbb{T}$;

(ii) *if* $p(t) \leqslant q(t)$ *for all* $t \geqslant s$, $t, s \in \mathbb{T}$, *then* $e_p(t, s) \leqslant e_q(t, s)$ *for all* $t \geqslant s$.

Lemma 2.3 ([**3**]). *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c, \cdot)]^\Delta = -p[e_p(c, \cdot)]^\sigma,$$

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Lemma 2.4 ([**3**]). *Let $p : \mathbb{T} \to \mathbb{R}$ be right-dense continuous and regressive, $a \in \mathbb{T}$ and $u_a \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$u^\Delta(t) = p(t)u(t) + f(t),$$
$$u(a) = u_a$$

is given by

$$u(t) = e_r(t, a)u_a + \int_a^t e_r(t, \sigma(s))f(s)\Delta s.$$

Definition 2.6. ([**12**]) A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t + \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.7. ([**12**]) Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{R})$ is said to be almost periodic on $\mathbb{T}$, if, for any $\varepsilon > 0$, the set

$$E(\varepsilon, f) = \{\tau \in \Pi : |f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$; that is, for any $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one $\tau \in E(\varepsilon, f)$ such that

$$|f(t + \tau) - f(t)| < \varepsilon, \ \forall t \in \mathbb{T}.$$

The set $E(\varepsilon, f)$ is called the $\varepsilon-$translation number of $f(t)$. We denote the set of all such functions by $AP(\mathbb{T})$.

Lemma 2.5 ([**12**]). *If $f \in C(\mathbb{T}, \mathbb{R})$ is an almost periodic function, then f is bounded on $\mathbb{T}$.*

Lemma 2.6 ([**12**]). *If $f, g \in C(\mathbb{T}, \mathbb{R})$ are almost periodic functions, then $f + g$, fg are also almost periodic.*

Lemma 2.7 ([**12**]). *If $f \in C(\mathbb{T}, \mathbb{R})$ is almost periodic, then $F(t) = \int_0^t f(s)\Delta s$ is almost periodic if and only if $F(t)$ is bounded.*

Lemma 2.8 ([**12**]). *If $f \in C(\mathbb{T}, \mathbb{R})$ is almost periodic and $F(\cdot)$ is uniformly continuous on the value field of $f(t)$, then $F \circ f$ is almost periodic.*

Definition 2.8. ([**17**]) Let $x \in \mathbb{R}^m$ and $A(t)$ be an $m \times m$ rd-continuous matrix on $\mathbb{T}$; the linear system

$$x^{\Delta}(t) = A(t)x(t), \ t \in \mathbb{T}, \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants k, α, projection P, and the fundamental solution matrix $x(t)$ of (2.1) satisfying

$$|x(t)Px^{-1}(\sigma(s))|_0 \leqslant ke_{\ominus\alpha}(t, \sigma(s)), \ s, t \in \mathbb{T}, \ t \geqslant s,$$

$$|x(t)(I - P)x^{-1}(\sigma(s))|_0 \leqslant ke_{\ominus\alpha}(\sigma(s), t), \ s, t \in \mathbb{T}, \ t \leqslant s,$$

where $|\cdot|_0$ is a matrix norm on $\mathbb{T}$; that is, if $A = (a_{ij})_{m\times m}$, then we can take $|A|_0 = (\sum_{i=1}^m \sum_{j=1}^m |a_{ij}|^2)^{1/2}$.

Lemma 2.9 ([**12**]). *If the linear system* (2.1) *admits an exponential dichotomy, then the following system $x^{\Delta}(t) = A(t)x(t) + f(t)$, $t \in \mathbb{T}$, has a solution as follows:*

$$x(t) = \int_{-\infty}^{t} x(t)Px^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} x(t)(I - P)x^{-1}(\sigma(s))f(s)\Delta s,$$

where $x(t)$ is the fundamental solution matrix of (2.1).

Lemma 2.10 ([**12**]). *Let $A(t)$ be a regressive $n \times n$ matrix-valued function on $\mathbb{T}$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$, then the initial value problem*

$$x^{\Delta}(t) = A(t)x(t), \ x(t_0) = x_0$$

has a unique solution $x(t)$ as follows

$$x(t) = e_A(t, t_0)x_0.$$

Lemma 2.11 ([**12**]). *Let $d_i(t) > 0$ be a function on $\mathbb{T}$ such that $-d_i(t) \in \mathcal{R}^+$ for all $t \in \mathbb{T}$ and $\min\limits_{1\leqslant i\leqslant m} \left\{ \inf\limits_{t\in\mathbb{T}} d_i(t) \right\} > 0$. Then the linear system*

$$x^{\Delta}(t) = diag\big(-d_1(t), -d_2(t), \cdots, -d_m(t)\big)x(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

3. Existence of the unique positive almost periodic solution

Let $X = \{u(t) : u \in C(\mathbb{T},\mathbb{R}),\ u(t)$ is almost periodic function $\}$ with norm $\|u\| = \sup_{t\in\mathbb{T}} |u(t)|$. Then X is Banach space.

For $w \in X$, consider the equation

$$u^{\Delta}(t) = -b(t)u(t) + \sum_{r=1}^{N} \frac{a_r(t)}{1 + \left(\frac{w(t-\tau_r(t))}{K(t)}\right)^{\gamma_r}}. \tag{3.1}$$

Since $\inf_{t\in\mathbb{T}} b(t) = b^L > 0$, then from Lemma 2.11 the linear equation $u^{\Delta}(t) = -b(t)u(t)$ admits exponential dichotomy on $\mathbb{T}$.

Hence, by Lemma 2.9, the equation (3.1) has exactly one almost periodic solution,

$$u_\omega(t) = \int_{-\infty}^{t} e_{-b}(t,\sigma(s)) \sum_{r=1}^{N} \frac{a_r(s)}{1 + \left(\frac{\omega(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} \Delta s.$$

Define the operator $F : X \to X$,

$$(F\omega)(t) = \int_{-\infty}^{t} e_{-b}(t,\sigma(s)) \sum_{r=1}^{N} \frac{a_r(s)}{1 + \left(\frac{\omega(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} \Delta s.$$

It is clear that, $\omega(t)$ is the almost periodic solution of equation (1.3) if and only if ω is the fixed point of the operator F.

We assume the following:

(H5) There exist two positive constants $M > m > 0$ such that

$$\frac{1}{b^L} \sum_{r=1}^{N} a_r^U \leqslant M \quad \text{and} \quad m \leqslant \frac{1}{b^U} \sum_{r=1}^{N} \frac{a_r^L}{1 + \left(\frac{M}{K^L}\right)^{\gamma_r}}.$$

THEOREM 3.1. *Assume that conditions (H5), $\gamma_r \geqslant 1$, $(r = 1, 2, 3, \cdots, N)$ and*

$$\sum_{r=1}^{N} a_r^U \gamma_r < b^L K^L$$

are satisfied, then equation (1.3) *has a unique almost periodic positive solution.*

PROOF. Firstly, we prove that F is self map on Ω, where

$$\Omega = \{w(t) \in X : m \leqslant w(t) \leqslant M,\ t \in \mathbb{T}\}.$$

Let $w \in \Omega$. Then,

$$\begin{aligned} (Fw)(t) &= \int_{-\infty}^{t} e_{-b}(t,\sigma(s)) \sum_{r=1}^{N} \frac{a_r(s)}{1 + \left(\frac{w(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} \Delta s \\ &\leqslant \int_{-\infty}^{t} e_{-b^L}(t,\sigma(s)) \sum_{r=1}^{N} \frac{a_r^U}{1 + \left(\frac{w(s-\tau_r(s))}{K^U}\right)^{\gamma_r}} \Delta s. \end{aligned} \tag{3.2}$$

The function

$$f_i(x) = \frac{1}{1+\left(\frac{x}{K^U}\right)^{r_i}},\ r_i > 0$$

is nonincreasing on $[0,+\infty)$. So $f_{i\max}(x) = 1$. It follows from (3.2) that

$$\begin{aligned}(F\omega)(t) &\leqslant \sum_{r=1}^{N} a_r^U \int_{-\infty}^{t} e_{-b^L}(t,\sigma(s))\Delta s \\ &\leqslant \frac{1}{b^L}\sum_{r=1}^{N} a_r^U \leqslant M.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}(F\omega)(t) &= \int_{-\infty}^{t} e_{-b}(t,\sigma(s))\sum_{r=1}^{N}\frac{a_r(s)}{1+\left(\frac{\omega(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}}\Delta s \\ &\geqslant \int_{-\infty}^{t} e_{-b^U}(t,\sigma(s))\sum_{r=1}^{N}\frac{a_r^L}{1+\left(\frac{\omega(s-\tau_r(s))}{K^L}\right)^{\gamma_r}}\Delta s.\end{aligned} \tag{3.4}$$

Since the function

$$f_i(x) = \frac{1}{1+\left(\frac{x}{K^U}\right)^{r_i}},\ r_i > 0$$

is nonincreasing on $[0,+\infty)$ and $m \leqslant \omega(t) \leqslant M$, we have $f_i(\omega(t)) \geqslant f_i(M)$ for $t \in \mathbb{T}$. Thus by (3.4) we obtain

$$\begin{aligned}(F\omega)(t) &\geqslant \int_{-\infty}^{t} e_{-b^U}(t,\sigma(s))\sum_{r=1}^{N}\frac{a_r^L}{1+\left(\frac{M}{K^L}\right)^{\gamma_r}}\Delta s \\ &\geqslant \frac{1}{b^U}\sum_{r=1}^{N}\frac{a_r^L}{1+\left(\frac{M}{K^L}\right)^{\gamma_r}} \geqslant m.\end{aligned} \tag{3.5}$$

Hence from (3.3) and (3.5), we have

$$m \leqslant (F\omega)(t) \leqslant M. \tag{3.6}$$

Next, by Lemma 2.4 for every $\omega \in \Omega$, the equation (3.1) has exactly one almost periodic solution

$$u_\omega(t) = \int_{-\infty}^{t} e_{-b}(t,\sigma(s))\sum_{r=1}^{N}\frac{a_r(s)}{1+\left(\frac{\omega(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}}\Delta s.$$

Since $u_\omega(t)$ is almost periodic, then $(F\omega)(t)$ is almost periodic. This, together with (3.6), implies $F\omega \in \Omega$. So we have $F\Omega \subset \Omega$. Finally, we prove that F is a contraction mapping on Ω. For $u, w \in \Omega$, consider

$$
\begin{aligned}
\|Fu - Fw\| &= \sup_{t\in\mathbb{T}} |(Fu)(t) - (Fw)(t)| \\
&\leqslant \sup_{t\in\mathbb{T}} \int_{-\infty}^{t} e_{-b}(t,\sigma(s)) \sum_{r=1}^{N} a_r^U \times \\
&\qquad \left| \frac{1}{1+\left(\frac{u(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} - \frac{1}{1+\left(\frac{w(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} \right| \Delta s
\end{aligned} \tag{3.7}
$$

By mean value theorem, we have

$$
\begin{aligned}
&\left| \frac{1}{1+\left(\frac{u(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} - \frac{1}{1+\left(\frac{w(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} \right| \\
&\qquad = \left(\frac{\frac{\gamma_r}{K(s)}\left(\frac{\xi}{K(s)}\right)^{\gamma_r-1}}{(1+\frac{\xi}{K(s)})^2} \right) |u(s-\tau_r(s)) - w(s-\tau_r(s))|
\end{aligned} \tag{3.8}
$$

where ξ lies between $u(s-\tau_r(s))$ and $w(s-\tau_r(s))$.

Note that the function

$$
f_r(x) = \frac{\frac{\gamma_r}{K(s)}\left(\frac{x}{K(s)}\right)^{\gamma_r-1}}{(1+\frac{x}{K(s)})^2} < \frac{\gamma_r}{K(s)} < \frac{\gamma_r}{K^L}
$$

for $x \in (0,\infty)$ and $\gamma_r \geqslant 1, (r = 1,2,3,\cdots,N)$. Thus, we have

$$
\frac{\frac{\gamma_r}{K(s)}\left(\frac{\xi}{K(s)}\right)^{\gamma_r-1}}{(1+\frac{\xi}{K(s)})^2} < \frac{\gamma_r}{K^L} \quad \text{for} \quad \gamma_r \geqslant 1.
$$

It follows from (3.8) that

$$
\left| \frac{1}{1+\left(\frac{u(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} - \frac{1}{1+\left(\frac{w(s-\tau_r(s))}{K(s)}\right)^{\gamma_r}} \right| \leqslant \frac{\gamma_r}{K^L} |u(s-\tau_r(s)) - w(s-\tau_r(s))|. \tag{3.9}
$$

Hence, from (3.7) and (3.9), we get

$$\begin{aligned}\|Fu - Fw\| &\leqslant \sup_{t\in\mathbb{T}}\left\{\frac{1}{K^L}\int_{-\infty}^{t} e_{-b}(t,\sigma(s))\times \right.\\ &\qquad\left.\sum_{r=1}^{N} a_r^U\gamma_r|u(s-\tau_r(s)) - w(s-\tau_r(s))|\Delta s\right\}\\ &\leqslant \sup_{t\in\mathbb{T}}\left\{\frac{1}{K^L}\int_{-\infty}^{t} e_{-b^L}(t,\sigma(s))\times \right.\\ &\qquad\left.\sum_{r=1}^{N} a_r^U\gamma_r|u(s-\tau_r(s)) - w(s-\tau_r(s))|\Delta s\right\}\\ &\leqslant \sup_{t\in\mathbb{T}}\left\{\frac{1}{K^L}\int_{-\infty}^{t} e_{-b^L}(t,\sigma(s))\sum_{r=1}^{N} a_r^U\gamma_r\|u-w\|\Delta s\right\}\\ &= \sup_{t\in\mathbb{T}}\left\{\frac{1}{K^L}\sum_{r=1}^{N} a_r^U\gamma_r\|u-w\|\int_{-\infty}^{t} e_{-b^L}(t,\sigma(s))\Delta s\right\}\\ &= \sup_{t\in\mathbb{T}}\left\{\frac{1}{K^L}\sum_{r=1}^{N} a_r^U\gamma_r\|u-w\|\frac{1}{b^L}\right\}\\ &= \frac{1}{b^L K^L}\sum_{r=1}^{N} a_r^U\gamma_r\|u-w\|.\end{aligned}$$

Since $\dfrac{1}{b^L K^L}\displaystyle\sum_{r=1}^{N} a_r^U\gamma_r < 1$, it follows that F is a contraction mapping. Thus, by the contraction mapping fixed point theorem, the operator F has a unique fixed point w^* in Ω. This implies that the equation (1.3) has a unique almost periodic positive solution $w^*(t)$ and $m \leqslant w^*(t) \leqslant M$. □

4. Stability analysis

In this section, we investigate the global exponential stability of the positive almost periodic solution of (1.3). We recall the Gronwall inequality on time scales, which can be found in [**3**].

THEOREM 4.1 (Gronwall inequality). *Let $y \in C_{rd}, p \in \mathcal{R}^+, p \geqslant 0$ and $k \in \mathbb{R}$. Then*

$$y(t) \leqslant k + \int_{t_0}^{t} y(s)p(s)ds \text{ for all } t \in \mathbb{T}$$

implies

$$y(t) \leqslant k e_p(t,t_0) \text{ for all } t \in \mathbb{T}.$$

THEOREM 4.2. *Assume that conditions (H5), $\gamma_r \geqslant 1$, $(r = 1,2,3,\cdots,N)$,*

$$\sum_{r=1}^{N} a_r^U\gamma_r < b^L K^L \quad \text{and} \quad b^L > \frac{1}{K^L}\sum_{r=1}^{N}\frac{a_r^U\gamma_r}{1-\mu(s)b(s)} = \delta$$

are satisfied. Then equation (1.3) *has a unique globally exponentially stable almost periodic positive solution.*

PROOF. From Theorem 3.1, equation (1.3) has a unique almost periodic positive solution $w^*(t)$ such that $m \leqslant w^*(t) \leqslant M$. Let ψ be the initial function of $w^*(t), w^*(t;\psi) = \psi(t)$ for $t \in [-\tau^*, 0]_{\mathbb{T}}$. Now we prove that w^* is globally exponentially stable.

Suppose that $u(t)$ is an arbitrary positive solution of equation (1.3) with the initial function $u(t;\varphi) = \varphi(t) > 0$, $t \in [-\tau^*, 0]_{\mathbb{T}}$. Let $v(t) = u(t) - w^*(t)$, then we have

$$\begin{aligned} v^{\Delta}(t) =& (u(t) - w^*(t))^{\Delta} \\ =& - b(t)(u(t) - w^*(t)) + \sum_{r=1}^{N} a_r(t)\Big[\frac{1}{1 + \left(\frac{u(t-\tau_k(t))}{K(s)}\right)^{\gamma_r}} \\ &- \frac{1}{1 + \left(\frac{w^*(t-\tau_k(t))}{K(s)}\right)^{\gamma_r}}\Big]. \end{aligned} \tag{4.1}$$

Let

$$p(t) = \sum_{r=1}^{N} a_r(t)\Big[\frac{1}{1 + \left(\frac{u(t-\tau_k(t))}{K(s)}\right)^{\gamma_r}} - \frac{1}{1 + \left(\frac{w^*(t-\tau_k(t))}{K(s)}\right)^{\gamma_r}}\Big],$$

then it follows from (4.1) that

$$v^{\Delta}(t) = -b(t)v(t) + p(t).$$

From Lemma 2.4, $v(t)$ can be expressed as follows

$$\begin{aligned} v(t) &= e_{-b}(t,t_0)v(t_0) + \int_{t_0}^{t} e_{-b}(t,\sigma(s))p(s)\Delta s, \ (t \geqslant t_0), \ t_0 \in [-\tau^*, 0]_{\mathbb{T}} \\ &= e_{-b}(t,t_0)v(t_0) + \int_{t_0}^{t} \frac{1}{1-\mu(s)b(s)}\, e_{-b}(t,s)p(s)\Delta s. \end{aligned}$$

Using initial functions, we obtain

$$v(t) = e_{-b}(t,t_0)(\varphi(t_0) - \psi(t_0)) + \int_{t_0}^{t} \frac{1}{1-\mu(s)b(s)}\, e_{-b}(t,s)p(s)\Delta s. \tag{4.2}$$

Note that, by mean value theorem

$$\begin{aligned} |p(t)| \leqslant& \sum_{r=1}^{N} a_r(t)\Big|\frac{1}{1 + \left(\frac{u(t-\tau_k(t))}{K(s)}\right)^{\gamma_r}} - \frac{1}{1 + \left(\frac{w^*(t-\tau_k(t))}{K(s)}\right)^{\gamma_r}}\Big| \\ \leqslant& \sum_{r=1}^{N} a_r^U \frac{\gamma_r}{K^L} |u(s - \tau_r(s)) - w^*(s - \tau_r(s))| \\ \leqslant& \frac{1}{K^L}\sum_{r=1}^{N} a_r^U \gamma_r \|u - w^*\|. \end{aligned} \tag{4.3}$$

So that

$$\|p(t)\| \leqslant \frac{1}{K^L}\sum_{r=1}^{N} a_r^U \gamma_r \|u - w^*\| = \frac{1}{K^L}\sum_{r=1}^{N} a_r^U \gamma_r \|v\|. \tag{4.4}$$

Taking norm on both sides of (4.2) and using (4.4), we have

$$\begin{aligned}\|v(t)\| &\leqslant e_{-b}(t,t_0)\|\varphi-\psi\| + \int_{t_0}^{t} \frac{e_{-b}(t,s)}{1-\mu(s)b(s)}\|p(s)\|\Delta s \\ &\leqslant e_{-b}(t,t_0)\|\varphi-\psi\| + \int_{t_0}^{t} \frac{e_{-b}(t,s)}{1-\mu(s)b(s)}\frac{1}{K^L}\sum_{r=1}^{N} a_r^U \gamma_r \|v\|\Delta s.\end{aligned} \tag{4.5}$$

From (4.5) and Lemma 2.1, we get

$$\frac{\|v(t)\|}{e_{-b}(t,t_0)} \leqslant \|\varphi-\psi\| + \int_{t_0}^{t} \frac{\|v\|}{e_{-b}(s,t_0)}\frac{1}{K^L}\sum_{r=1}^{N}\frac{a_r^U\gamma_r}{1-\mu(s)b(s)}\Delta s.$$

By Gronwall's inequality,

$$\frac{\|v(t)\|}{e_{-b}(t,t_0)} \leqslant \|\varphi-\psi\| e_\delta(t,t_0),$$

which implies that

$$\begin{aligned}\|v(t)\| &\leqslant \|\varphi-\psi\| e_\delta(t,t_0)e_{-b}(t,t_0)\\ &\leqslant \|\varphi-\psi\| e_\delta(t,t_0)e_{-b^L}(t,t_0)\\ &\leqslant \|\varphi-\psi\| e_{-(b^L-\delta)}(t,t_0).\end{aligned}$$

That is $\|u(t)-w^*(t)\| \leqslant \|\varphi-\psi\| e_{-(b^L-\delta)}(t,t_0)$, $b^L > \delta$, which means that $w^*(t)$ is globally exponentially stable. The proof is complete. □

5. Examples

In this section we give examples to illustrate our results.

Example 5.1. The first example we consider is the fishing model with multiple time varying variable delays described by

$$\begin{aligned}u^\Delta(t) = &-9u(t) + \left(3+\frac{1}{20}|\sin\sqrt{5}t|\right)\times\frac{1}{1+\left(\frac{u(t-\cos^2(t/40))}{3}\right)^2}\\ &+\left(2+\frac{1}{10}|\cos\sqrt{3}t|\right)\times\frac{1}{1+\left(\frac{u(t-\cos^2(t/40))}{3}\right)^2}.\end{aligned} \tag{5.1}$$

It is easy to see that all the assumptions of Theorem 3.1 are satisfied. Therefore, there exists a unique positive almost periodic solution for (5.1).

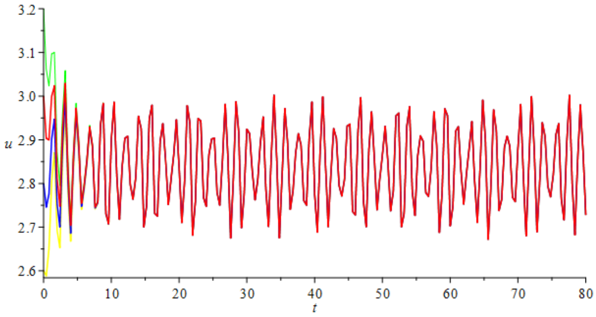

FIGURE 1. Numerical solution $u(t)$ of equation (5.2) for initial value $\varphi(s) = 2.6, 2.8, 3, 3.2$ $s \in [-3e, 0]$.

EXAMPLE 5.2. In this example we show global exponential stability of the following fishery model with time varying variable delays

$$\begin{aligned} u^{\Delta}(t) = & -0.6u(t) + \left(\frac{1}{4} + \frac{1}{2}|\cos\sqrt{5}t|\right) \times \frac{1}{1 + \left(\frac{u(t-3e^{\sin t})}{65}\right)^5} \\ & + \left(\frac{1}{2} + |\sin\sqrt{3}t|\right) \times \frac{1}{1 + \left(\frac{u(t-3e^{\sin t})}{65}\right)^5}. \end{aligned} \tag{5.2}$$

Here

$$b(t) = 0.6,\ K(t) = 65,\ \gamma_r = 5,\ a_1(t) = \frac{1}{4} + \frac{1}{2}|\cos\sqrt{5}t|,\ a_2(t) = \frac{1}{2} + |\sin\sqrt{3}t|.$$

On simple calculations, we get

$$a_1^L = \frac{1}{4},\ a_1^U = \frac{3}{4},\ a_2^L = \frac{1}{2},\ a_2^U = \frac{3}{2}.$$

Letting $m = 1$ and $M = 4$, then (C1), (C2) holds.
If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and hence

$$b^L = 0.6 > \delta = \frac{1}{K^L}\sum_{r=1}^{N}\frac{a_r^U\gamma_r}{1-\mu(s)b(s)} = 0.03462.$$

If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$ and hence

$$b^L = 0.6 > \delta = \frac{1}{K^L}\sum_{r=1}^{N}\frac{a_r^U\gamma_r}{1-\mu(s)b(s)} = 0.43270.$$

which implies that (5.2) satisfies the assumptions of Theorem 4.2. Therefore, equation (5.2) has a unique positive almost periodic solution $w^*(t)$, which is globally exponentially stable. The numerical simulations in Fig. 1 strongly support the conclusion.

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