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# LACEABILITY PROPERTIES IN EDGE TOLERANT SHADOW GRAPHS

**P. Gomathi and R. Murali**

Abstract. A connected graph $G$ is termed Hamiltonian-$t$-laceable ($t$*-laceable) if there exists in it a Hamiltonian path between every pair of distinct vertices (at least one pair of vertices) $u$ and $v$ with the property $d(u, v) = t, 1 \leqslant t \leqslant diam(G)$, where $t$ is a positive integer. The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$ and joining each vertex $u$ in $G'$ to the neighbors of the corresponding $u'$ in $G''$. In this paper, we establish laceability properties in the edge tolerant shadow graph of the path graph $P_n$ and the wheel graph $W_{1,n}$.

## 1. Introduction

Let $G$ is a finite, simple, connected and undirected graph. Let $u$ and $v$ be two vertices in $G$. The distance between $u$ and $v$ denoted by $d(u, v)$ is the length of a shortest path in $G$. $G$ is hamiltonian laceable if there exists in it a hamiltonian path for every pair of vertices at an odd distance. $G$ is hamiltonian-$t$-laceable ($t$*-laceable) if there exists in $G$ a hamiltonian path between every pair of vertices (at least one pair of vertices) $u$ and $v$ with the property $d(u, v) = t, 1 \leqslant t \leqslant diam(G)$, where $t$ is a positive integer. Throughout this paper, $P_n$ and $W_{1,n}$ denote the path graph and wheel graph with $n$ and $n + 1$ vertices respectively.

Laceability in the brick products of odd cycles was explored by Alspach et.al. in [**1**]. A characterization for a 1-connected graph to be hamiltonian-$t$-laceable for $t = 1, 2$ and 3 is given in [**9**] and this was extended to $t = 4$ and

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5 by Thimmaraju and Murali [**14**]. Leena Shenoy [**13**] studied hamiltonian laceability properties in product graphs involving cycles and paths. More results in the laceability properties of product graphs can be found in [**5**], [**6**], [**7**], [**10**], [**11**] and [**4**].

The concept of total magic cordial labelling of shadow graphs was studied by Parameswari [**12**]. Vaidya and Pandit [**15**] explored the edge domination of shadow graphs. Lekha and Vaidya showed that the shadow graph of path is an odd graceful graph, [**16**]. Rainbow coloring of shadow graphs was explored by Arputhamarya in [**2**].

In this paper, we explore the laceability properties of the shadow graph of the path graph and wheel graph.

DEFINITION 1.1. The shadow graph of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ of $G'$ to the neighbors of the corresponding vertex $u''$ of $G''$. The shadow graph of $G$ is denoted by $D_2(G)$.

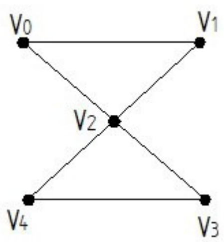


FIGURE 1. Friendship graph $F_2$

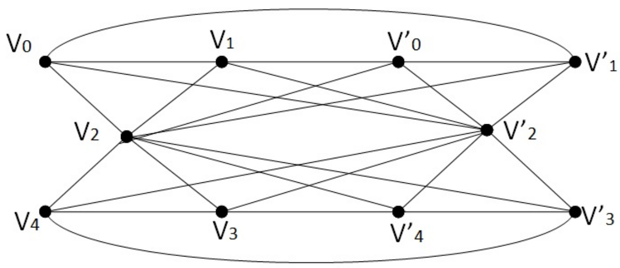


FIGURE 2. Shadow graph $D_2(F_2)$

DEFINITION 1.2. A graph $G^*$ is k-edge fault tolerant with respect to a graph $G$ if the graph obtained by removing any k edges from $G^*$ contains $G$, where k is a positive integer.

DEFINITION 1.3. Let $P$ be a path between the vertices $v_i$ to $v_j$ in a graph $G$ and let $P'$ be a path between the vertices $v_j$ and $v_k$. Then, the path $P \cup P'$ is the path obtained by extending the path $P$ from $v_i$ to $v_j$ to $v_k$ through the common vertex $v_j$ (*i.e.* if $P : v_i....v_j$ and $P' : v_j....v_k$ then $P \cup P' : v_i....v_j.....v_k$)

## 2. The Results

The results related to hamiltonian-laceable properties of the shadow graphs of some classes of graphs are presented in this section.

Following terminologies are defined from the inspiring work of Alspach *et. al.,* [**1**]. We use these terminologies to prove our results.

**Terminologies:**

- $xP[n] = x(x+1)(x+2)......(x+n-1)$.
- $xP^{-1}[n] = x(x-1)(x-2)......(x-n+1)$.
- $xZ[n] = x(x'+1)(x+2)(x'+3).........(x+n-2)(x'+n-1) or (x+n-1)$ (moving from one vertex to another vertex in the adjacent level *i.e.,* moving from left to right).
- $xZ^{-1}[n] = x(x'-1)(x-2)(x'-3).........(x-n+2)(x'-n+1) or (x-n+1)$ (moving from one vertex to another vertex in the adjacent level *i.e.,* moving from right to left).

THEOREM 2.1. *If $n \geqslant 4$ is even, the $2-q$ edge-fault-tolerant graph $G = D_2(P_n)$ is hamiltonian-$t^*$- laceable where $q = \begin{cases} 0 & t \text{ is even} \\ 1 & t \text{ is odd} \end{cases}$ for all $t$ such that $1 \leqslant t < diam(G)$.*

PROOF. Consider two copies of $P_n$, say $P_n'$ and $P_n''$. Let $a_1, a_2, a_3, ....., a_n$ and $a_1', a_2', a_3'......, a_n'$ be the vertices of $P_n'$ and $P_n''$ respectively. Clearly $G$ has $2n$ vertices, $4(n-1)$ edges and $diam(G) = n-1$.

Now in $G$, $d(a_1, a_{t+1}) = t$ and the path

$$P : (a_1)P[t]Z[3]\Big(P[2]Z^{-1}[2]P[2]Z[2]\Big)^{\frac{n-t-q-2}{2}}\Big(a_n', a_n\Big)^{1-q}\Big(P[2]Z^{-1}[2]P[2]\Big)^{q} \Big(a_n', a_1'\Big)P[t]Z[2](a_{t+1})$$

in the $2-q$ edge fault tolerant graph $G^*$, where $q = \begin{cases} 0 & t \text{ is even} \\ 1 & t \text{ is odd} \end{cases}$ is a hamiltonian path between the vertices $a_1$ and $a_{t+1}$.

Hence the proof. □

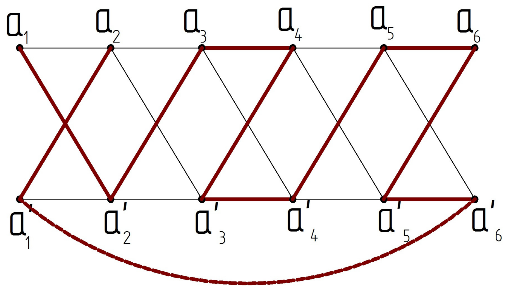


FIGURE 3. Hamiltonian path in $D_2(P_6)$ ($d(a_1, a_2) = 1$)

REMARK 2.1. In $G$, $d(a_1, a_n) = n - 1 = diam(G)$. In this case $G$ is zero edge fault tolerant since the path

$$P : (a_1)\Big(Z[2]P^{-1}[2]Z[2]\Big)\Big(Z[2]P[2]Z^{-1}[2]P[2]\Big)^{\frac{n-2}{2}}(a_n)$$

is a hamiltonian path between the vertices $a_1$ and $a_n$.

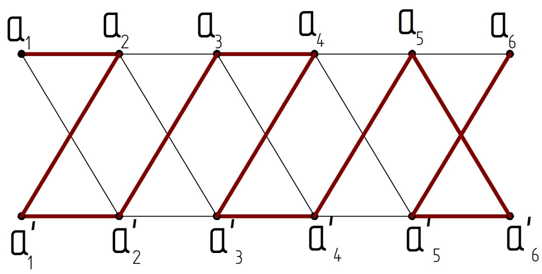


FIGURE 4. Hamiltonian path in $D_2(P_6)$ ($d(a_1, a_6) = 5$)

THEOREM 2.2. *If $n \geqslant 5$ is odd, the $1+q$ edge fault tolerant graph $G = D_2(P_n)$ is hamiltonian-$t^*$-laceable where* $q = \begin{cases} 0 & t \text{ is even} \\ 1 & t \text{ is odd} \end{cases}$ *for all $t$ such that $1 \leqslant t < diam(G)$.*

PROOF. The order of the graph $G$ is same as in Theorem 2.1.

Now in $G$, $d(a_1, a_{t+1}) = t$ and the path

$$P : (a_1)P[t]Z[3]\Big(P[2]Z^{-1}[2]P[2]Z[2]\Big)^{\frac{n-t+q-3}{2}}\Big(a'_n, a_n\Big)^q\Big(P[2]Z^{-1}[2]P[2]\Big)^{1-q}\Big(a'_n, a'_1\Big)P[t]Z[2](a_{t+1})$$

in the $1+q$ edge fault tolerant graph $G^*$, where $q = \begin{cases} 0 & t \text{ is even} \\ 1 & t \text{ is odd} \end{cases}$ is a hamiltonian path between the vertices $a_1$ and $a_{t+1}$.

Hence the proof. □

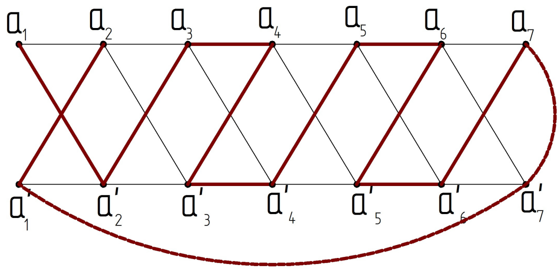


FIGURE 5. Hamiltonian path in $D_2(P_7)$ $(d(a_1, a_2) = 1)$

REMARK 2.2. In $G$, $d(a_1, a_n) = n - 1 = diam(G)$. In this case $G$ is 1-edge fault tolerant since the path

$$P : (a_1)\Big(Z[2]P^{-1}[2]Z[2]\Big)\Big(Z[2]P[2]Z^{-1}[2]P[2]\Big)^{\frac{t-2}{2}}Z[2](a'_n, a_n)(a_{t+1})$$

is a hamiltonian path between the vertices $a_1$ and $a_n$.

In our next result, we show that the shadow graph of the wheel graph $W_{1,n}$ is hamiltonian connected for all $n \geqslant 3$. In this graph, we call the vertex of degree $n-1$ as the center vertex and denote this vertex by $v_c$. The remaining $n-1$ vertices of degree 3 are called the rim vertices.

DEFINITION 2.1. The closure $C(G)$ of $G$ as the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n$ until no such pair remains.

We recall the following results from Bondy and Murty [**3**].

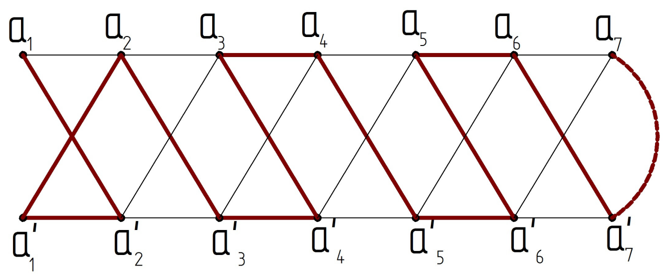

FIGURE 6. Hamiltonian path in $D_2(P_7)$ $(d(a_1, a_7) = 6)$

THEOREM 2.3. [**3**] *Let $G$ be a simple graph with at least 3 vertices. If $u$ and $v$ are distinct non adjacent vertices of $G$ with $d(u) + d(v) \geqslant n$, then $G$ is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

THEOREM 2.4. [**3**] *Let $G$ be a simple graph with at least 3 vertices. If $u$ and $v$ are distinct non adjacent vertices of $G$ with $d(u) + d(v) > n$, then $G$ is Hamiltonian-connected if and only if $G + uv$ is Hamiltonian-connected.*

We shall first prove the following theorems.

THEOREM 2.5. *For $3 \leqslant n \leqslant 6$ the graph $G = D_2(W_{1,n})$ is Hamiltonian-connected.*

PROOF. In $W_{1,n}$, every vertex has three neighbours whereas the center vertex $v_c$ has $(n-1)$ neighbours. After constructing the shadow graph $G$, every vertex is of degree 6 and the bicenter vertices $v_c$ and $v'_c$ are of degree $2n-2$.

From $G$, it is clear that all the corresponding vertices *i.e.,* $v_i$ and $v'_i$ are non adjacent and $deg(v_i) + deg(v'_i) = 12$ for $i \leqslant n$, which satisfies the requirement of the Theorem 2.4. With respect to bicenter vertices also, $deg(v_c) + deg(v'_c) = 4n - 4 > 2n - 2$.

Now, let us construct a closure graph $C(G)$ by iteratively adding edge joining pairs of nonadjacent vertices $u$ and $v$ whose degree sum is atleast $n$ until no such pair remains and this closure graph results in a complete graph with $2n$ vertices.

Since a complete graph is hamiltonian connected, the closure graph is also hamilton-connected.

Thus, $G$ is a hamiltonian connected *i.e.,* hamiltonian-$t$-laceable for all $t$. Hence the proof. □

THEOREM 2.6. *For $n \geqslant 6$ the graph $G = D_2(W_{1,n})$ is Hamiltonian-1-laceable.*

PROOF. Let $v_1$,$v_2$,.....,$v_{n-1}$ be the rim vertices in $W_{1,n}$ (taken in clockwise fashion) with the center vertex as $v_c$. Let $v'_1$,$v'_2$,.....,$v'_{n-1}$ and $v'_c$ be the correponding rim vertices and center vertex repectively of the graph $W'_{1,n}$ in $G$.

CLAIM 2.6.1. *The vertices $v_i$ and $v_{i+1}$ are attainable.*

Let $P_1$ be a path starting at $v_i$. From $v_i$, we visit $v_c$ and come back on the cycle at $v_{n-i}$. Hence the path $P_1$ is $P_1 : v_i - v_c - v_{n+i-1} - ....... - v_{i+2}$ (which is obtained by traversing the cycle of the wheel anticlockwise). By using the shadow edge $(v_{i+2}, v'_c)$, we traverse from the wheel cycle $W_{1,n}$ to its shadow $W'_{1,n}$ and back on the cycle at $v'_{i+1}$.

Consider the path $P_2$ traversing the cycle of the wheel $W'_{1,n}$ clockwise from $v'_{i+1}$ to $v'_i$ that contains all the vertices of the cycle $W'_{1,n}$. Again, using the shadow edge $(v'_i, v_{i+1})$, we move from $W'_{1,n}$ to $W_{1,n}$.

Thus, the hamiltonian path is of the form $P_1 \cup (v_{i+2}, v'_c) \cup P_2 \cup (v'_i, v_{i+1})$.

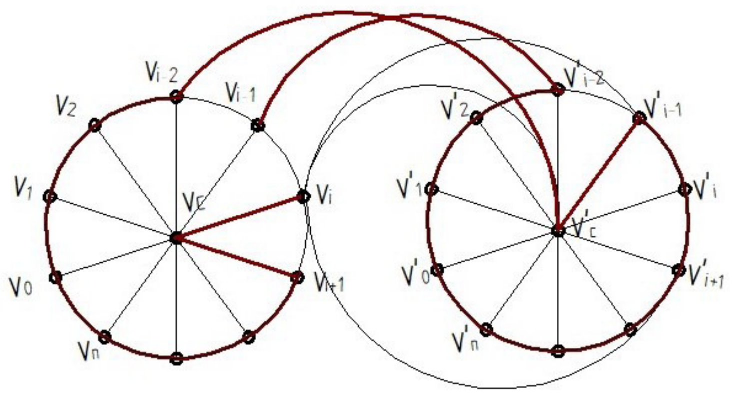


FIGURE 7. Hamiltonian path between the vertices $v_i$ and $v_{i+1}$

CLAIM 2.6.2. *The vertices $v_i$ and $v_c$ are attainable.*

Let $P_1$ be a path traversing cycle of the wheel $W_{1,n}$ clockwise from $v_i$ to $v_{n+i-1}$. By using the shadow edge $(v_{n+i-1}, v'_c)$, we traverse from the wheel cycle $W_{1,n}$ to its shadow $W'_{1,n}$ and back on the cycle at $v'_i$.

Now, $P_2$ is a path on the wheel cycle traversed clockwise from $v'_i$ to $v'_{n+i-1}$ *i.e.,* $P_2 : v'_i - v'_{i+1} .... - v'_{n+i-1}$. Again, using the shadow edge $(v'_{n+i-1}, v_c)$, we move from $W'_{1,n}$ to $W_{1,n}$.

Thus, the hamiltonianwhitevpath of the form $P_1 \cup (v_{n+i-1}, v'_c) \cup P_2 \cup (v'_{n+i-1}, v_c)$.

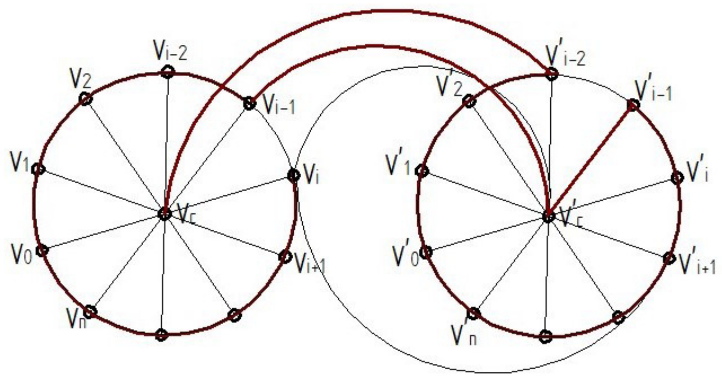


FIGURE 8. Hamiltonian path between the vertices $v_i$ and $v_c$

CLAIM 2.6.3. *The vertices $v_i$ and $v'_{i-1}$ are attainable.*

Let $P_1$ be a path traversing cycle of the wheel $W_{1,n}$ clockwise from $v_i$ to $v_{n+i-1}$ and then visit $v_c$. Using the shadow edge $(v_c, v'_i)$ we move from $W_{1,n}$ to $W'_{1,n}$.

Now $P_2$ is a path traversed clockwise on the cycle $W'_{1,n}$ from $v'_{i+2}$ to $v'_{n+i}$ and then visit $v'_c$ and back on the cycle of the wheel at $v'_{i+1}$ *i.e.,* $P_2 : v'_{i+2} - ... - v'_{n+i} - v'_c - v'_{i+1}$

Thus, the hamiltonian path of the form $P_1 \cup (v_c, v'_{i+2}) \cup P_2$.

Hence the proof. □

THEOREM 2.7. *For $n \geqslant 6$ the shadow graph $G = D_2(W_{1,n})$ is Hamiltonian-2-laceable.*

PROOF. We consider the vertex set of $G$ as in Theorem 2.6.

CLAIM 2.7.1. *The two non adjacent vertices $v_i$ and $v_{i-2}$ are attainable.*

Let $P_1$ be a path traversing cycle of the wheel $W_{1,n}$ anticlockwise from $v_i$ to $v_{i+3}$. From $v_{i+3}$, we visit $v_c$ and come back on the cycle at $v_{i+1}$. Using the shadow edge $(v_{i+1}, v'_c)$ we move from $W_{1,n}$ to $W'_{1,n}$.

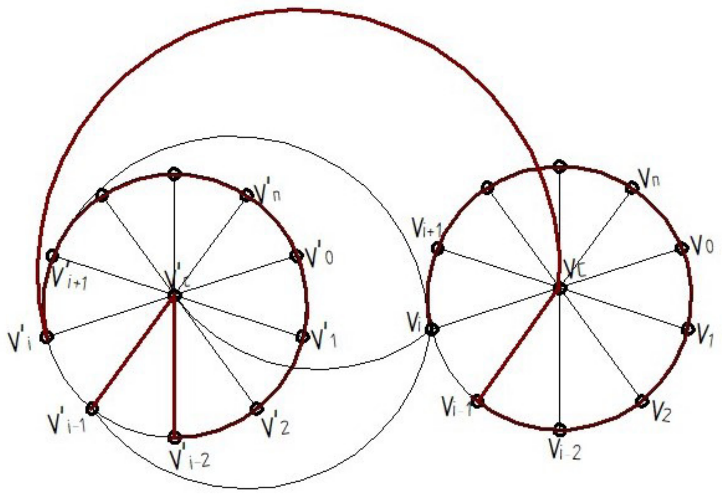


FIGURE 9. Hamiltonian path between the vertices $v_i$ and $v'_{i+1}$

Now $P_2$ is a path traversed on the wheel cycle $W'_{1,n}$ anticlockwise from $v'_i$ to $v'_{i+1}$. Again, using shadow edge $(v'_{i+1}, v_{i+2})$ we move from $W'_{1,n}$ to $W_{1,n}$.

Thus, the hamiltonian path of the form $P_1 \cup (v_{i+1}, v'_c) \cup P_2 \cup (v'_{i+1}, v_{i+2)}$.

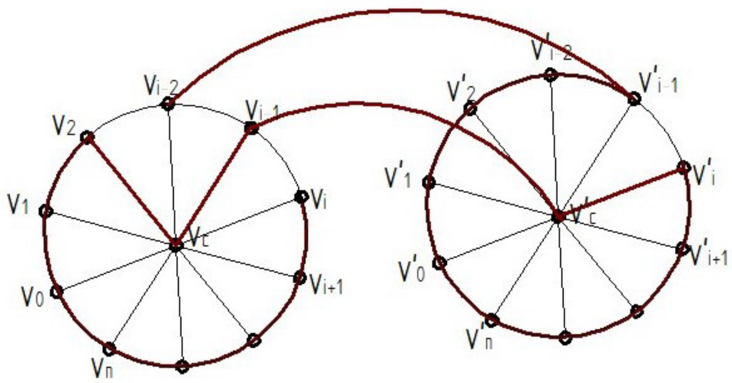


FIGURE 10. Hamiltonian path between the vertices $v_i$ and $v_{i-2}$

CLAIM 2.7.2. *The vertices $v_i$ and $v'_i$ are attainable.*

Let $P_1$ be a path traversing cycle of the wheel $W_{1,n}$ clockwise from $v_i$ to $v_{n+i-1}$ and then visit $v_c$. Using the shadow edge $(v_c, v'_{i+1})$ we move from $W_{1,n}$ to $W'_{1,n}$.

Consider the path $P_2$ traversed on the cycle $W'_{1,n}$ clockwise from $v'_{i+1}$ to $v'_{n+i-1}$ and then visit $v'_c$ and back on cycle at $v'_i$ *i.e.,* $P_2 : v'_{i+1} - ... - v'_{i-1} - v'_c - v'_i$.

Thus, the hamiltonian path of the form $P_1 \cup (v_c, v'_{i+1}) \cup P_2$.

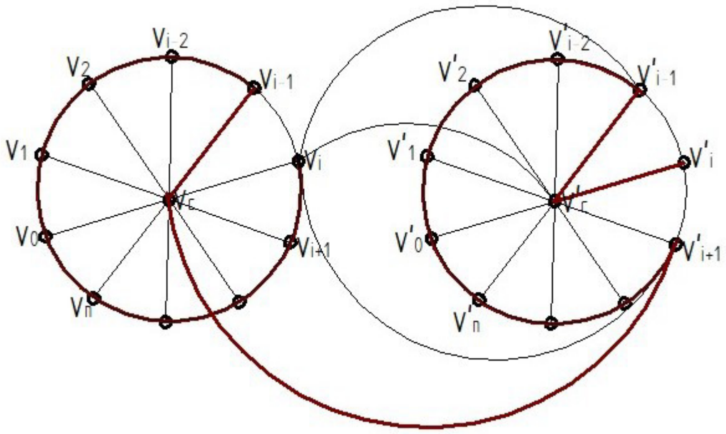


FIGURE 11. Hamiltonian path between the vertices $v_i$ and $v'_i$

CLAIM 2.7.3. *The vertices $v_c$ and $v'_c$ are attainable.*

Let $P_1$ be a path from $v_c$ to $v_i$ and then traverse on the cycle from $v_i$ to $v_{n+i-1}$ clockwise *i.e.,* $P_1 : v_c - v_i - ... - v_{n+i-1}$. By the shadow edge $(v_{n+i-1}, v'_i)$ we traverse from $W_{1,n}$ to $W'_{1,n}$.

Consider the path $P_2$ from $v'_i$ to $v'_{n+i-1}$ traversing clockwise on the cycle of the wheel and then visit $v'_c$ *i.e.,* $P_2 : v'_i - ... - v'_{n+i-1} - v'_c$.

Thus, the hamiltonian path of the form $P_1 \cup (v_{n+i-1}, v'_i) \cup P_2$.

Hence the proof. □

Theorems 2.6 and 2.7 lead to the following result.

THEOREM 2.8. *The graph $D_2(W_{1,n})$ is hamiltonian connected for all values of $n \geqslant 3$.*

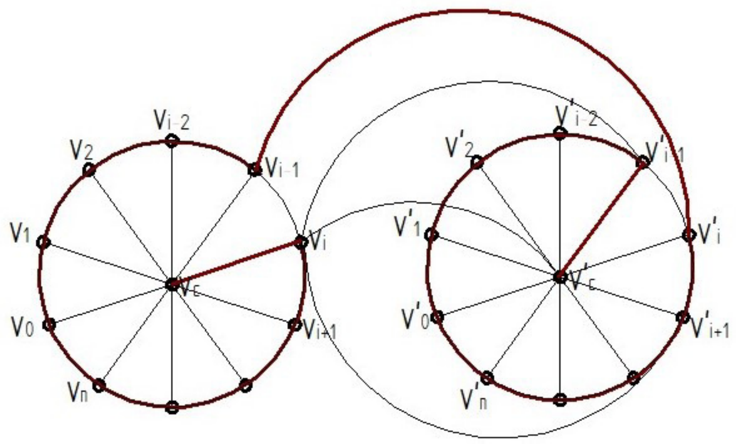


FIGURE 12. Hamiltonian path between the vertices $v_i$ and $v'_c$

## 3. Conclusion

The laceability properties of the shadow graph of path and wheel graph have been studied in this chapter. In particular, it is shown that the shadow graph of the wheel graph $W_{1,n}$ is hamiltonian connected.

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