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STRONG WEAK DOMINATION: A MATHEMATICAL PROGRAMMING STRATEGY

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Abstract. Let *G=(V,E)* be a graph. A subset $S \subseteq V$ of vertices is a dominating set if every vertex in *V-S* is adjacent to at least one vertex of *S*. The domination number is the minimum cardinality of a dominating set. Let $u, v \in V$. Then, u strongly dominates v and v weakly dominates u if (i) $uv \in E$ and (ii) $deg(u) \geqslant deg(v)$. A subset D of V is a strong (weak) dominating set of G if every vertex in *V-D* is strongly (weakly) dominated by at least one vertex in *D*. The strong (weak) domination number of G is the minimum cardinality of a strong (weak) dominating set. In this paper, mathematical models are developed for the problems of domination and strong (weak) domination of a graph. Then test problems are solved by the GAMS software, the optima and execution times are implemented. To the best of our knowledge, this is the first mathematical programming formulations for the problems, and computational results show that the proposed models are capable of finding optimal solutions within a reasonable amount of time.

1. Introduction

Graph theoretic techniques provide a convenient tool for the investigation of networks. It is well-known that an interconnection network can be modeled by a graph with vertices representing sites of the network and edges representing links between sites of the network. Therefore various problems in networks can be studied by graph theoretical methods. The study of domination in graphs is an important research area, perhaps also the fastest-growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real-world problems. For instance, dominating sets

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in graphs are natural models for facility location problems in operations research. Research on domination in graphs has not only important theoretical signification, but also varied application in such fields as computer science, communication networks, wireless and ad hoc networks, biological and social networks, distributed computing, coding theory, and web graphs.

Domination and its variations have been extensively studied [**3, 4, 1, 7, 8**]. In general, the concept of dominating sets in graph theory finds wide applications in different types of communication networks. A broadcast from a communication vertex is received by all its neighbors. This is captured by the notion of domination in a graph. The minimum dominating set of sites plays an important role in the network for it dominates the whole network with the minimum cost. A thorough study of domination appears in [**7, 8**]. In some sense, one could say that the domination based parameters reveal an underlying efficient and stable communication network. Among the domination-type parameters that have been studied, the strong and weak domination numbers are the fundamental ones. A set $D \subseteq V$ is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. A set $D \subseteq V$ is a *strong dominating* set if every vertex u not in D is adjacent to a vertex v in D where $deg(v) \geqslant deg(u)$. A set $D \subseteq V$ is a *weak dominating* set if every vertex u not in D is adjacent to a vertex v in D where $deg(v) \leqslant deg(u)$. The *domination number* of G (*strong domination number*, *weak domination number*), denoted $\gamma(G)$ ($\gamma_{st}(G)$, $\gamma_w(G)$, respectively) is the minimum size of a dominating set (strong dominating set, weak dominating set, respectively) of G. The concepts of strong and weak domination were introduced by Sampathkumar and Pushpa Latha in [**9**] by the following motivation. Consider a network of roads connecting a number of locations. In such a network, the degree of a vertex v is the number of roads meeting at v. Suppose $deg(u) \geqslant deg(v)$. Naturally, the traffic at u is heavier than that at v. If we consider the traffic between u and v, preference should be given to the vehicles going from u to v. Thus, in some sense, u strongly dominates v and v weakly dominates u. In [**5, 6**], it is shown that the problems of computing γ_{st} and γ_w are NP-hard. Since computing the (strong, weak, respectively) domination of a graph is NP-hard in general, it becomes an interesting question to implement the mathematical formulations for these problems. In the following section we will deal with this question.

In this paper, we consider finite undirected graphs without loops and multiple edges. The order of G is the number of vertices in G. The open neighborhood of v is $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. The degree of a vertex v is $d_v = |N(v)|$ [**2**].

The rest of this paper is organized as follows. Section 2 represents the model formulations and the computational results are discussed in Section 3.

2. Model formulations

The binary decision variables used in the presented models are as defined as follows:

$$x_i = \begin{cases} 1, & if\ vertex\ i\ belongs\ to\ minimum\ dominating\ set \\ 0, & otherwise \end{cases}$$

Followings are the parameters used in the presented mathematical models. n is the number of vertices of graph G, that is, $|V(G)| = n$. A is the adjacency matrix of G that is used to store the neighbors of each vertex but the problem is solved under the assumption that the adjacency matrix is modified by changing the all zero elements of main diagonal to one, that is $[a_{ij}] = 1$, in order to ensure that the vertex i should dominate itself by the way. The vector $d[n]$ is the vector of degrees of vertices, where

$$d_i = \sum_{j=1, j\neq i}^{n} a_{ij}.$$

2.1. Domination. Now, the domination problem can be mathematically formulated as follows:

$$min \sum_{i=1}^{n} x_i \tag{2.1}$$

s.t.

$$\sum_{j=1, a_{ij}=1, j\neq i}^{n} x_j \geqslant 1, \quad \forall i = 1, ..., n \tag{2.2}$$

$$x_i \in \{0, 1\}, \quad \forall i = 1, ..., n \tag{2.3}$$

In this formulation, the objective function (2.1) is clear and minimizes the total number of vertices included in a dominating set of minimum cardinality. Constraint (2.2) ensures that every vertex i of G is dominated by itself or at least by one of the vertices of its open neighborhood. Constraint (2.3) defines variables of model.

2.2. Strong domination. The mathematical model of strong domination problem can be formulated as follows:

$$min \sum_{i=1}^{n} x_i \tag{2.4}$$

s.t.

$$\sum_{j=1, a_{ij}=1, j\neq i}^{n} x_j \geqslant 1, \quad \forall i = 1, ..., n \tag{2.5}$$

$$(1 - x_i)d_i \leqslant max_{j, a_{ij}=1, j\neq i}\{x_j d_j\}, \quad \forall i = 1, ..., n \tag{2.6}$$

$$x_i \in \{0, 1\}, d_i \geqslant 0, \quad \forall i = 1, ..., n \tag{2.7}$$

In this formulation, the objective function (2.4) minimizes the total number of vertices included in a strong dominating set of G with minimum cardinality. Constraint (2.5) is for the dominating vertices and is functioning same as constraint (2.2). Constraint (2.6) indicates that for all dominated vertices i, that is $(1-x_i) = 1$, among the vertices j that dominate vertex i, the degree of at least one of them should be greater than or equal to the degree of vertex i. This constraint ensures that for the neighboring vertices j of vertex i which do not dominate vertex i, $x_j = 0$. Constraint (2.7) defines variables of model.

2.3. Weak domination. The mathematical model of weak domination problem is as follows:

$$min \sum_{i=1}^{n} x_i \tag{2.8}$$

s.t.

$$\sum_{j=1, a_{ij}=1, j\neq i}^{n} x_j \geqslant 1, \quad \forall i = 1, ..., n \tag{2.9}$$

$$(1 - x_i)d_i \geqslant max_{j,a_{ij}=1,j\neq i}\{n^{(1-x_i)} d_j\}(1 - x_i), \ \forall i = 1, ..., n \tag{2.10}$$

$$x_i \in \{0, 1\}, d_i \geqslant 0, \quad \forall i = 1, ..., n \tag{2.11}$$

In this formulation, the objective function (2.8) minimizes the total number of vertices included in a weak dominating set of G with minimum cardinality. Constraint (2.9) ensures the dominating set. Constraint (2.10) indicates that for all dominated vertices i, that is $(1 - x_i) = 1$, among the vertices j that dominate vertex i, the degree of at least one of them should be less than or equal to the degree of vertex i. More clearly, if $x_i = 1$, then the constraint is satisfied as equality implying that the vertex i dominates itself. Otherwise, if $x_i = 0$, then there exists at least one neighboring vertex j of vertex i that dominates vertex i. If the neighboring vertex j of vertex i does not dominate the vertex i, that is if $x_j = 0$, then the contribution nd_j of this vertex to min function yields no change. Otherwise, if the neighboring vertex j of vertex i dominates the vertex i, that is if $x_j = 1$, then the contribution of this vertex to min function is d_j, and the minimum value among overall contributions remains less than the degree of dominated vertex i, thus the weak dominating constraint is satisfied. Constraint (2.11) defines variables of model.

3. Computational results

The proposed implemented programming models are coded and solved by GAMS 23.9.4 64 bit IDE software by using SCIP (Solving Constraint Integer Programs) solver for all specific types of graphs. All the test problems are run using the Intel i5-2400 CPU and 4 GB RAM. The optimal solutions for the test problems of (strong and weak, respectively) domination numbers are all shown in Table 1,

n	P_n			S_n			C_n			W_n			K_n			$K_{n,m}$		
	γ	γ_{st}	γ_w	γ	γ_{st}	γ_w	γ	γ_{st}	γ_w	γ	γ_{st}	γ_w	γ	γ_{st}	γ_w	γ	γ_{st}	γ_w
10	4	4	4	1	1	9	4	4	4	1	1	3	1	1	1	2	9	10
25	9	9	9	1	1	24	9	9	9	1	1	8	1	1	1	2	24	25
50	17	17	17	1	1	49	17	17	17	1	1	17	1	1	1	2	49	50
75	25	25	25	1	1	74	25	25	25	1	1	25	1	1	1	2	74	75
100	34	34	34	1	1	99	34	34	34	1	1	33	1	1	1	2	99	100

TABLE 1. The optima for (strong, weak, respectively) domination problem

where P_n, S_n, C_n, W_n, K_n are path, star, cycle, wheel, complete graph on n vertices, respectively, and $K_{n,m}$ (where $(m = n - 1)$) is the complete bipartite graph on $n + m$ vertices. The obtained execution times are gathered from solver reports of GAMS and denote the total job times of test problems as shown in Table 2, 3 and 4. All sample problems and related GAMS codes are available in the following link: http://kisi.deu.edu.tr/murat.berberler/swd/.

n	P_n	S_n	C_n	W_n	K_n	$K_{n,m}$
10	0,110	0,113	0,118	0,131	0,150	0,169
25	0,136	0,145	0,151	0,167	0,195	0,225
50	0,151	0,160	0,169	0,178	0,272	0,308
75	0,163	0,178	0,182	0,194	0,299	0,317
100	0,386	0,409	0,431	0,465	0,562	0,695

TABLE 2. The execution times for domination problem

n	P_n	S_n	C_n	W_n	K_n	$K_{n,m}$
10	0,117	0,119	0,122	0,140	0,156	0,171
25	0,163	0,168	0,169	0,171	0,203	0,228
50	0,172	0,175	0,178	0,183	0,294	0,316
75	0,187	0,191	0,193	0,201	0,304	0,332
100	0,515	0,518	0,520	0,531	0,684	0,717

TABLE 3. The execution times for strong domination problem

n	P_n	S_n	C_n	W_n	K_n	$K_{n,m}$
10	0,126	0,131	0,143	0,151	0,169	0,185
25	0,170	0,179	0,187	0,194	0,218	0,241
50	0,188	0,196	0,218	0,221	0,323	0,362
75	0,197	0,219	0,232	0,242	0,319	0,389
100	0,543	0,569	0,574	0,597	0,751	0,903

TABLE 4. The execution times for weak domination problem

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