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ON Λ_g-NORMAL AND Λ_g-REGULAR IN IDEAL TOPOLOGICAL SPACES

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Abstract. The aim of this paper, we introduce I_{Λ_g}-normal, ${}_{\Lambda_g}I$-normal and I_{Λ_g}-regular spaces using I_{Λ_g}-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, Λ_g-normal and regular spaces are given.

1. Introduction and Preliminaries

In 1986, Maki [**9**] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (= saturated set) i.e to the intersection of all open supersets of A. Arenas et al [**1**] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. Caldas et al [**2**] introduced and investigated the notion of Λ_g-closed sets in topological spaces and established several properties of such sets.

Quite Recently, Ravi et al [**17**] introduced and investigated the notions of I_{Λ_g}-closed sets and I_{Λ_g}-open sets in ideal topological spaces.

In this paper, we define I_{Λ_g}-normal, ${}_{\Lambda_g}I$-normal and I_{Λ_g}-regular spaces using I_{Λ_g}-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, Λ_g-normal and regular spaces are given.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X, τ).

Definition 1.1. A subset A of a space (X, τ) is said to be

(1) regular open ([**20**]) if $A = int(cl(A))$.
(2) an α-open ([**14**]) if $A \subseteq int(cl(int(A)))$.

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(3) preopen ([**11**]) if $A \subseteq int(cl(A)))$.

The complement of above sets are called their respective closed sets.

The α-closure of a subset A of X, denoted by $\alpha cl(A)$, is defined to be the intersection of all α-closed sets containing A. The α-interior of a subset A of X, denoted by $\alpha int(A)$, is defined to be the union of all α-open sets contained in A. The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The interior of a subset A in (X, τ^α) is denoted by $\mathrm{int}_\alpha(A)$. The closure of a subset A in (X, τ^α) is denoted by $cl_\alpha(A)$.

Definition 1.2. A subset A of a space (X, τ) is said to be

(1) g-closed ([**8**]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
(2) rg-closed ([**16**]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open.
(3) αg-closed ([**10**]) if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
(4) λ-closed ([**1**]) if $A = L \cap D$, where L is a Λ-set and D is a closed set.
(5) Λ_g-closed ([**2**]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open.

The complement of above sets are called their respective open sets.

Definition 1.3. A subset A of a space (X, τ) is said to be

(1) $\Lambda_{g\alpha}$-closed ([**17**]) if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open.
(2) $r\alpha g$-closed ([**15**]) if $cl_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open.
(3) $\Lambda_{g\alpha}$-open (resp. $r\alpha g$-open) if $X - A$ is $\Lambda_{g\alpha}$-closed (resp. $r\alpha g$-closed).

Remark 1.1. A subset A of a space (X, τ) is said to be

(1) every closed set is Λ_g-closed but not conversely ([**2**]).
(2) every Λ_g-closed set is g-closed but not conversely ([**2**]).
(3) every closed set is λ-closed but not conversely ([**1, 2**]).

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in I$ and $B \subseteq A$ imply $B \in I$ and
(2) $A \in I$ and $B \in I$ imply $A \cup B \in I$ ([**7**]).

Given a topological space $(X, \tau)$with an ideal I on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a local function ([**7**]) of A with respect to τ and I is defined as follows: for $A \subseteq X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. We will make use of the basic facts about the local functions ([**6**], Theorem 2.3) without mentioning it explicitly.

A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(I, \tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ ([**21**]). When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$. $int^\star(A)$ will denote the interior of A in $(X, \tau^\star)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ).

A subset A of an ideal topological space (X, τ, I) is

$\tau^\star$-closed ([**6**]) or $\star$-closed (resp. $\star$-dense in itself [**5**]) if $A^\star \subseteq A$ (resp. $A \subseteq A^\star$).

An ideal I is said to be codense ([**4**]) if $\tau \cap I = \{\phi\}$. I is said to be completely codense ([**18**]) if $PO(X) \cap I = \{\phi\}$, where $PO(X)$ is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not conversely ([**18**]).

The following lemmas and Definitions will be useful in the sequel.

Definition 1.4. ([**19**]) A space (X, τ) is said to be a mildly normal space if disjoint regular closed sets are separated by disjoint open sets.

Definition 1.5. A subset A of an ideal topological space (X, τ, I) is said to be

(1) I_g-closed ([**3**]) if $A^\star \subseteq U$ whenever $A \subseteq U$ and U is open.
(2) I_{rg}-closed ([**13**]) if $A^\star \subseteq U$ whenever $A \subseteq U$ and U is regular open.
(3) I_{Λ_g}-closed ([**17**]) if $A^\star \subseteq U$ whenever $A \subseteq U$ and U is λ-open.

The complement of above sets are called their respective open sets.

Remark 1.2. ([**17**])

(1) every $\star$-closed set is I_{Λ_g}-closed.
(2) every closed set is I_{Λ_g}-closed.

Theorem 1.1 ([**18**]). *Let (X, τ, I) be an ideal topological space. If I is completely codense, then $\tau^\star \subseteq \tau^\alpha$.*

Theorem 1.2 ([**17**]). *Let (X, τ, I) be an ideal topological space where I is completely codense. Then the following are equivalent.*

(1) *X is normal.*
(2) *For any disjoint closed sets A and B, there exist disjoint I_{Λ_g}-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
(3) *For any closed set A and open set V containing A, there exists an I_{Λ_g}-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*

Lemma 1.1 ([**17**]). *If (X, τ, I) is an ideal topological space and $A \subseteq X$, then the following hold.*

(1) *If $I = \{\phi\}$, then A is I_{Λ_g}-closed $\iff$ A is Λ_g-closed.*
(2) *If $I = \mathcal{N}$, then A is I_{Λ_g}-closed $\iff$ A is $\Lambda_{g\alpha}$-closed.*

Theorem 1.3 ([**17**]). *If (X, τ, I) is an ideal topological space and $A \subseteq X$, then the following are equivalent.*

(1) *A is I_{Λ_g}-closed.*
(2) *$cl^\star(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open in X.*

Theorem 1.4 ([**17**]). *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then A is I_{Λ_g}-open $\iff$ $F \subseteq int^\star(A)$ whenever F is λ-closed and $F \subseteq A$.*

Theorem 1.5 ([**17**]). *Let (X, τ, I) be an ideal topological space. Then every subset of X is I_{Λ_g}-closed $\iff$ every λ-open set is $\star$-closed.*

Lemma 1.2 ([**13**]). *Let (X, τ, I) be an ideal topological space. A subset $A \subseteq X$ is I_{rg}-open $\iff$ $F \subseteq int^\star(A)$ whenever F is regular closed and $F \subseteq A$.*

2. On I_{Λ_g}-normal spaces

Definition 2.1. A space (X, τ) is said to be Λ_g-normal, if for every disjoint Λ_g-closed sets A and B, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Definition 2.2. An ideal topological space (X, τ, I) is said to be an I_{Λ_g}-normal space if for every pair of disjoint closed sets A and B, there exist disjoint I_{Λ_g}-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Remark 2.1. (1) Every open set is an I_{Λ_g}-open set. (2) Every normal space is an I_{Λ_g}-normal.

Example 2.1. (1) Let

$$X = \{a, b, c\},\ \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\} \text{ and } I = \{\phi, \{b\}\}.$$

Then

$$\phi^\star = \phi,\ (\{a, b\})^\star = \{a\}, (\{b, c\})^\star = \{c\}, (\{b\})^\star = \phi \text{ and } X^\star = \{a, c\}.$$

Here every λ-open set is $\star$-closed and so, by Theorem 1.5, every subset of X is I_{Λ_g}-closed and hence every subset of X is I_{Λ_g}-open. This implies that (X, τ, I) is I_{Λ_g}-normal. Now $\{a\}$ and $\{c\}$ are disjoint closed subsets of X which are not separated by disjoint open sets and so (X, τ) is not normal.

Theorem 2.1. *Let (X, τ, I) be an ideal topological space. Then the following are equivalent.*

(1) *X is I_{Λ_g}-normal.*
(2) *For every closed set A and an open set V containing A, there exists an I_{Λ_g}-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*

Proof. (1) $\Rightarrow$ (2). Let A be a closed set and V be an open set containing A. Since A and $X - V$ are disjoint closed sets, there exist disjoint I_{Λ_g}-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Again, $U \cap W = \phi$ implies that $U \cap int^\star(W) = \phi$ and so $cl^\star(U) \subseteq X - int^\star(W)$. Since $X - V$ is λ-closed and W is I_{Λ_g}-open, $X - V \subseteq W$ implies that $X - V \subseteq int^\star(W)$ and so $X - int^\star(W) \subseteq V$. Thus, we have $A \subseteq U \subseteq cl^\star(U) \subseteq X - int^\star(W) \subseteq V$ which proves (2).

(2) $\Rightarrow$ (1). Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an I_{Λ_g}-open set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq X - B$. If $W = X - cl^\star(U)$, then U and W are the required disjoint I_{Λ_g}-open sets containing A and B respectively. So, (X, τ, I) is I_{Λ_g}-normal. □

Theorem 2.2. *Let (X, τ, I) be an ideal topological space where I is completely codense. If (X, τ, I) is I_{Λ_g}-normal, then it is a normal space.*

Proof. It is obvious from Theorem 2.1 and Theorem 1.2. □

By Theorem 2.1 gives a characterizations of I_{Λ_g}-normal spaces. Theorem 2.2 shows that the two concepts coincide for completely codense an ideal topological spaces.

THEOREM 2.3. *Let (X, τ, I) be an I_{Λ_g}-normal space. If F is closed and A is a Λ_g-closed set such that $A \cap F = \phi$, then there exist disjoint I_{Λ_g}-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

PROOF. Since $A \cap F = \phi$, $A \subseteq X - F$ where $X - F$ is λ-open. Therefore, by hypothesis, $cl(A) \subseteq X - F$. Since $cl(A) \cap F = \phi$ and X is I_{Λ_g}-normal, there exist disjoint I_{Λ_g}-open sets U and V such that $cl(A) \subseteq U$ and $F \subseteq V$. Thus $A \subseteq U$ and $F \subseteq V$. □

COROLLARY 2.1. *Let (X, τ) be a normal space with $I = \{\phi\}$. If F is a closed set and A is a Λ_g-closed set disjoint from F, then there exist disjoint Λ_g-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

COROLLARY 2.2. *Let (X, τ, I) be a normal ideal topological space where $I = \mathcal{N}$. If F is a closed set and A is a Λ_g-closed set disjoint from F, then there exist disjoint $\Lambda_{g\alpha}$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.*

The Corollaries 2.1 and 2.2 give properties of normal spaces. If $I = \{\phi\}$ in Theorem 2.3, then we have the Corollary 2.1, the proof of which follows from Theorem 2.2 and Lemma 1.1, since $\{\phi\}$ is a completely codense ideal. If $I = \mathcal{N}$ in Theorem 2.3, then we have the Corollary 2.2, since $\tau^\star(\mathcal{N}) = \tau^\alpha$ and I_{Λ_g}-open sets coincide with $\Lambda_{g\alpha}$-open sets.

THEOREM 2.4. *Let (X, τ, I) be an ideal topological space which is I_{Λ_g}-normal. Then the following hold.*

(1) *For every closed set A and every Λ_g-open set B containing A, there exists an I_{Λ_g}-open set U such that $A \subseteq int^\star(U) \subseteq U \subseteq B$.*
(2) *For every Λ_g-closed set A and every open set B containing A, there exists an I_{Λ_g}-closed set U such that $A \subseteq U \subseteq cl^\star(U) \subseteq B$.*

PROOF. (1) Let A be a closed set and B be a Λ_g-open set containing A. Then $A \cap (X - B) = \phi$, where A is closed and $X - B$ is Λ_g-closed. By Theorem 2.3, there exist disjoint I_{Λ_g}-open sets U and V such that $A \subseteq U$ and $X - B \subseteq V$. Since $U \cap V = \phi$, we have $U \subseteq X - V$. By Theorem 1.4, $A \subseteq int^\star(U)$. Therefore, $A \subseteq int^\star(U) \subseteq U \subseteq X - V \subseteq B$. This proves (1).

(2) Let A be a Λ_g-closed set and B be an open set containing A. Then $X-B$ is a closed set contained in the Λ_g-open set $X-A$. By (1), there exists an I_{Λ_g}-open set V such that $X-B \subseteq int^\star(V) \subseteq V \subseteq X-A$. Therefore, $A \subseteq X-V \subseteq cl^\star(X-V) \subseteq B$. If $U = X-V$, then $A \subseteq U \subseteq cl^\star(U) \subseteq B$ and so U is the required I_{Λ_g}-closed set. □

COROLLARY 2.3. *Let (X, τ) be a normal space with $I=\{\phi\}$. Then the following hold.*

(1) *For every closed set A and every Λ_g-open set B containing A, there exists a Λ_g-open set U such that $A \subseteq int(U) \subseteq U \subseteq B$.*
(2) *For every Λ_g-closed set A and every open set B containing A, there exists a Λ_g-closed set U such that $A \subseteq U \subseteq cl(U) \subseteq B$.*

COROLLARY 2.4. *Let (X, τ) be a normal space with $I = \mathcal{N}$. Then the following hold.*

(1) *For every closed set A and every Λ_g-open set B containing A, there exists an $\Lambda_{g\alpha}$-open set U such that $A \subseteq int_\alpha(U) \subseteq U \subseteq B$.*
(2) *For every Λ_g-closed set A and every open set B containing A, there exists an $\Lambda_{g\alpha}$-closed set U such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq B$.*

The Corollaries 2.3 and 2.4 give a some properties of normal spaces. If $I = \{\phi\}$ in Theorem 2.4, then we've the Corollary 2.3. If $I = \mathcal{N}$ in Theorem 2.4, then we've the Corollary 2.4.

3. On ${}_{\Lambda_g}I$-normal spaces

Definition 3.1. An ideal topological space (X, τ, I) is said to be ${}_{\Lambda_g}I$-normal if for each pair of disjoint I_{Λ_g}-closed sets A and B, there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

Remark 3.1. (1) Every closed set is I_{Λ_g}-closed. (2) Every ${}_{\Lambda_g}I$-normal space is normal.

The next Example 3.1 show that the reverse direction of the above Remark 3.1(2) is not true.

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $I = \wp(X)$. Every λ-open set is $\star$-closed and so every subset of X is I_{Λ_g}-closed. Now $A = \{a, b\}$ and $B = \{c\}$ are disjoint I_{Λ_g}-closed sets, but they are not separated by disjoint open sets. So (X, τ, I) is not ${}_{\Lambda_g}I$-normal. But (X, τ, I) is normal.

Theorem 3.1. *In an ideal topological space (X, τ, I), the following are equivalent.*

(1) *X is ${}_{\Lambda_g}I$-normal.*
(2) *For every I_{Λ_g}-closed set A and every I_{Λ_g}-open set B containing A, there exists an open set U of X such that $A \subseteq U \subseteq cl(U) \subseteq B$.*

Proof. It is similar to the proof of Theorem 2.1. □

If $I = \{\phi\}$, then ${}_{\Lambda_g}I$-normal spaces coincide with Λ_g-normal spaces and so if we take I=$\{\phi\}$, in Theorem 3.1, then we have the characterization for Λ_g-normal spaces.

Corollary 3.1. *In a space (X, τ), the following are equivalent.*

(1) *X is Λ_g-normal.*
(2) *For every Λ_g-closed set A and every Λ_g-open set B containing A, there exists an open set U of X such that $A \subseteq U \subseteq cl(U) \subseteq B$.*

Theorem 3.2. *In an ideal topological space (X, τ, I), the following are equivalent.*

(1) *X is ${}_{\Lambda_g}I$-normal.*
(2) *For each pair of disjoint I_{Λ_g}-closed subsets A and B of X, there exists an open set U of X containing A such that $cl(U) \cap B = \phi$.*

(3) *For each pair of disjoint I_{Λ_g}-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that $cl(U) \cap cl(V) = \phi$.*

PROOF. (1) $\Rightarrow$ (2). Suppose that A and B are disjoint I_{Λ_g}-closed subsets of X. Then the I_{Λ_g}-closed set A is contained in the I_{Λ_g}-open set $X - B$. By Theorem 3.1, there exists an open set U such that $A \subseteq U \subseteq cl(U) \subseteq X - B$. Therefore, U is the required open set containing A such that $cl(U) \cap B = \phi$.

(2) $\Rightarrow$ (3). Let A and B be two disjoint I_{Λ_g}-closed subsets of X. By hypothesis, there exists an open set U of X containing A such that $cl(U) \cap B = \phi$. Also, $cl(U)$ and B are disjoint I_{Λ_g}-closed sets of X. By hypothesis, there exists an open set V of X containing B such that $cl(U) \cap cl(V) = \phi$.

(3) $\Rightarrow$ (1). The proof is clear. □

If $I = \{\phi\}$, in Theorem 3.2, then we have a characterizations for Λ_g-normal spaces.

The Theorems 3.1 and 3.2 give a characterizations of ${}_{\Lambda_g}I$-normal spaces.

COROLLARY 3.2. *Let (X, τ) be a space. Then the following are equivalent.*

(1) *X is Λ_g-normal.*
(2) *For each pair of disjoint Λ_g-closed subsets A and B of X, there exists an open set U of X containing A such that $cl(U) \cap B = \phi$.*
(3) *For each pair of disjoint Λ_g-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that $cl(U) \cap cl(V) = \phi$.*

THEOREM 3.3. *Let (X, τ, I) be an ${}_{\Lambda_g}I$-normal space. If A and B are disjoint I_{Λ_g}-closed subsets of X, then there exist disjoint open sets U and V such that $cl^\star(A) \subseteq U$ and $cl^\star(B) \subseteq V$.*

PROOF. Suppose that A and B are disjoint I_{Λ_g}-closed sets. By Theorem 3.2(3), there exist an open set U containing A and an open set V containing B such that $cl(U) \cap cl(V) = \phi$. Since A is I_{Λ_g}-closed, $A \subseteq U$ implies that $cl^\star(A) \subseteq U$. Similarly $cl^\star(B) \subseteq V$. □

If $I = \{\phi\}$, in Theorem 3.3, then we have a property of disjoint Λ_g-closed sets in Λ_g-normal spaces.

COROLLARY 3.3. *Let (X, τ) be a Λ_g-normal space. If A and B are disjoint Λ_g-closed subsets of X, then there exist disjoint open sets U and V such that $cl(A) \subseteq U$ and $cl(B) \subseteq V$.*

THEOREM 3.4. *Let (X, τ, I) be an ${}_{\Lambda_g}I$-normal space. If A is an I_{Λ_g}-closed set and B is an I_{Λ_g}-open set containing A, then there exists an open set U such that $A \subseteq cl^\star(A) \subseteq U \subseteq int^\star(B) \subseteq B$.*

PROOF. Suppose A is an I_{Λ_g}-closed set and B is an I_{Λ_g}-open set containing A. Since A and $X - B$ are disjoint I_{Λ_g}-closed sets, by Theorem 3.3, there exist disjoint open sets U and V such that $cl^\star(A) \subseteq U$ and $cl^\star(X - B) \subseteq V$. Now,

$X - int^\star(B) = cl^\star(X - B) \subseteq V$ implies that $X - V \subseteq int^\star(B)$. Again, $U \cap V = \phi$ implies $U \subseteq X - V$ and so $A \subseteq cl^\star(A) \subseteq U \subseteq X - V \subseteq int^\star(B) \subseteq B$. □

COROLLARY 3.4. *Let (X, τ) be a Λ_g-normal space. If A is a Λ_g-closed set and B is a Λ_g-open set containing A, then there exists an open set U such that $A \subseteq cl(A) \subseteq U \subseteq int(B) \subseteq B$.*

If $I = \{\phi\}$, in Theorem 3.4, then the Corollary 3.4.

THEOREM 3.5. *Let (X, τ) be a space. Then the following are equivalent.*

(1) *X is normal.*
(2) *For any disjoint closed sets A and B, there exist disjoint Λ_g-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
(3) *For any closed set A and open set V containing A, there exists a Λ_g-open set U such that $A \subseteq U \subseteq cl(U) \subseteq V$.*

The Theorem 3.5 gives a characterization of normal spaces in terms of Λ_g-open sets which a Theorem 1.2 if $I = \{\phi\}$.

The rest of the section is devoted to the study of mildly normal spaces in terms of I_{Λ_g}-open sets, I_g-open sets and I_{rg}-open sets.

REMARK 3.2. (1) Every I_{Λ_g}-closed set is I_g-closed. (2) Every I_g-closed set is I_{rg}-closed.

THEOREM 3.6. *Let (X, τ, I) be an ideal topological space where I is completely codense. Then the following are equivalent.*

(1) *X is mildly normal.*
(2) *For disjoint regular closed sets A and B, there exist disjoint I_{Λ_g}-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
(3) *For disjoint regular closed sets A and B, there exist disjoint I_g-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
(4) *For disjoint regular closed sets A and B, there exist disjoint I_{rg}-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
(5) *For a regular closed set A and a regular open set V containing A, there exists an I_{rg}-open set U of X such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*
(6) *For a regular closed set A and a regular open set V containing A, there exists an $\star$-open set U of X such that $A \subseteq U \subseteq cl^\star(U) \subseteq V$.*
(7) *For disjoint regular closed sets A and B, there exist disjoint $\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

PROOF. (1) ⇒ (2). Suppose that A and B are disjoint regular closed sets. Since X is mildly normal, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. But every open set is an I_{Λ_g}-open set. This proves (2).

(2) ⇒ (3). The proof follows from the fact that every I_{Λ_g}-open set is an I_g-open set.

(3) ⇒ (4). The proof follows from the fact that every I_g-open set is an I_{rg}-open set.

(4) ⇒ (5). Suppose A is a regular closed and B is a regular open set containing A. Then A and $X - B$ are disjoint regular closed sets. By hypothesis, there

exist disjoint I_{rg}-open sets U and V such that $A \subseteq U$ and $X - B \subseteq V$. Since $X - B$ is regular closed and V is I_{rg}-open, by Lemma 1.2, $X - B \subseteq int^\star(V)$ and so $X - int^\star(V) \subseteq B$. Again, $U \cap V = \phi$ implies that $U \cap int^\star(V) = \phi$ and so $cl^\star(U) \subseteq X - int^\star(V) \subseteq B$. Hence U is the required I_{rg}-open set such that $A \subseteq U \subseteq cl^\star(U) \subseteq B$.

(5) $\Rightarrow$ (6). Let A be a regular closed set and V be a regular open set containing A. Then there exists an I_{rg}-open set G of X such that $A \subseteq G \subseteq cl^\star(G) \subseteq V$. By Lemma 1.2, $A \subseteq int^\star(G)$. If $U = int^\star(G)$, then U is an $\star$-open set and $A \subseteq U \subseteq cl^\star(U) \subseteq cl^\star(G) \subseteq V$. Therefore, $A \subseteq U \subseteq cl^\star(U) \subseteq V$.

(6) $\Rightarrow$ (7). Let A and B be disjoint regular closed subsets of X. Then $X - B$ is a regular open set containing A. By hypothesis, there exists an $\star$-open set U of X such that $A \subseteq U \subseteq cl^\star(U) \subseteq X - B$. If $V = X - cl^\star(U)$, then U and V are disjoint $\star$-open sets of X such that $A \subseteq U$ and $B \subseteq V$.

(7) $\Rightarrow$ (1). Let A and B be disjoint regular closed sets of X. Then there exist disjoint $\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since I is completely codense, by Theorem 1.1, $\tau^\star \subseteq \tau^\alpha$ and so $U, V \in \tau^\alpha$. Hence $A \subseteq U \subseteq int(cl(int(U))) = G$ and $B \subseteq V \subseteq int(cl(int(V))) = H$. G and H are the required disjoint open sets containing A and B respectively. This proves (1). □

Corollary 3.5. *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint $\Lambda_{g\alpha}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For disjoint regular closed sets A and B, there exist disjoint αg-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
4. *For disjoint regular closed sets A and B, there exist disjoint $r\alpha g$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
5. *For a regular closed set A and a regular open set V containing A, there exists an $r\alpha g$-open set U of X such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*
6. *For a regular closed set A and a regular open set V containing A, there exists an α-open set U of X such that $A \subseteq U \subseteq cl_\alpha(U) \subseteq V$.*
7. *For disjoint regular closed sets A and B, there exist disjoint α-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

If $I = \mathcal{N}$, in the above Theorem 3.6, then I_{rg}-closed sets coincide with $r\alpha g$-closed sets and so we've the Corollary 3.5.

Corollary 3.6. *Let (X, τ) be a space. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets A and B, there exist disjoint Λ_g-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
3. *For disjoint regular closed sets A and B, there exist disjoint g-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*
4. *For disjoint regular closed sets A and B, there exist disjoint rg-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

(5) *For a regular closed set A and a regular open set V containing A, there exists an rg-open set U of X such that* $A \subseteq U \subseteq cl(U) \subseteq V$.
(6) *For a regular closed set A and a regular open set V containing A, there exists an open set U of X such that* $A \subseteq U \subseteq cl(U) \subseteq V$.
(7) *For disjoint regular closed sets A and B, there exist disjoint open sets U and V such that* $A \subseteq U$ *and* $B \subseteq V$.

If $I = \{\phi\}$ in the above Theorem 3.6, we get the Corollary 3.6.

The Theorem 3.6 gives a characterizations of mildly normal spaces. Corollary 3.5 gives a characterizations of mildly normal spaces in terms of $\Lambda_{g\alpha}$-open, αg-open and $r\alpha g$-open sets. Corollary 3.6 gives a characterizations of mildly normal spaces in terms of Λ_g-open, g-open and rg-open sets. The Lemma 1.2 is essential to prove Theorem 3.6.

4. On I_{Λ_g}-regular spaces

DEFINITION 4.1. An ideal topological space (X, τ, I) is said to be an I_{Λ_g}-regular space if for each pair consisting of a point x and a closed set B not containing x, there exist disjoint I_{Λ_g}-open sets U and V such that $x \in U$ and $B \subseteq V$.

REMARK 4.1. (1) Every regular space is I_{Λ_g}-regular. (2) Every open set is I_{Λ_g}-open.

The next Example 4.1 show that the reverse direction of the above Remark 4.1(1) is not true.

EXAMPLE 4.1. Consider the ideal topological space (X, τ, I) of Example 2.1. Then $\phi^\star = \phi$, $(\{b\})^\star = \phi$, $(\{a, b\})^\star = \{a\}, (\{b, c\})^\star = \{c\}$ and $X^\star = \{a, c\}$. Since every λ-open set is $\star$-closed, every subset of X is I_{Λ_g}-closed and so every subset of X is I_{Λ_g}-open. This implies that (X, τ, I) is I_{Λ_g}-regular. Now, $\{c\}$ is a closed set not containing $a \in X$, $\{c\}$ and a are not separated by disjoint open sets. So (X, τ, I) is not regular.

THEOREM 4.1. *In an ideal topological space* (X, τ, I)*, the following are equivalent.*
(1) X *is* I_{Λ_g}*-regular.*
(2) *For every open set V containing* $x \in X$*, there exists an* I_{Λ_g}*-open set U of X such that* $x \in U \subseteq cl^\star(U) \subseteq V$.

PROOF. (1) ⇒ (2). Let V be an open subset such that $x \in V$. Then $X - V$ is a closed set not containing x. Therefore, there exist disjoint I_{Λ_g}-open sets U and W such that $x \in U$ and $X - V \subseteq W$. Now, $X - V \subseteq W$ implies that $X - V \subseteq int^\star(W)$ and so $X - int^\star(W) \subseteq V$. Again, $U \cap W = \phi$ implies that $U \cap int^\star(W) = \phi$ and so $cl^\star(U) \subseteq X - int^\star(W)$. Therefore, $x \in U \subseteq cl^\star(U) \subseteq V$. This proves (2).

(2) ⇒ (1). Let B be a closed set not containing x. By hypothesis, there exists an I_{Λ_g}-open set U such that $x \in U \subseteq cl^\star(U) \subseteq X - B$. If $W = X - cl^\star(U)$, then U and W are disjoint I_{Λ_g}-open sets such that $x \in U$ and $B \subseteq W$. This proves (1). □

The Theorem 4.1 gives a characterization of I_{Λ_g}-regular spaces.

THEOREM 4.2. *If (X, τ, I) is an I_{Λ_g}-regular, T_1-space where I is completely codense, then X is regular.*

PROOF. Let B be a closed set not containing $x \in X$. By Theorem 4.1, there exists an I_{Λ_g}-open set U of X such that $x \in U \subseteq cl^\star(U) \subseteq X - B$. Since X is a T_1-space, $\{x\}$ is λ-closed and so $\{x\} \subseteq int^\star(U)$, by Theorem 1.4. Since I is completely codense, $\tau^\star \subseteq \tau^\alpha$ and so $int^\star(U)$ and $X - cl^\star(U)$ are α-open sets. Now, $x \in int^\star(U) \subseteq int(cl(int(int^\star(U)))) = G$ and $B \subseteq X - cl^\star(U) \subseteq int(cl(int(X - cl^\star(U)))) = H$. Then G and H are disjoint open sets containing x and B respectively. Therefore, X is regular. □

COROLLARY 4.1. *If (X, τ) is a T_1-space, then the following are equivalent.*

(1) *X is regular.*
(2) *For every open set V containing $x \in X$, there exists an $\Lambda_{g\alpha}$-open set U of X such that $x \in U \subseteq cl_\alpha(U) \subseteq V$.*

If $I = \phi$ in Theorem 4.1, then the Corollary 4.1 which gives characterizations of regular spaces, the proof of Theorem 4.2.

COROLLARY 4.2. *If (X, τ) is a T_1-space, then the following are equivalent.*

(1) *X is regular.*
(2) *For every open set V containing $x \in X$, there exists a Λ_g-open set U of X such that $x \in U \subseteq cl(U) \subseteq V$.*

THEOREM 4.3. *If every λ-open subset of an ideal topological space (X, τ, I) is $\star$-closed, then (X, τ, I) is I_{Λ_g}-regular.*

PROOF. Suppose every λ-open subset of X is $\star$-closed. Then by Theorem 1.5, every subset of X is I_{Λ_g}-closed and hence every subset of X is I_{Λ_g}-open. If B is a closed set not containing x, then $\{x\}$ and B are the required disjoint I_{Λ_g}-open sets containing x and B respectively. Therefore, (X, τ, I) is I_{Λ_g}-regular. □

The next Example 4.2 shows that the reverse direction of the above Theorem 4.3 is not true.

EXAMPLE 4.2. Consider the real line $\mathcal{R}$ with the usual topology with $I = \{\phi\}$. Since $\mathcal{R}$ is regular, $\mathcal{R}$ is I_{Λ_g}-regular. Obviously $U = (0, 1)$ is λ-open being open in $\mathcal{R}$. But U is not $\star$-closed because, when $I = \{\phi\}$, $cl \star (U) = cl(U) = [0, 1] \neq U$.

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