ON $\lambda_g$-NORMAL AND $\lambda_g$-REGULAR IN IDEAL TOPOLOGICAL SPACES

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Abstract. The aim of this paper, we introduce $I\lambda_g$-normal, $\lambda_g I$-normal and $I\lambda_g$-regular spaces using $I\lambda_g$-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, $\lambda_g$-normal and regular spaces are given.

1. Introduction and Preliminaries

In 1986, Maki [9] introduced the notion of $\Lambda$-sets in topological spaces. A $\Lambda$-set is a set $A$ which is equal to its kernel (= saturated set) i.e to the intersection of all open supersets of $A$. Arenas et al [1] introduced and investigated the notion of $\lambda$-closed sets by involving $\Lambda$-sets and closed sets. Caldas et al [2] introduced and investigated the notion of $\lambda_g$-closed sets in topological spaces and established several properties of such sets.

Quite Recently, Ravi et al [17] introduced and investigated the notions of $I\lambda_g$-closed sets and $I\lambda_g$-open sets in ideal topological spaces.

In this paper, we define $I\lambda_g$-normal, $\lambda_g I$-normal and $I\lambda_g$-regular spaces using $I\lambda_g$-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, $\lambda_g$-normal and regular spaces are given.

By a space, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subseteq X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of $A$ in $(X, \tau)$.

Definition 1.1. A subset $A$ of a space $(X, \tau)$ is said to be
(1) regular open ([20]) if $A = int(cl(A))$.
(2) an $\alpha$-open ([14]) if $A \subseteq int(cl(int(A)))$.

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(3) preopen ([11]) if \( A \subseteq \text{int}(\text{cl}(A)) \).

The complement of above sets are called their respective closed sets.

The \( \alpha \)-closure of a subset \( A \) of \( X \), denoted by \( \alpha cl(A) \), is defined to be the intersection of all \( \alpha \)-closed sets containing \( A \). The \( \alpha \)-interior of a subset \( A \) of \( X \), denoted by \( \alpha int(A) \), is defined to be the union of all \( \alpha \)-open sets contained in \( A \).

The family of all \( \alpha \)-open sets in \( (X, \tau) \), denoted by \( \alpha \tau \), is a topology on \( X \) finer than \( \tau \). The interior of a subset \( A \) in \( (X, \tau^\alpha) \) is denoted by \( \text{int}_\alpha(A) \). The closure of a subset \( A \) in \( (X, \tau^\alpha) \) is denoted by \( \text{cl}_\alpha(A) \).

**Definition 1.2.** A subset \( A \) of a space \( (X, \tau) \) is said to be

1. \( g \)-closed ([8]) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open.
2. \( rg \)-closed ([16]) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular open.
3. \( og \)-closed ([10]) if \( \text{cl}_\alpha(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open.
4. \( \lambda \)-closed ([1]) if \( A = L \cap D \), where \( L \) is an \( \Lambda \)-set and \( D \) is a closed set.
5. \( \Lambda_g \)-closed ([2]) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \lambda \)-open.

The complement of above sets are called their respective open sets.

**Definition 1.3.** A subset \( A \) of a space \( (X, \tau) \) is said to be

1. \( \Lambda_g \)-closed but not conversely ([2]).
2. every \( \Lambda_g \)-closed set is \( g \)-closed but not conversely ([2]).
3. every closed set is \( g \)-closed but not conversely ([2]).

An ideal \( I \) on a topological space \( (X, \tau) \) is a nonempty collection of subsets of \( X \) which satisfies

1. \( A \in I \) and \( B \subseteq A \) imply \( B \in I \) and
2. \( A \in I \) and \( B \in I \) imply \( A \cup B \in I \) ([7]).

Given a topological space \( (X, \tau) \) with an ideal \( I \) on \( X \) and if \( \varphi(X) \) is the set of all subsets of \( X \), a set operator \((,) : \varphi(X) \rightarrow \varphi(X)\), called a local function ([7]) of \( A \) with respect to \( \tau \) and \( I \) is defined as follows: for \( A \subseteq X \), \( A^*(I, \tau) = \{ x \in X : U \cap A \notin I \text{ for every } U \in \tau(x) \} \) where \( \tau(x) = \{ U \in \tau : x \in U \} \). We will make use of the basic facts about the local functions ([6], Theorem 2.3) without mentioning it explicitly.

A Kuratowski closure operator \( cl^*(.) \) for a topology \( \tau^*(I, \tau) \), called the \(*\)-topology, finer than \( \tau \) is defined by \( cl^*(A) = A \cup A^*(I, \tau) \) ([21]). When there is no choice for confusion, we will simply write \( A^* \) for \( A^*(I, \tau) \) and \( \tau^* \) for \( \tau^*(I, \tau) \). \( \text{int}^*(A) \) will denote the interior of \( A \) in \( (X, \tau^*) \). If \( I \) is an ideal on \( X \), then \( (X, \tau, I) \) is called an ideal topological space. \( \mathcal{N} \) is the ideal of all nowhere dense subsets in \( (X, \tau) \).

A subset \( A \) of an ideal topological space \( (X, \tau, I) \) is \( \tau^* \)-closed ([6]) or \(*\)-closed (resp. \(*\)-dense in itself [5]) if \( A \subseteq A \) (resp. \( A \subseteq A^* \)).
An ideal $I$ is said to be codense ([4]) if $\tau \cap I = \{\phi\}$. $I$ is said to be completely codense ([18]) if $PO(X) \cap I = \{\phi\}$, where $PO(X)$ is the family of all preopen sets in $(X, \tau)$. Every completely codense ideal is codense but not conversely ([18]).

The following lemmas and Definitions will be useful in the sequel.

**Definition 1.4.** ([19]) A space $(X, \tau)$ is said to be a mildly normal space if disjoint regular closed sets are separated by disjoint open sets.

**Definition 1.5.** A subset $A$ of an ideal topological space $(X; \langle ; I\rangle)$ is said to be
(1) $I$-closed ([3]) if $A \in U$ whenever $A \subseteq U$ and $U$ is open.
(2) $I$rg-open ([13]) if $A^{\ast} \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open.
(3) $I$rg-open ([17]) if $A^{\ast} \subseteq U$ whenever $A \subseteq U$ and $U$ is $\lambda$-open.

The complement of above sets are called their respective open sets.

**Remark 1.2.** ([17])
(1) every $\ast$-closed set is $I_{\lambda_{g}}$-closed.
(2) every closed set is $I_{\lambda_{g}}$-closed.

**Theorem 1.1 ([18]).** Let $(X, \tau, I)$ be an ideal topological space. If $I$ is completely codense, then $\tau^{\ast} \subseteq \tau_{\alpha}$.

**Theorem 1.2 ([17]).** Let $(X, \tau, I)$ be an ideal topological space where $I$ is completely codense. Then the following are equivalent.
(1) $X$ is normal.
(2) For any disjoint closed sets $A$ and $B$, there exist disjoint $I_{\lambda_{g}}$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
(3) For any closed set $A$ and open set $V$ containing $A$, there exists an $I_{\lambda_{g}}$-open set $U$ such that $A \subseteq U \subseteq \text{cl}^{\ast}(U) \subseteq V$.

**Lemma 1.1 ([17]).** If $(X, \tau, I)$ is an ideal topological space and $A \subseteq X$, then the following hold.
(1) If $I = \{\phi\}$, then $A$ is $I_{\lambda_{g}}$-closed $\iff$ $A$ is $\lambda_{g}$-closed.
(2) If $I = \ast$, then $A$ is $I_{\lambda_{g}}$-closed $\iff$ $A$ is $\lambda_{g_{a}}$-closed.

**Theorem 1.3 ([17]).** If $(X, \tau, I)$ is an ideal topological space and $A \subseteq X$, then the following are equivalent.
(1) $A$ is $I_{\lambda_{g}}$-closed.
(2) $\text{cl}^{\ast}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\lambda$-open in $X$.

**Theorem 1.4 ([17]).** Let $(X, \tau, I)$ be an ideal topological space and $A \subseteq X$. Then $A$ is $I_{\lambda_{g}}$-open $\iff F \subseteq \text{int}^{\ast}(A)$ whenever $F$ is $\lambda$-closed and $F \subseteq A$.

**Theorem 1.5 ([17]).** Let $(X, \tau, I)$ be an ideal topological space. Then every subset of $X$ is $I_{\lambda_{g}}$-closed $\iff$ every $\lambda$-open set is $\ast$-closed.

**Lemma 1.2 ([13]).** Let $(X, \tau, I)$ be an ideal topological space. A subset $A \subseteq X$ is $I_{rg}$-open $\iff F \subseteq \text{int}^{\ast}(A)$ whenever $F$ is regular closed and $F \subseteq A$. 
2. On $I_{\lambda_g}$-normal spaces

**Definition 2.1.** A space $(X, \tau)$ is said to be $\lambda_g$-normal, if for every disjoint $\lambda_g$-closed sets $A$ and $B$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Definition 2.2.** An ideal topological space $(X, \tau, I)$ is said to be an $I_{\lambda_g}$-normal space if for every pair of disjoint closed sets $A$ and $B$, there exist disjoint $I_{\lambda_g}$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Remark 2.1.** (1) Every open set is an $I_{\lambda_g}$-open set. (2) Every normal space is an $I_{\lambda_g}$-normal.

**Example 2.1.** (1) Let

$$X = \{a, b, c\}, \quad \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\} \text{ and } I = \{\phi, \{b\}\}.$$ 

Then

$$\phi^* = \phi, \quad ((a, b))^* = \{a\}, ((b, c))^* = \{c\}, ((b))^* = \phi \text{ and } X^* = \{a, c\}.$$ 

Here every $\lambda$-open set is $\ast$-closed and so, by Theorem 1.5, every subset of $X$ is $I_{\lambda_g}$-closed and hence every subset of $X$ is $I_{\lambda_g}$-open. This implies that $(X, \tau, I)$ is $I_{\lambda_g}$-normal. Now $\{a\}$ and $\{c\}$ are disjoint closed subsets of $X$ which are not separated by disjoint open sets and so $(X, \tau)$ is not normal.

**Theorem 2.1.** Let $(X, \tau, I)$ be an ideal topological space. Then the following are equivalent.

1. $X$ is $I_{\lambda_g}$-normal.
2. For every closed set $A$ and an open set $V$ containing $A$, there exists an $I_{\lambda_g}$-open set $U$ such that $A \subseteq U \subseteq cl^*(U) \subseteq V$.

**Proof.** (1) $\Rightarrow$ (2). Let $A$ be a closed set and $V$ be an open set containing $A$. Since $A$ and $X - V$ are disjoint closed sets, there exist disjoint $I_{\lambda_g}$-open sets $U$ and $W$ such that $A \subseteq U$ and $X - V \subseteq W$. Again, $U \cap W = \phi$ implies that $U \cap int^*(W) = \phi$ and so $cl^*(U) \subseteq X - int^*(W)$. Since $X - V$ is $\lambda$-closed and $W$ is $I_{\lambda_g}$-open, $X - V \subseteq W$ implies that $X - V \subseteq int^*(W)$ and so $X - int^*(W) \subseteq V$. Thus, we have $A \subseteq U \subseteq cl^*(U) \subseteq X - int^*(W) \subseteq V$ which proves (2).

(2) $\Rightarrow$ (1). Let $A$ and $B$ be two disjoint closed subsets of $X$. By hypothesis, there exists an $I_{\lambda_g}$-open set $U$ such that $A \subseteq U \subseteq cl^*(U) \subseteq X - B$. If $W = X - cl^*(U)$, then $U$ and $W$ are the required disjoint $I_{\lambda_g}$-open sets containing $A$ and $B$ respectively. So, $(X, \tau, I)$ is $I_{\lambda_g}$-normal.

**Theorem 2.2.** Let $(X, \tau, I)$ be an ideal topological space where $I$ is completely codense. If $(X, \tau, I)$ is $I_{\lambda_g}$-normal, then it is a normal space.

**Proof.** It is obvious from Theorem 2.1 and Theorem 1.2.

By Theorem 2.1 gives a characterization of $I_{\lambda_g}$-normal spaces. Theorem 2.2 shows that the two concepts coincide for completely codense an ideal topological spaces.
Theorem 2.3. Let \((X, \tau, I)\) be an \(I_{\Lambda_{\sigma}}\)-normal space. If \(F\) is closed and \(A\) is a \(\Lambda_{\sigma}\)-closed set such that \(A \cap F = \phi\), then there exist disjoint \(I_{\Lambda_{\sigma}}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(F \subseteq V\).

Proof. Since \(A \cap F = \phi\), \(A \subseteq X - F\) where \(X - F\) is \(\lambda\)-open. Therefore, by hypothesis, \(cl(A) \subseteq X - F\). Since \(cl(A) \cap F = \phi\) and \(X\) is \(I_{\Lambda_{\sigma}}\)-normal, there exist disjoint \(I_{\Lambda_{\sigma}}\)-open sets \(U\) and \(V\) such that \(cl(A) \subseteq U\) and \(F \subseteq V\). Thus \(A \subseteq U\) and \(F \subseteq V\).

Corollary 2.1. Let \((X, \tau, I)\) be a normal space with \(I = \{\phi\}\). If \(F\) is a closed set and \(A\) is a \(\Lambda_{\sigma}\)-closed set disjoint from \(F\), then there exist disjoint \(\Lambda_{\sigma}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(F \subseteq V\).

Corollary 2.2. Let \((X, \tau, I)\) be a normal ideal topological space where \(I = \mathcal{N}\). If \(F\) is a closed set and \(A\) is a \(\Lambda_{\sigma}\)-closed set disjoint from \(F\), then there exist disjoint \(\Lambda_{\sigma\alpha}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(F \subseteq V\).

The Corollaries 2.1 and 2.2 give properties of normal spaces. If \(I = \{\phi\}\) in Theorem 2.3, then we have the Corollary 2.1, the proof of which follows from Theorem 2.2 and Lemma 1.1, since \(\{\phi\}\) is a completely codense ideal. If \(I = \mathcal{N}\) in Theorem 2.3, then we have the Corollary 2.2, since \(\tau^*(\mathcal{N}) = \tau^\alpha\) and \(I_{\Lambda_{\sigma}}\)-open sets coincide with \(\Lambda_{\sigma\alpha}\)-open sets.

Theorem 2.4. Let \((X, \tau, I)\) be an ideal topological space which is \(I_{\Lambda_{\sigma}}\)-normal. Then the following hold.

1. For every closed set \(A\) and every \(\Lambda_{\sigma}\)-open set \(B\) containing \(A\), there exists an \(I_{\Lambda_{\sigma}}\)-open set \(U\) such that \(A \subseteq int^*(U) \subseteq U \subseteq B\).
2. For every \(\Lambda_{\sigma}\)-closed set \(A\) and every open set \(B\) containing \(A\), there exists an \(I_{\Lambda_{\sigma}}\)-closed set \(U\) such that \(A \subseteq U \subseteq cl^*(U) \subseteq B\).

Proof. (1) Let \(A\) be a closed set and \(B\) be a \(\Lambda_{\sigma}\)-open set containing \(A\). Then \(A \cap (X - B) = \phi\), where \(A\) is closed and \(X - B\) is \(\Lambda_{\sigma}\)-closed. By Theorem 2.3, there exist disjoint \(I_{\Lambda_{\sigma}}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(X - B \subseteq V\). Since \(U \cap V = \phi\), we have \(U \subseteq X - V\). By Theorem 1.4, \(A \subseteq int^*(U)\). Therefore, \(A \subseteq int^*(U) \subseteq U \subseteq X - V \subseteq B\). This proves (1).

(2) Let \(A\) be a \(\Lambda_{\sigma}\)-closed set and \(B\) be an open set containing \(A\). Then \(X - B\) is a closed set contained in the \(\Lambda_{\sigma}\)-open set \(X - A\). By (1), there exists an \(I_{\Lambda_{\sigma}}\)-open set \(V\) such that \(X - B \subseteq int^*(V) \subseteq V \subseteq X - A\). Therefore, \(A \subseteq X - V \subseteq cl^*(X - V) \subseteq B\). If \(U = X - V\), then \(A \subseteq U \subseteq cl^*(U) \subseteq B\) and so \(U\) is the required \(I_{\Lambda_{\sigma}}\)-closed set.

Corollary 2.3. Let \((X, \tau)\) be a normal space with \(I = \{\phi\}\). Then the following hold.

1. For every closed set \(A\) and every \(\Lambda_{\sigma}\)-open set \(B\) containing \(A\), there exists a \(\Lambda_{\sigma}\)-open set \(U\) such that \(A \subseteq int(U) \subseteq U \subseteq B\).
2. For every \(\Lambda_{\sigma}\)-closed set \(A\) and every open set \(B\) containing \(A\), there exists a \(\Lambda_{\sigma}\)-closed set \(U\) such that \(A \subseteq U \subseteq cl(U) \subseteq B\).

Corollary 2.4. Let \((X, \tau)\) be a normal space with \(I = \mathcal{N}\). Then the following hold.
(1) For every closed set $A$ and every $\Lambda_I$-open set $B$ containing $A$, there exists an $\Lambda_{\alpha_I}$-open set $U$ such that $A \subseteq \text{int}_\alpha(U) \subseteq U \subseteq B$.

(2) For every $\Lambda_I$-closed set $A$ and every open set $B$ containing $A$, there exists an $\Lambda_{\alpha_I}$-closed set $U$ such that $A \subseteq U \subseteq \text{cl}_\alpha(U) \subseteq B$.

The Corollaries 2.3 and 2.4 give some properties of normal spaces. If $I = \{\phi\}$ in Theorem 2.4, then we've the Corollary 2.3. If $I = \mathcal{N}$ in Theorem 2.4, then we've the Corollary 2.4.

3. On $\Lambda_I$-normal spaces

Definition 3.1. An ideal topological space $(X, \tau, I)$ is said to be $\Lambda_I$-normal if for each pair of disjoint $\Lambda_I$-closed sets $A$ and $B$, there exist disjoint open sets $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$.

Remark 3.1. (1) Every closed set is $I$-closed. (2) Every $\Lambda_I$-normal space is normal.

The next Example 3.1 show that the reverse direction of the above Remark 3.1(2) is not true.

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $I = \phi(X)$. Every $\Lambda$-open set is $*$-closed and so every subset of $X$ is $I_{\Lambda^*}$-closed. Now $A = \{a, b\}$ and $B = \{c\}$ are disjoint $I_{\Lambda^*}$-closed sets, but they are not separated by disjoint open sets. So $(X, \tau, I)$ is not $\Lambda_I$-normal. But $(X, \tau, I)$ is normal.

Theorem 3.1. In an ideal topological space $(X, \tau, I)$, the following are equivalent.

(1) $X$ is $\Lambda_I$-normal.

(2) For every $I_{\Lambda^*}$-closed set $A$ and every $I_{\Lambda^*}$-open set $B$ containing $A$, there exists an open set $U$ of $X$ such that $A \subseteq U \subseteq \text{cl}(U) \subseteq B$.

Proof. It is similar to the proof of Theorem 2.1. \[\Box\]

If $I = \{\phi\}$, then $\Lambda_I$-normal spaces coincide with $\Lambda_I$-normal spaces and so if we take $I = \{\phi\}$, in Theorem 3.1, then we have the characterization for $\Lambda_I$-normal spaces.

Corollary 3.1. In a space $(X, \tau)$, the following are equivalent.

(1) $X$ is $\Lambda^*_I$-normal.

(2) For every $\Lambda^*_I$-closed set $A$ and every $\Lambda^*_I$-open set $B$ containing $A$, there exists an open set $U$ of $X$ such that $A \subseteq U \subseteq \text{cl}(U) \subseteq B$.

Theorem 3.2. In an ideal topological space $(X, \tau, I)$, the following are equivalent.

(1) $X$ is $\Lambda_I$-normal.

(2) For each pair of disjoint $I_{\Lambda^*}$-closed subsets $A$ and $B$ of $X$, there exists an open set $U$ of $X$ containing $A$ such that $\text{cl}(U) \cap B = \phi$. 
(3) For each pair of disjoint $I_{\Lambda_g}$-closed subsets $A$ and $B$ of $X$, there exist an open set $U$ containing $A$ and an open set $V$ containing $B$ such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $A$ and $B$ are disjoint $I_{\Lambda_g}$-closed subsets of $X$. Then the $I_{\Lambda_g}$-closed set $A$ is contained in the $I_{\Lambda_g}$-open set $X - B$. By Theorem 3.1, there exists an open set $U$ such that $A \subseteq U \subseteq \text{cl}(U) \subseteq X - B$. Therefore, $U$ is the required open set containing $A$ such that $\text{cl}(U) \cap B = \emptyset$.

(2) $\Rightarrow$ (3). Let $A$ and $B$ be two disjoint $I_{\Lambda_g}$-closed subsets of $X$. By hypothesis, there exists an open set $U$ of $X$ containing $A$ such that $\text{cl}(U) \cap B = \emptyset$. Also, $\text{cl}(U)$ and $B$ are disjoint $I_{\Lambda_g}$-closed sets of $X$. By hypothesis, there exists an open set $V$ of $X$ containing $B$ such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

(3) $\Rightarrow$ (1). The proof is clear. $\square$

If $I = \{\emptyset\}$, in Theorem 3.2, then we have a characterization for $\Lambda_g$-normal spaces.

**Theorem 3.3.** Let $(X, \tau)$ be a space. Then the following are equivalent.

(1) $X$ is $\Lambda_g$-normal.

(2) For each pair of disjoint $\Lambda_g$-closed subsets $A$ and $B$ of $X$, there exists an open set $U$ of $X$ containing $A$ such that $\text{cl}(U) \cap B = \emptyset$.

(3) For each pair of disjoint $\Lambda_g$-closed subsets $A$ and $B$ of $X$, there exists an open set $U$ containing $A$ and an open set $V$ containing $B$ such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

**Proof.** Suppose that $A$ and $B$ are disjoint $I_{\Lambda_g}$-closed sets. By Theorem 3.2(3), there exist an open set $U$ containing $A$ and an open set $V$ containing $B$ such that $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Since $A$ is $I_{\Lambda_g}$-closed, $A \subseteq U$ implies that $\text{cl}^*(A) \subseteq U$. Similarly $\text{cl}^*(B) \subseteq V$. $\square$

If $I = \{\emptyset\}$, in Theorem 3.3, then we have a property of disjoint $\Lambda_g$-closed sets in $\Lambda_g$-normal spaces.

**Corollary 3.3.** Let $(X, \tau)$ be a $\Lambda_g$-normal space. If $A$ and $B$ are disjoint $I_{\Lambda_g}$-closed subsets of $X$, then there exist disjoint open sets $U$ and $V$ such that $\text{cl}^*(A) \subseteq U$ and $\text{cl}^*(B) \subseteq V$.

**Theorem 3.4.** Let $(X, \tau)$ be an $\Lambda_g$-normal space. If $A$ is an $I_{\Lambda_g}$-closed set and $B$ is an $I_{\Lambda_g}$-open set containing $A$, then there exists an open set $U$ such that $A \subseteq \text{cl}^*(A) \subseteq U \subseteq \text{int}^*(B) \subseteq B$.

**Proof.** Suppose $A$ is an $I_{\Lambda_g}$-closed set and $B$ is an $I_{\Lambda_g}$-open set containing $A$. Since $A$ and $X - B$ are disjoint $I_{\Lambda_g}$-closed sets, by Theorem 3.3, there exist disjoint open sets $U$ and $V$ such that $\text{cl}^*(A) \subseteq U$ and $\text{cl}^*(X - B) \subseteq V$. Now,
set.

\( X - \text{int}^*(B) = \text{cl}^*(X - B) \subseteq V \) implies that \( X - V \subseteq \text{int}^*(B) \). Again, \( U \cap V = \emptyset \) implies \( U \subseteq X - V \) and so \( A \subseteq \text{cl}^*(A) \subseteq U \subseteq X - V \subseteq \text{int}^*(B) \subseteq B \).

\[ \square \]

**Corollary 3.4.** Let \((X, \tau)\) be a \(\Lambda_{\gamma}\)-normal space. If \(A\) is a \(\Lambda_{\gamma}\)-closed set and \(B\) is a \(\Lambda_{\gamma}\)-open set containing \(A\), then there exists an open set \(U\) such that \(A \subseteq \text{cl}(A) \subseteq U \subseteq \text{int}(B) \subseteq B\).

If \(I = \{\emptyset\}\), in Theorem 3.4, then the Corollary 3.4.

**Theorem 3.5.** Let \((X, \tau)\) be a space. Then the following are equivalent.

1. \(X\) is normal.
2. For any disjoint closed sets \(A\) and \(B\), there exist disjoint \(\Lambda_{\gamma}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).
3. For any closed set \(A\) and open set \(V\) containing \(A\), there exists a \(\Lambda_{\gamma}\)-open set \(U\) such that \(A \subseteq U \subseteq \text{cl}(U) \subseteq V\).

The Theorem 3.5 gives a characterization of normal spaces in terms of \(\Lambda_{\gamma}\)-open sets which a Theorem 1.2 if \(I = \{\emptyset\}\).

The rest of the section is devoted to the study of mildly normal spaces in terms of \(\Lambda_{\gamma}\)-open sets, \(I_{\gamma}\)-open sets and \(I_{r\gamma}\)-open sets.

**Remark 3.2.** (1) Every \(I_{\Lambda_{\gamma}}\)-closed set is \(I_{\gamma}\)-closed. (2) Every \(I_{\gamma}\)-closed set is \(I_{r\gamma}\)-closed.

**Theorem 3.6.** Let \((X, \tau, I)\) be an ideal topological space where \(I\) is completely codense. Then the following are equivalent.

1. \(X\) is mildly normal.
2. For disjoint regular closed sets \(A\) and \(B\), there exist disjoint \(I_{\Lambda_{\gamma}}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).
3. For disjoint regular closed sets \(A\) and \(B\), there exist disjoint \(I_{\gamma}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).
4. For disjoint regular closed sets \(A\) and \(B\), there exist disjoint \(I_{r\gamma}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).
5. For a regular closed set \(A\) and a regular open set \(V\) containing \(A\), there exists an \(I_{\gamma}\)-open set \(U\) of \(X\) such that \(A \subseteq U \subseteq \text{cl}^*(U) \subseteq V\).
6. For a regular closed set \(A\) and a regular open set \(V\) containing \(A\), there exists an \(\ast\)-open set \(U\) of \(X\) such that \(A \subseteq U \subseteq \text{cl}^*(U) \subseteq V\).
7. For disjoint regular closed sets \(A\) and \(B\), there exist disjoint \(\ast\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

**Proof.** (1) \(\Rightarrow\) (2). Suppose that \(A\) and \(B\) are disjoint regular closed sets. Since \(X\) is mildly normal, there exist disjoint open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\). But every open set is an \(I_{\Lambda_{\gamma}}\)-open set. This proves (2).

(2) \(\Rightarrow\) (3). The proof follows from the fact that every \(I_{\Lambda_{\gamma}}\)-open set is an \(I_{\gamma}\)-open set.

(3) \(\Rightarrow\) (4). The proof follows from the fact that every \(I_{\gamma}\)-open set is an \(I_{r\gamma}\)-open set.

(4) \(\Rightarrow\) (5). Suppose \(A\) is a regular closed and \(B\) is a regular open set containing \(A\). Then \(A\) and \(X - B\) are disjoint regular closed sets. By hypothesis, there
exist disjoint $I_{rg}$-open sets $U$ and $V$ such that $A \subseteq U$ and $X - B \subseteq V$. Since $X - B$ is regular closed and $V$ is $I_{rg}$-open, by Lemma 1.2, $X - B \subseteq \text{int}^*(V)$ and so $X - \text{int}^*(V) \subseteq B$. Again, $U \cap V = \emptyset$ implies that $U \cap \text{int}^*(V) = \emptyset$ and so $\text{cl}^*(U) \subseteq X - \text{int}^*(V) \subseteq B$. Hence $U$ is the required $I_{rg}$-open set such that $A \subseteq U \subseteq \text{cl}^*(U) \subseteq B$.

(5) $\Rightarrow$ (6). Let $A$ be a regular closed set and $V$ be a regular open set containing $A$. Then there exists an $I_{rg}$-open set $G$ of $X$ such that $A \subseteq G \subseteq \text{cl}^*(G) \subseteq V$. By Lemma 1.2, $A \subseteq \text{int}^*(G)$. If $U = \text{int}^*(G)$, then $U$ is an $*$-open set and $A \subseteq U \subseteq \text{cl}^*(U) \subseteq \text{cl}^*(G) \subseteq V$. Therefore, $A \subseteq U \subseteq \text{cl}^*(U) \subseteq V$.

(6) $\Rightarrow$ (7). Let $A$ and $B$ be disjoint regular closed subsets of $X$. Then $X - B$ is a regular open set containing $A$. By hypothesis, there exists an $*$-open set $U$ of $X$ such that $A \subseteq U \subseteq \text{cl}^*(U) \subseteq X - B$. If $V = X - \text{cl}^*(U)$, then $U$ and $V$ are disjoint $*$-open sets of $X$ such that $A \subseteq U$ and $B \subseteq V$.

(7) $\Rightarrow$ (1). Let $A$ and $B$ be disjoint regular closed sets of $X$. Then there exist disjoint $*$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$. Since $I$ is completely codense, by Theorem 1.1, $\tau^* \subseteq \tau^a$ and so $U, V \in \tau^a$. Hence $A \subseteq U \subseteq \text{int}(\text{cl}(\text{int}(U))) = G$ and $B \subseteq V \subseteq \text{int}(\text{cl}(\text{int}(V))) = H$. $G$ and $H$ are the required disjoint open sets containing $A$ and $B$ respectively. This proves (1).

**Corollary 3.5.** Let $(X, \tau)$ be a space. Then the following are equivalent.

1. $X$ is mildly normal.
2. For disjoint regular closed sets $A$ and $B$, there exist disjoint $\Lambda_{ga}$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
3. For disjoint regular closed sets $A$ and $B$, there exist disjoint $\alpha q$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
4. For disjoint regular closed sets $A$ and $B$, there exist disjoint $r\alpha q$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
5. For a regular closed set $A$ and a regular open set $V$ containing $A$, there exists an $r\alpha q$-open set $U$ of $X$ such that $A \subseteq U \subseteq \text{cl}_a(U) \subseteq V$.
6. For a regular closed set $A$ and a regular open set $V$ containing $A$, there exists an $\alpha$-open set $U$ of $X$ such that $A \subseteq U \subseteq \text{cl}_a(U) \subseteq V$.
7. For disjoint regular closed sets $A$ and $B$, there exist disjoint $\alpha$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

If $I = N$, in the above Theorem 3.6, then $I_{rg}$-closed sets coincide with $rq$-closed sets and so we’ve the Corollary 3.5.

**Corollary 3.6.** Let $(X, \tau)$ be a space. Then the following are equivalent.

1. $X$ is mildly normal.
2. For disjoint regular closed sets $A$ and $B$, there exist disjoint $\Lambda_g$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
3. For disjoint regular closed sets $A$ and $B$, there exist disjoint $g$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.
4. For disjoint regular closed sets $A$ and $B$, there exist disjoint $rg$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$. 
(5) For a regular closed set $A$ and a regular open set $V$ containing $A$, there exists an $rg$-open set $U$ of $X$ such that $A \subseteq U \subseteq \text{cl}(U) \subseteq V$.

(6) For a regular closed set $A$ and a regular open set $V$ containing $A$, there exists an open set $U$ of $X$ such that $A \subseteq U \subseteq \text{cl}(U) \subseteq V$.

(7) For disjoint regular closed sets $A$ and $B$, there exist disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

If $I = \{\emptyset\}$ in the above Theorem 3.6, we get the Corollary 3.6.

The Theorem 3.6 gives a characterizations of mildly normal spaces. Corollary 3.5 gives a characterizations of mildly normal spaces in terms of $\Lambda g_{rg}$-open, $og$-open and $rgg$-open sets. Corollary 3.6 gives a characterizations of mildly normal spaces in terms of $\Lambda g_{rg}$-open, $g$-open and $rg$-open sets. The Lemma 1.2 is essential to prove Theorem 3.6.

4. On $I_{\Lambda_{rg}}$-regular spaces

**Definition 4.1.** An ideal topological space $(X, \tau, I)$ is said to be an $I_{\Lambda_{rg}}$-regular space if for each pair consisting of a point $x$ and a closed set $B$ not containing $x$, there exist disjoint $I_{\Lambda_{rg}}$-open sets $U$ and $V$ such that $x \in U$ and $B \subseteq V$.

**Remark 4.1.** (1) Every regular space is $I_{\Lambda_{rg}}$-regular. (2) Every open set is $I_{\Lambda_{rg}}$-open.

The next Example 4.1 show that the reverse direction of the above Remark 4.1(1) is not true.

**Example 4.1.** Consider the ideal topological space $(X, \tau, I)$ of Example 2.1. Then $\phi^* = \phi$, $\{(b)\}^* = \phi$, $\{(a, b)\}^* = \{a\}$, $\{(b, c)\}^* = \{c\}$ and $X^* = \{a, c\}$. Since every $\Lambda g_{rg}$-open set is $*$-closed, every subset of $X$ is $I_{\Lambda_{rg}}$-closed and so every subset of $X$ is $I_{\Lambda_{rg}}$-open. This implies that $(X, \tau, I)$ is $I_{\Lambda_{rg}}$-regular. Now, $\{c\}$ is a closed set not containing $a \in X$, $\{c\}$ and $a$ are not separated by disjoint open sets. So $(X, \tau, I)$ is not regular.

**Theorem 4.1.** In an ideal topological space $(X, \tau, I)$, the following are equivalent.

1. $X$ is $I_{\Lambda_{rg}}$-regular.
2. For every open set $V$ containing $x \in X$, there exists an $I_{\Lambda_{rg}}$-open set $U$ of $X$ such that $x \in U \subseteq \text{cl}^*(U) \subseteq V$.

**Proof.** (1) $\Rightarrow$ (2). Let $V$ be an open subset such that $x \in V$. Then $X - V$ is a closed set not containing $x$. Therefore, there exist disjoint $I_{\Lambda_{rg}}$-open sets $U$ and $W$ such that $x \in U$ and $X - V \subseteq W$. Now, $X - V \subseteq W$ implies that $X - V \subseteq \text{int}^*(W)$ and so $X - \text{int}^*(W) \subseteq V$. Again, $U \cap W = \emptyset$ implies that $U \cap \text{int}^*(W) = \emptyset$ and so $\text{cl}^*(U) \subseteq X - \text{int}^*(W)$. Therefore, $x \in U \subseteq \text{cl}^*(U) \subseteq V$. This proves (2).

(2) $\Rightarrow$ (1). Let $B$ be a closed set not containing $x$. By hypothesis, there exists an $I_{\Lambda_{rg}}$-open set $U$ such that $x \in U \subseteq \text{cl}^*(U) \subseteq X - B$. If $W = X - \text{cl}^*(U)$, then $U$ and $W$ are disjoint $I_{\Lambda_{rg}}$-open sets such that $x \in U$ and $B \subseteq W$. This proves (1).
The Theorem 4.1 gives a characterization of $I_{\Lambda_p}$-regular spaces.

**Theorem 4.2.** If $(X, \tau, I)$ is an $I_{\Lambda_p}$-regular, $T_1$-space where $I$ is completely codense, then $X$ is regular.

**Proof.** Let $B$ be a closed set not containing $x \in X$. By Theorem 4.1, there exists an $I_{\Lambda_p}$-open set $U$ of $X$ such that $x \in U \subseteq cl^*(U) \subseteq X - B$. Since $X$ is a $T_1$-space, $\{x\}$ is $\lambda$-closed and so $\{x\} \subseteq int^*(U)$, by Theorem 1.4. Since $I$ is completely codense, $\tau^* \subseteq \tau^I$ and so $int^*(U)$ and $X - cl^*(U)$ are $\alpha$-open sets. Now, $x \in int^*(U) \subseteq int(cl(int(int^*(U)))) = G$ and $B \subseteq X - cl^*(U) \subseteq int(cl(int(X - cl^*(U)))) = H$. Then $G$ and $H$ are disjoint open sets containing $x$ and $B$ respectively. Therefore, $X$ is regular. \qed

**Corollary 4.1.** If $(X, \tau)$ is a $T_1$-space, then the following are equivalent.

1. $X$ is regular.
2. For every open set $V$ containing $x \in X$, there exists an $\Lambda_{\gamma_0}$-open set $U$ of $X$ such that $x \in U \subseteq cl_\alpha(U) \subseteq V$.

If $I = \phi$ in Theorem 4.1, then the Corollary 4.1 which gives characterizations of regular spaces, the proof of Theorem 4.2.

**Corollary 4.2.** If $(X, \tau)$ is a $T_1$-space, then the following are equivalent.

1. $X$ is regular.
2. For every open set $V$ containing $x \in X$, there exists a $\Lambda_\gamma$-open set $U$ of $X$ such that $x \in U \subseteq cl(U) \subseteq V$.

**Theorem 4.3.** If every $\lambda$-open subset of an ideal topological space $(X, \tau, I)$ is $\ast$-closed, then $(X, \tau, I)$ is $I_{\Lambda_p}$-regular.

**Proof.** Suppose every $\lambda$-open subset of $X$ is $\ast$-closed. Then by Theorem 1.5, every subset of $X$ is $I_{\Lambda_\gamma}$-closed and hence every subset of $X$ is $I_{\Lambda_p}$-open. If $B$ is a closed set not containing $x$, then $\{x\}$ and $B$ are the required disjoint $I_{\Lambda_p}$-open sets containing $x$ and $B$ respectively. Therefore, $(X, \tau, I)$ is $I_{\Lambda_p}$-regular. \qed

The next Example 4.2 shows that the reverse direction of the above Theorem 4.3 is not true.

**Example 4.2.** Consider the real line $\mathcal{R}$ with the usual topology with $I = \{\phi\}$. Since $\mathcal{R}$ is regular, $\mathcal{R}$ is $I_{\Lambda_p}$-regular. Obviously $U = (0, 1)$ is $\lambda$-open being open in $\mathcal{R}$. But $U$ is not $\ast$-closed because, when $I = \{\phi\}$, $cl \ast(U) = cl(U) = [0, 1] \neq U$.

**References**


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