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SOME RESULTS ON ANALYTIC ODD MEAN LABELING OF GRAPH

P. Jeyanthi, R.Gomathi, and Gee-Choon Lau

Abstract. Let $G = (V, E)$ be a graph with p vertices and q edges. A graph G is analytic odd mean if there exist an injective function $f : V \to \{0, 1, 3, 5 \ldots, 2q - 1\}$ with an induce edge labeling $f^* : E \to Z$ such that for each edge uv with $f(u) < f(v)$,

$$f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2-(f(u)+1)^2}{2} \right\rceil, & \text{if } f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil, & \text{if } f(u) = 0. \end{cases}$$

is injective. We say that f is an analytic odd mean labeling of G. In this paper we prove that quadrilateral snake $Q(n)$, double quadrilateral snake $DQ(n)$, coconut tree, fire cracker graph, splitting graph $spl(G)$, $P_n(1, 2, 3, \ldots, n)$, and the graph $C_k \odot \bar{K_n}$ are analytic odd mean graphs.

1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one.The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively.Terms and notations not defined here are used in the sense of Harary [**1**]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling. An excellent survey of graph labeling is available in [**5**]. The concept of analytic mean labeling was introduced in [**6**]. A (p, q) graph $G(V, E)$ is said to be an analytic mean graph if it is possible to label the vertices in V with distinct elements from $0, 1, 2, \ldots, p-1$ in such a way that when each edge $e = uv$ is labelled with $f^*(uv) = \left|(f(u))^2 - f(v)^2\right|/2$ if $\left|(f(u))^2 - f(v)^2\right|$ is even and $(\left|(f(u))^2 - f(v)^2\right| + 1)/2$ if $\left|(f(u))^2 - f(v)^2\right|$ is odd

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and the edge labels are distinct. Motivated by the results in [**6**], we introduced a new mean labeling called analytic odd mean labeling in [**2**]. We proved that cycle C_n, path P_n, n-bistar, comb $P_n \odot K_1$, graph $L_n \odot K_1$, wheel graph W_n, flower graph Fl_n, some splitting graphs, multiple of graphs, the square graph of P_n, C_n, $B_{n,n}$, H-graph and $H \odot mK_1$ fan F_n, double fan $D(F_n)$, double wheel $D(W_n)$, closed helm CH_n, total graph of cycle $T(C_n)$, total graph of path $T(P_n)$, armed crown $C_n \Theta P_m$, generalized peterson graph $GP(n,2)$ are analytic odd mean graphs in [**2**], [**3**] and [**4**].

We use the following definitions in the subsequent section.

Definition 1.1. Let $G(V,E)$ be a graph with p vertices and q edges. We say G is an analytic odd mean graph if there exist an injective function f from the vertex set to $0, 1, 3, 5, \ldots, 2q-1$ in such a way that when each edge $e = uv$ such that $f(u) < f(v)$ is labeled with $f^*(uv) = \lceil f(v)^2 - (f(u)+1)^2/2 \rceil$ if $f(u) \neq 0$ and $f^*(uv) = \lceil f(v)^2/2 \rceil$ if $f(u) = 0$ all edge labels are odd and distinct. In this case f is called an analytic odd mean labeling.

Definition 1.2. Let$Q(n)$ be the quadrilateral snake obtained from the path $v_1, v_2, \ldots, v_{n+1}$ by joining v_i and v_{i+1} to the new vertices u_i and w_i for $1 \leqslant i \leqslant n$. That is, every edge of a path is replaced by a cycle C_4.

Definition 1.3. Let $DQ(n)$ be the quadrilateral snake obtained from the path $v_1, v_2, v_3, \ldots, v_{n+1}$. The double quadrilateral snake $DQ(n)$ is obtained from $Q(n)$ by adding the vertices $s_1, s_2, s_3, \ldots, s_n; t_1, t_2, t_3, \ldots, t_n$ and the edges $v_i s_i$, $t_i v_{i+1}, s_i t_i$ for $1 \leqslant i \leqslant n$.

Definition 1.4. The splitting graph $spl(G)$ is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 1.5. The graph $P_n(1,2,3,\ldots,n)$ is a graph obtained from a path of vertices $v_1, v_2, v_3, \ldots, v_n$ by joining i pendent vertices at each i^{th} vertex $1 \leqslant i \leqslant n$. The pendent vertices are labeled as u_{ij} for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant i$.

Definition 1.6. The fire cracker is constructed as follows: Let $a_0, a_1, a_2, \ldots, a_{k-1}$ be the vertices of the path P_k and b_j be the vertex adjacent to a_j for $1 \leqslant j \leqslant k$. Let $b_{j1}, b_{j2}, b_{j3}, \ldots, b_{jn}$ be the pendent vertices adjacent to b_j for $1 \leqslant j \leqslant k$.

Definition 1.7. The coconut tree is having the vertices $v_0, v_1, v_2, \ldots, v_i$ of a path $(i \geqslant 1)$ and the pendent vertices $v_{i+1}, v_{i+2}, v_{i+3}, \ldots, v_{i+n}$, being adjacent with v_0.

Definition 1.8. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the the i^{th} copy of G_2 for $1 \leqslant i \leqslant p$.

2. Main Results

In this section we prove that quadrilateral snake $Q(n)$, double quadrilateral snake $DQ(n)$, coconut tree, fire cracker graph and some special star graphs, splitting graph $spl(G)$, $P_n(1,2,3,\cdots n)$ and the graph $C_k \odot \bar{K}_n$ are analytic odd mean graphs.

THEOREM 2.1. *The splitting graph $spl(P_n)$ is an analytic odd mean graph.*

PROOF. Let $V(G) = \{v_i, v_i' : 0 \leqslant i \leqslant n-1\}$ and

$$E(G) = \{v_{i-1}v_i, v_{i-1}'v_i, v_{i-1}v_i' : 1 \leqslant i \leqslant n-1\}.$$

Now $|V(G)| = 2n$ and $|E(G)| = 3n-3$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 6n-7\}$ by

$f(v_0) = 0, f(v_i) = 2i-1$ for $1 \leqslant i \leqslant n-1$ and $f(v_{i-1}') = 2n-3+2i$ for $1 \leqslant i \leqslant n$.

Let f^* be the induced edge labeling of f. The induced edge labels are as follows:
$f^*(v_{i-1}v_i) = 2i-1$ for $1 \leqslant i \leqslant n-1$
$f^*(v_{i-1}'v_i) = 2n(n-3) + 2i(2n-3) + 5$ for $1 \leqslant i \leqslant n-1$
$f^*(v_iv_{i+1}') = 2n(n+1) + 2i(2n+1) + 1$ for $1 \leqslant i \leqslant n-2$ and
$f^*(v_0v_1') = 2n^2 + 2n + 1$.
Clearly all the edge labels are odd. We observe that the edge labels $v_{i-1}v_i$ increase by 2 from 1 to 2n-3 as i increases from 0 to n-1. Also the edge labels of $v_{i-1}'v_i$ increase by 4n-6 from $2n^2-2n-1$ to $6n^2-16n+11$ as i increases from 1 to n-1 and the edge labels of v_iv_{i+1}' increase by 4n+2 from $2n^2+2n+1$ to $6n^2-4n-3$ as i increases from 1 to n-2. Moreover, $f^*(v_{i-1}'v_i) \neq f^*(v_iv_{i+1}')$ so all the edge labels are distinct. Hence the splitting graph *spl*(P_n) admits an analytic odd mean labeling. An analytic odd mean labeling of splitting graph *spl*(P_6) is shown in Figure 1. □

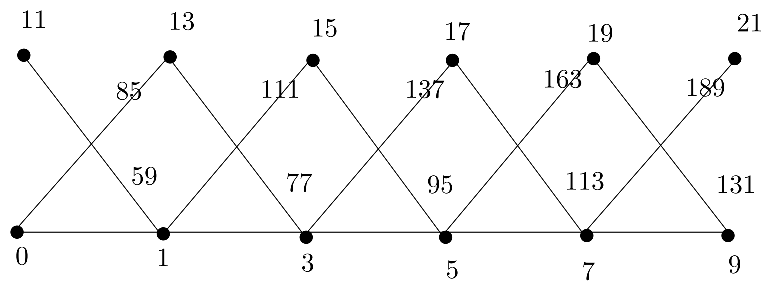

Figure 1

THEOREM 2.2. *The splitting graph $spl(C_n)$ is an analytic odd mean graph.*

PROOF. Let $V(G) = \{v_i, v_i' : 0 \leqslant i \leqslant n-1\}$ and

$$E(G) = \{v_{i-1}v_i : 1 \leqslant i \leqslant n-1\} \cup \{v_i'v_{i+1} : 1 \leqslant i \leqslant n-2\}$$
$$\cup\{v_i'v_{i-1} : 1 \leqslant i \leqslant n-1\}\{v_{n-1}'v_0, v_0'v_{n-1}\}.$$

Now $|V(G)| = 2n$ and $|E(G)| = 3n$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 6n-1\}$ by

$$f(v_0) = 0 \text{ and } f(v_i) = 2i-1 \text{ for } 1 \leqslant i \leqslant n-1,$$
$$f(v_i') = 2n+2i-1 \text{ for } 0 \leqslant i \leqslant n-1.$$

Let f^* be the induced edge labeling of f. The induced edge labels are as follows:

$f^*(v_{i-1}v_i) = 2i - 1$ for $1 \leqslant i \leqslant n-1$
$f^*(v_i'v_{i+1}) = 2n(n-1) + 2i(2n-3) - 1$ for $1 \leqslant i \leqslant n-2$
$f^*(v_i'v_{i-1}) = 2n(n-1) + 2i(2n+1) - 1$ for $1 \leqslant i \leqslant n-1$
$f^*(v_{n-1}'v_0) = 8n^2 - 12n + 5$ and
$f^*(v_{n-1}v_0') = 2n - 1$.

It is easy to show that the edge labels are odd and distinct. Hence the splitting graph $spl(C_n)$ admits an analytic odd mean labeling. An analytic odd mean labeling of splitting graph $spl(C_8)$ is shown in Figure 2.

□

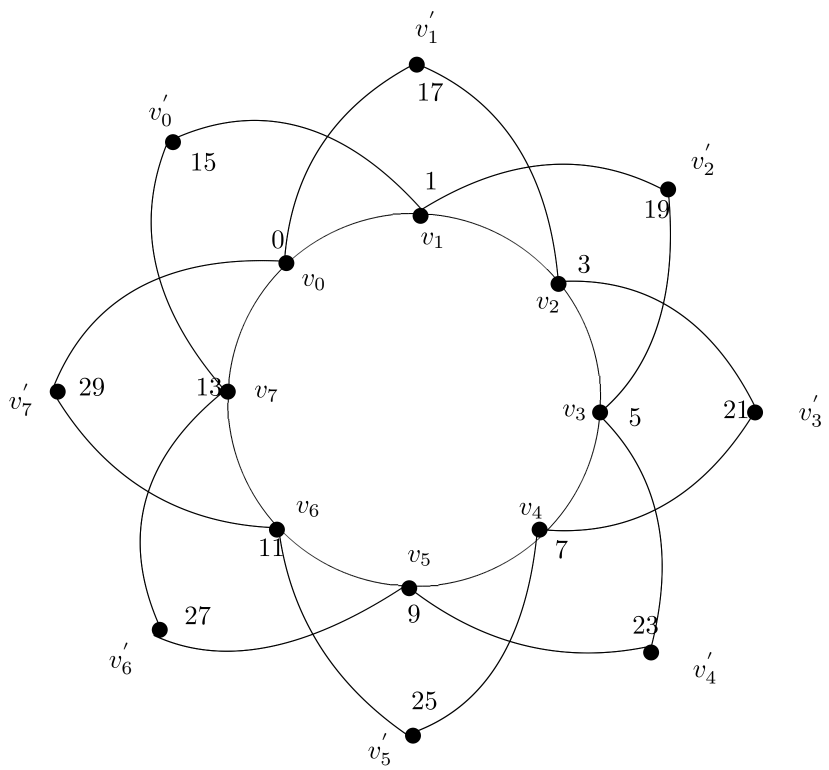

Figure 2

THEOREM 2.3. *The splitting graph $spl(K_{m,n})$ for any integer $n \geqslant m \geqslant 1$ is an analytic odd mean graph.*

PROOF. Let
$V(G) = \{u_i, u_i', u_{n+j}, u_{n+j}' : \ 0 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant m\}$ and
$E(G) = \{u_iu_{n+j}, u_iu_{n+j}', u_i'u_{n+j} : \ 0 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant m\}$

be the vertex set and edge set of the graph $G = spl(K_{m,n})$. Now $|V(G)| = 2(m+n)$ and $|E(G)| = 3mn$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 6mn-1\}$ by

$f(u_0) = 0$,
$f(u_i) = 2i - 1$ for $1 \leqslant i \leqslant n-1$ and
$f(u_i') = 2n + 2i - 1$ for $0 \leqslant i \leqslant n-1$.

We label m vertices u_{n+j} by 6mn-1,6mn-3,6mn-5,6mn-2m+1. That is $f(u_{n+j}) = 6mn - 2j + 1$ for $1 \leqslant j \leqslant m$.

Also we label m vertices u'_{n+j} by 6mn-2m-1,6mn-2m-3,6mn-2m-5,6mn-4m+1. That is $f(u'_{n+j}) = 6mn - 2m - 2j + 1$ for $1 \leqslant j \leqslant m$. Then the induced edge labels are as follows:

$f^*(u_{n+j}u_i) = [(6mn-2j+1)^2 - (2i)^2 + 1]2 = 6mn(3mn+1) + 1 - 2j(6mn + 1) + 2j^2 - 2i^2$ for $0 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant m$.

$f^*(u'_{n+j}u_i) = [(6mn-2m-2j+1)^2 - (2i)^2 + 1]2 = [(6mn-2m+1)^2 + 1] \div 2 - 2j(6mn-2m+1) + 2j^2 - 2i^2)$ for $0 \leqslant i \leqslant n-1 \& 1 \leqslant j \leqslant m$.

$f^*(u_{n+j}u_i') = [(6mn-2j+1)^2 - (2n+2i)^2 + 1]2 = 6mn(3mn+1) + 1 - 2j(6mn + 1) + 2j^2 - 2i^2 - 2n^2 - 4ni$ for $0 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant m$.

We observe that the vertices u_{n+1} to u_{n+m} and $u_0, u_1, \ldots, u_{n-1}, u_0', u_1', \ldots, u_{n-1}'$ induce $K_{m,2n}$ whereas the vertices u_0 to u_{n-1} and $u_{n+1}, u_{n+2}, \ldots, u_{n+m}, u'_{n+1}, u'_{n+2}, \ldots, u'_{n+m}$ induced a $K_{n,2m}$. By an argument similar to that for $K_{m,n}$, we can see the labeling is analytic odd mean labeling . The analytic odd mean labeling of splitting graph $spl(K_{3,4})$ is shown in Figure 3.

□

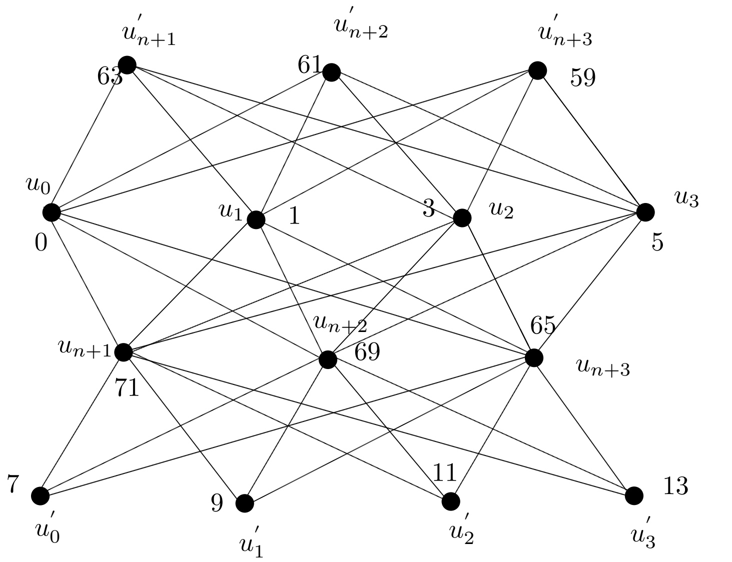

Figure 3

THEOREM 2.4. *The graph $P_n(1, 2, 3, \ldots, n)$ is an analytic odd mean graph.*

PROOF. Let $G = P_n(1, 2, 3, \ldots n)$. Let
$V(G) = \{v_i, v_i^j : 0 \leqslant i \leqslant n-1 \text{ and } 1 \leqslant j \leqslant i+1\}$ and

$E(G) = \{v_{i-1}v_i : 1 \leqslant i \leqslant n-1\} \cup \{v_i v_i^j : 0 \leqslant i \leqslant n-1 \text{ and } 1 \leqslant j \leqslant i+1\}$.

Now $|V(G)| = n + n(n+1) \div 2$ and $|E(G)| = n - 1 + n(n+1) \div 2$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, n^2 + 3n - 3\}$ by

$f(v_0) = 0, f(v_i) = 2i - 1$ for $1 \leqslant i \leqslant n-1, f(v_0^1) = 2n - 1$ and

$f(v_i^j) = 2n - 3 + 2(i+j) + \sum_{k=1}^{i-1} 2(i-k)$ for $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant i+1$.

Let f^* be the induced edge labeling of f. The induced edge labels are as follows:

$f^*(v_{i-1}v_i) = 2i - 1$ for $1 \leqslant i \leqslant n-1$

$f^*(v_0 v_0^1) = 2n^2 - 2n + 1$ and

$f^*(v_i v_i^j) = 2(n^2 - i^2) - 6n + 5 + 2[(i+j) + \sum_{k=1}^{i-1}(i-k)][(i+j) + \sum_{k=1}^{i-1}(i-k) + 2n - 3]$ for $1 \leqslant j \leqslant m$ and $1 \leqslant i \leqslant n-1$.

Clearly the edge labels are odd and We observe that the edge labels are increase by 2 as i increases from 0 to n-1. For fix i, the difference of $f^*(v_i v_i^j)$ and $f^*(v_i v_i^{j+1})$ is $4[i + j + \sum_{k=1}^{i-1}(i-k) + n - 1]$. So all the edge labels are distinct. Hence the graph G admits an analytic odd mean labeling. An analytic odd mean labeling of $P_5(1, 2, 3, \ldots, 5)$ is shown in Figure 4.

□

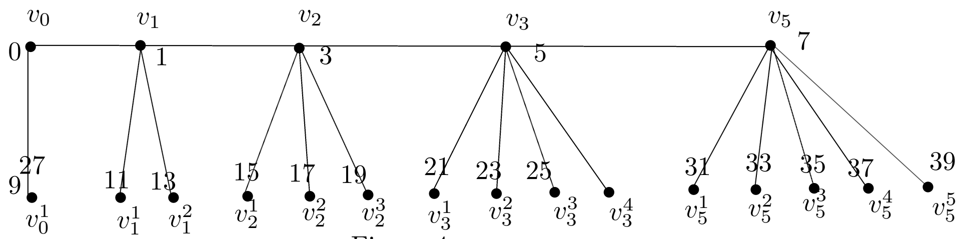

Figure 4

Theorem 2.5. *The quadrilateral snake $Q(n)$ is an analytic odd mean graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_{n+1}; u_1, u_2, u_3, \ldots, u_n; w_1, w_2, w_3, \ldots, w_n$ be the vertices of $V(Q_n)$ and $E(W_n) = \{v_i v_{i+1}, v_i u_i, u_i w_i, w_i v_{i+1} : 1 \leqslant i \leqslant n\}$. Let $G = Q(n)$ and hence $|V(G)| = 3n + 1$ and $|E(G)| = 4n$. We define f on the vertex set of Q_n as follows :

$f(v_i) = 2i - 1$ for $1 \leqslant i \leqslant n+1$

$f(u_i) = 2n + 1 + 2i$ for $1 \leqslant i \leqslant n$

$f(w_i) = 4n + 1 + 2i$ for $1 \leqslant i \leqslant n$.

Let f^* be the induced edge labeling of f. The induced edge labels are as follows :

$f^*(v_i v_{i+1}) = 2i + 1$ for $1 \leqslant i \leqslant n$

$f^*(v_i u_i) = 2n(n+1) + 2i(2n+1) + 1$ for $1 \leqslant i \leqslant n$

$f^*(u_i w_i) = 6n^2 + 2i(2n-1) - 1$ for $1 \leqslant i \leqslant n$ and

$f^*(w_i v_{i+1}) = 4n(2n+1) + 2i(4n-1) - 1$ for $1 \leqslant i \leqslant n$.

Clearly the edge labels are odd and we observe that the edge labels of path increase by 2 , the edge labels of $v_i u_i$ increase by 4n+2, the edge labels of $u_i w_i$ increase by 4n-2 and the edge labels of $w_i v_{i+1}$ increase by 8n-2 as i increases. So all the edge labels are distinct. Therefore f is an analytic odd mean labeling and hence $Q(n)$ is an analytic odd mean graph. An analytic odd mean labeling of $Q(5)$ is shown in Figure 5.

□

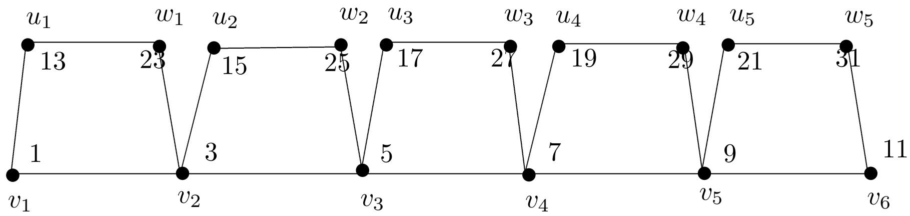

Figure 5

THEOREM 2.6. *The double quadrilateral snake $D(Q(n))$ is an analytic odd mean graph.*

PROOF. Let
$v_1, v_2, v_3, \ldots, v_{n+1}; u_1, u_2, u_3, \ldots, u_n; w_1, w_2, w_3, \ldots, w_n; s_1, s_2, s_3, \ldots, s_n; t_1, t_2, t_3, \ldots, t_n$ be the vertices of $V(Q_n)$ .Let $G = D(Q(n))$. Hence $|V(G)| = 5n + 1$ and $|E(G)| = 7n$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 14n - 1\}$ by

$f(v_i) = 2i - 1$ for $1 \leqslant i \leqslant n + 1$
$f(u_i) = 2n + 1 + 2i$ for $1 \leqslant i \leqslant n$
$f(w_i) = 4n + 1 + 2i$ for $1 \leqslant i \leqslant n$
$f(s_i) = 6n + 1 + 2i$ for $1 \leqslant i \leqslant n$ and
$f(t_i) = 8n + 1 + 2i$ for $1 \leqslant i \leqslant n$.

Let f^* be the induced edge labeling of f. Then the induced edge labels are as follows:

$f^*(v_iv_{i+1}) = 2i + 1$ for $1 \leqslant i \leqslant n$
$f^*(v_iu_i) = 2n(n + 1) + 2i(2n + 1) + 1$ for $1 \leqslant i \leqslant n$
$f^*(u_iw_i) = 6n^2 + 2i(n - 1) - 1$ for $1 \leqslant i \leqslant n$
$f^*(u_is_i) = 6n(3n + 1) + 2i(6n + 1) + 1$ for $1 \leqslant i \leqslant n$
$f^*(w_iv_{i+1}) = 4n(2n + 1) + 2i(4n - 1) - 1$ for $1 \leqslant i \leqslant n - 1$
$f^*(t_iv_{i+1}) = 8n(4n + 1) + 2i(8n - 1) - 1$ for $1 \leqslant i \leqslant n$ and
$f^*(s_it_i) = 2n(7n - 2) + 2i(2n - 1) - 1$ for $1 \leqslant i \leqslant n$.

We observe that the edge labels are odd and distinct. Therefore f is an analytic odd mean labeling and hence $D(Q(n))$ is an analytic odd mean graph. An analytic odd mean labeling of $D(Q(4))$ is shown in Figure 6.

□

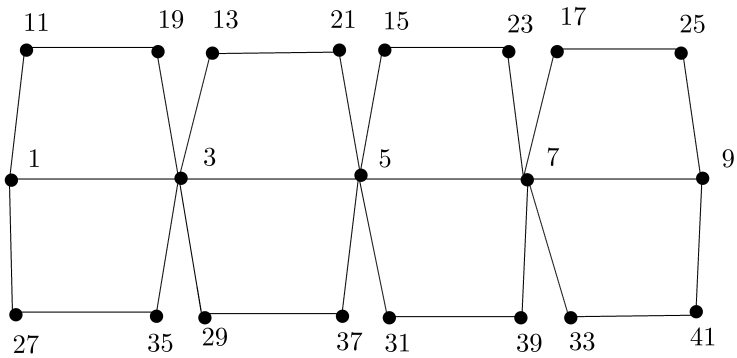

Figure 6

THEOREM 2.7. *The graph $C_k \odot \bar{K_n}$ is an analytic odd mean graph.*

PROOF. Let $G = C_k \odot \bar{K_n}$. Let

$$V(G) = \{v_i, v_i^j : \ 0 \leqslant i \leqslant k-1 \,\text{and}\, 1 \leqslant j \leqslant n\}$$

and

$$E(G) = \{v_i v_{i+1} : \ 0 \leqslant i \leqslant k-2\} \cup \{v_i v_i^j : \ 0 \leqslant i \leqslant k-1 \,\text{and}\, 1 \leqslant j \leqslant n\} \cup \{v_{k-1} v_0\}.$$

Then there are k(n+1) vertices and $deg(v_i) = n+2$ for $0 \leqslant i \leqslant k-1$. We define f on the vertex set as follows :

$f(v_0) = 0, f(v_i) = 2i-1$ for $1 \leqslant i \leqslant k-1$and

$f(v_i^j) = 2k-3+2j+2ni$ for $0 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant n.$

Let f^* be the induced edge labeling of f.Then the induced edge labels are as follows:

$f^*(v_i v_{i+1} = 2i-1$ for $1 \leqslant i \leqslant k-2$

$f^*(v_{k-1} v_0) = 2k^2 - 6k + 5$

$f^*(v_i v_i^j) = 2k(k-3)+2(2k+ni-3)(ni+j)+2nij+5+2(j^2-i^2)$ for $1 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant n$ and

$f^*(v_0 v_0^j) = 2(k+j)(k+j-3)+5$ for $1 \leqslant j \leqslant n.$

It can be easily verified that f is an analytic odd mean labeling and hence graph $C_k \odot \bar{K_n}$ is an analytic odd mean graph. An analytic odd mean labeling of $C_5 \odot \bar{K_3}$ is shown in Figure 7.

□

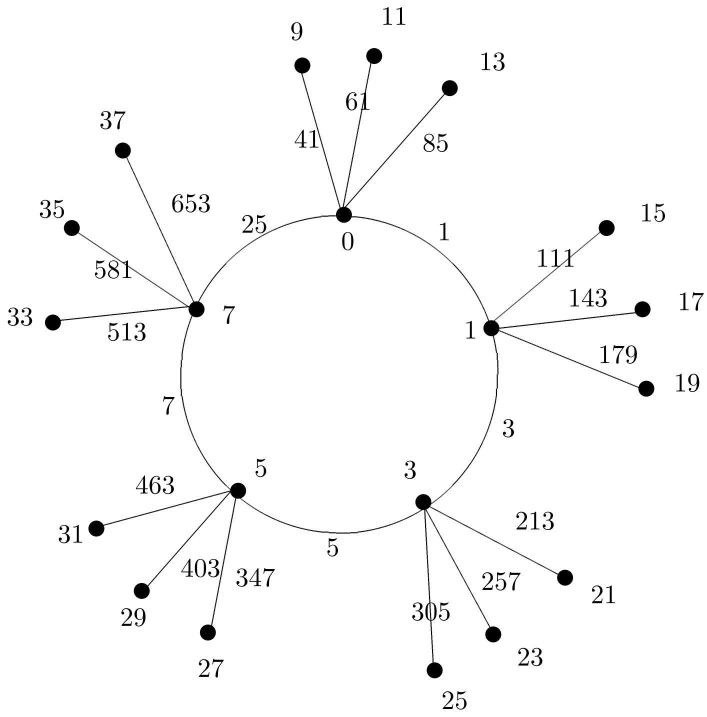

Figure 7

THEOREM 2.8. *Let $S_1, S_2, S_3, \ldots S_k$ be the disjoint copies of the k-star $K_{1,k}$ with vertex set $V(S_i) = \{v_i, v_{i,r} : 1 \leqslant r \leqslant k\}$ and the edge set $E(S_i) = \{v_iv_{i,r} : 1 \leqslant r \leqslant k\}$ for $1 \leqslant i \leqslant k$ and G be the graph obtained by joining a new vertex v with $v_{1,1}, v_{2,1}, v_{3,1}, \ldots v_{k,1}$. Then G is an analytic odd mean graph.*

PROOF. Let $V(G) = \{v, v_i, v_{i,r} : 1 \leqslant i, r \leqslant k\}$ and $E(G) = \{vv_{i,1}, v_iv_{i,r} : 1 \leqslant i, r \leqslant k\}$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 2k^2 + 2k - 1\}$ by

$f(v) = 0, f(v_i) = 2i - 1$ for $1 \leqslant i \leqslant k$ and

$f(v_{i,r}) = 2ki + 2r - 1$ for $1 \leqslant i, r \leqslant k$.

Then the induced edge labeling of f are

$f^*(vv_{i,1}) = 2k^2i^2 + 2ki + 1$ for $1 \leqslant i \leqslant k$

$f^*(v_iv_{i,r}) = 2i(i(k^2 - 1) + kr) + 2(r - 1)(ki + r) + 1$ for $1 \leqslant i, r \leqslant k$.

Clearly the edge labels are odd and distinct. Hence the graph admits G an analytic odd mean labeling. An analytic odd mean labelling of the above graph G with $k = 6$ is shown in Figure 8.

□

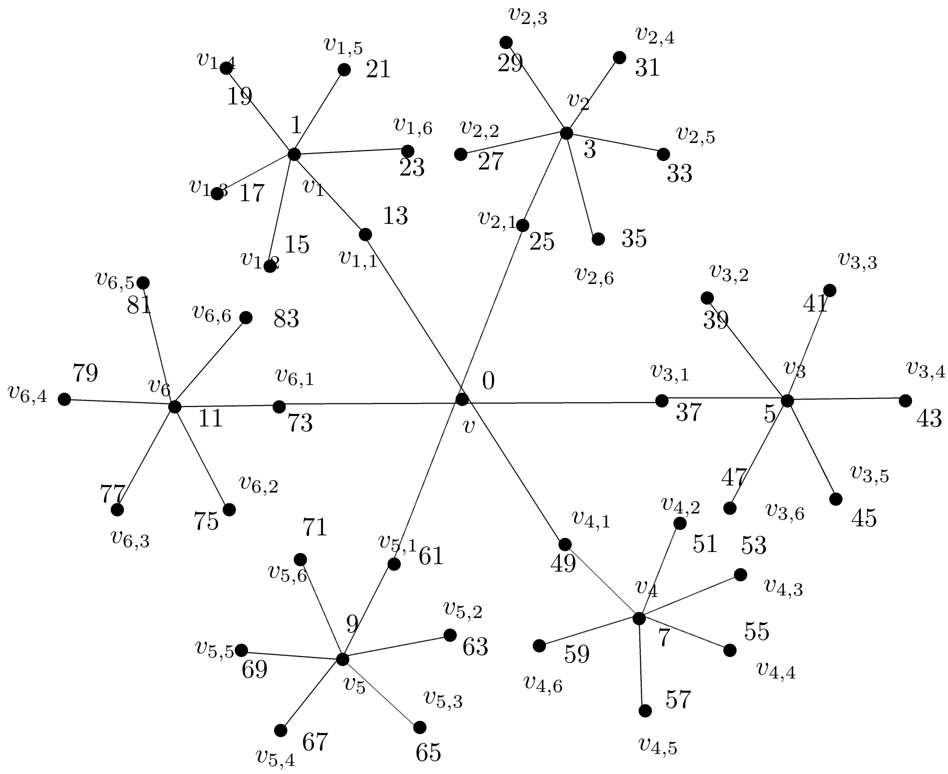

Figure 8

THEOREM 2.9. *Let $S_1, S_2, S_3, \ldots, S_{k+1}$ be the disjoint copies of the k-star $K_{1,k}$ with vertex set $V(S_i) = \{v_i, v_i^r :\ 1 \leqslant r \leqslant k\}$ and the edge set $E(S_i) = \{v_i v_i^r :\ 1 \leqslant r \leqslant k\}$ for $1 \leqslant i \leqslant k+1$. Let G be the graph obtained by joining a new vertex v to the centre of each $k+1$ stars. Then G is an analytic odd mean graph.*

PROOF. Let $V(G) = \{v, v_i, v_i^r :\ 1 \leqslant i \leqslant k+1 \text{ and } 1 \leqslant r \leqslant k\}$ and $E(G) = \{vv_i, v_i v_i^r :\ 1 \leqslant i \leqslant k+1 \text{ and } 1 \leqslant r \leqslant k\}$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 2k^2 + 4k + 1\}$ by

$f(v) = 0, f(v_i) = 2i - 1$ for $1 \leqslant i \leqslant k+1$ and

$f(v_i^r) = 2ki + 2r + 1$ for $1 \leqslant i \leqslant k+1$ and $1 \leqslant r \leqslant k$.

Then the induced edge labeling of f are

$f^*(vv_i) = 2i - 1$ for $1 \leqslant i \leqslant k+1$ and

$f^*(v_i v_i^r) = 2i(i(k^2 - 1) + kr) + 2(r-1)(ki + r) + 1$ for $1 \leqslant i \leqslant k+1$ and $1 \leqslant r \leqslant k$.

Clearly the edge labels are odd and distinct. Hence the graph admits G an analytic odd mean labeling. An analytic odd mean labeling of the graph G with $k = 4$ is shown in Figure 9.

□

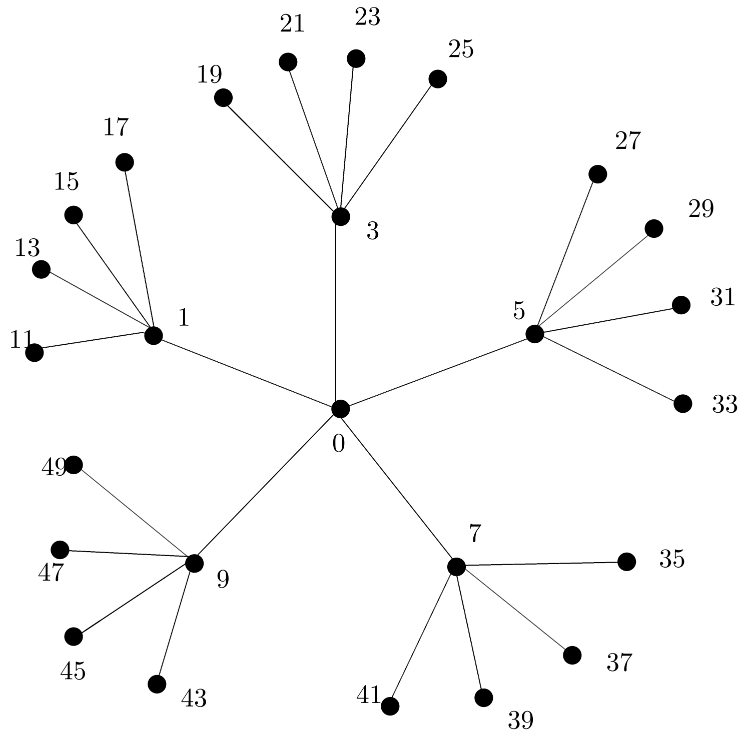

Figure 9

THEOREM 2.10. *The fire cracker is an analytic odd mean graph.*

PROOF. Let
$V(G) = \{a_i, b_j, b_j^k :\ 0 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m\}$ and
$E(G) = \{a_i a_{i+1} :\ 0 \leqslant i \leqslant n-2\} \cup \{a_i b_j :\ 0 \leqslant i \leqslant n-2\ 1 \leqslant j \leqslant n\} \cup \{b_j b_j^k :\ 1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m\}$.

We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 2n + nm - 1\}$ by
$f(a_0) = 0, f(a_i) = 2i - 1$ for $1 \leqslant i \leqslant n-1$
$f(b_j) = 2n - 3 + 2j$ for $1 \leqslant j \leqslant n$ and
$f(b_j^k) = 4n - 3 + 2m(j-1) + 2k$ for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$.

Let f^* be the induced edge labeling of f.Then the induced edge labels are as follow:

$f^*(a_{i-1}a_i) = 2i - 1$ for $1 \leqslant i \leqslant n$

$f^*(a_0 b_1) = 2n^2 - 2n + 1$

$f^*(a_i b_j) = 2(n-1)(n+i) + 2ni + 1$ for $0 \leqslant i < j \leqslant n$ $f^*(b_j b_j^k) = 2(3n^2 + k^2 - j^2) - 4n(j+2) + 2k(4n-3) + 2m(j-1)[m(j-1) + 4n - 3 + 2k] + 4j + 3$ for $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$.

Clearly the edge labels are odd and distinct. Hence the fire cracker admits analytic odd mean labeling. An analytic odd mean labeling of fire cracker for $n, m = 3$ is shown in Figure 10.

□

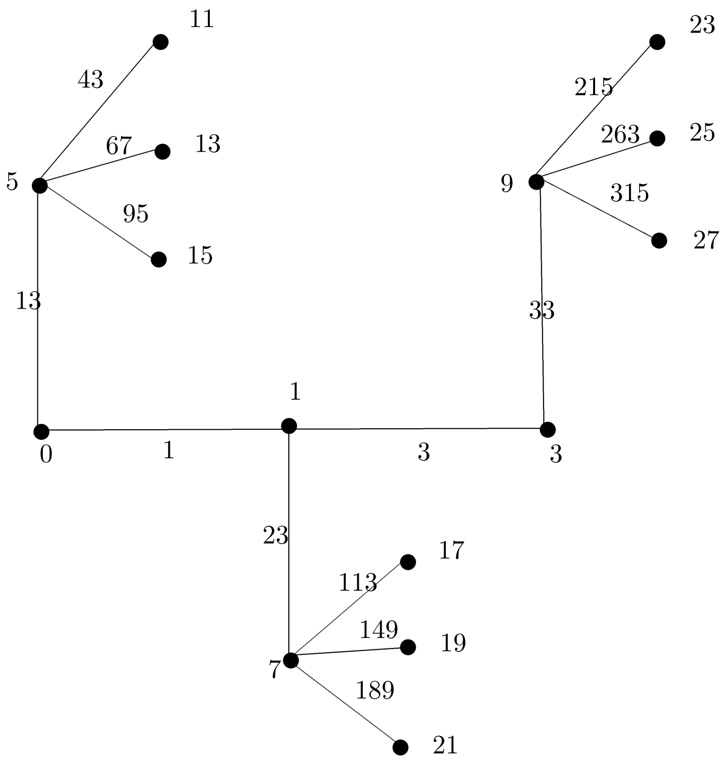

Figure 10

Theorem 2.11. *The coconut tree G is an analytic odd mean graph.*

Proof. Let
$V(G) = \{v_i, v_{n+j} : 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m\}$ and
$E(G) = \{v_{i-1}v_i : 1 \leqslant i \leqslant n-1\} \cup \{v_0 v_{n+j} : 1 \leqslant j \leqslant m\}$.
Now $|V(G)| = n + m$ and $|E(G)| = m + n - 1$. We define an injective map $f : V(G) \to \{0, 1, 3, 5, \ldots, 2(m+n) - 3\}$ by
$f(v_0) = 0, f(v_i) = 2i - 1$ for $1 \leqslant i \leqslant n-1$ and
$f(v_{n+j}) = 2n - 3 + 2j$ for $1 \leqslant j \leqslant m$.
Let f^* be the induced edge labeling of f. The induced edge labels are as follows:
$f^*(v_{i-1}v_i) = 2i - 1$ for $1 \leqslant i \leqslant n-1$ and
$f^*(v_0 v_{n+j}) = 2n(n-3) + 2j(2n-3) + 2j^2 + 5$ for $1 \leqslant j \leqslant m$.
Clearly the edge labels are odd and we observe that the edge labels of path increase by 2 as i increases and the pendent edge labels increase by

$$4n, 4(n+1), 4(n+2), 4(n+3), \ldots$$

as j increases from 1 to m. So all the edge labels are distinct. Hence the coconut tree admits an analytic odd mean labeling. An analytic odd mean labeling of coconut tree with $n = 6, m = 9$ is shown in Figure 11.

□

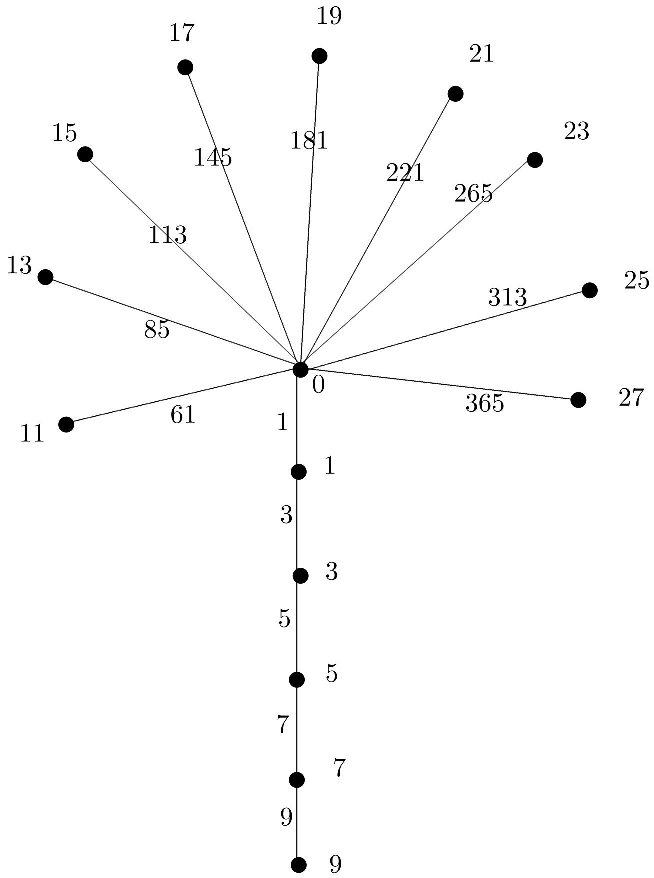

Figure 11

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