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RESIDUATED MAPPINGS ON ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of residuated mapping on an Almost Distributive Lattice(ADL) L and its residual. We give certain properties of residuated mappings on ADL's. If P and Q are two ADL's, then we have characterized residuated maps in terms of principal initial segments. If $f^+ : Q \to P$ is a residual of $f : P \to Q$ and $g^+ : R \to Q$ is a residual of $g : Q \to R$, then we prove that $f^+ \circ g^+$ is a residual of $g \circ f$.

1. Introduction

Swamy and Rao [**3**] introduced the concept of an Almost Distributive Lattice [ADL] as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, $p-$rings, biregular rings, associate rings, P_1-rings etc.) on one hand and distributive lattices on the other. In [**2**], Swamy and Murty has discussed certain properties of residuated mappings of a partially ordered set P into another partially ordered set Q.

In this paper, we introduce the concept of a residuated mapping on an ADL and its residual and study their basic properties.

In Section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL. These are taken from Swamy and Rao [**3**] and Rao [**1**].

In Section 3, we give the definition of a residuated map on an ADL and its residual and derive equivalent conditions for an isotone map to be residuated. If P and Q are two ADL's, then we have characterized residuated maps in terms of principal initial segments. We deduce that a map $f : P \to Q$ to be residuated

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it is necessary and sufficient that there exists an isotone $g : Q \to P$ such that $g(y) \wedge x = x$ if and only if $y \wedge f(x) = f(x)$, for any $x \in P, y \in Q$. If $f^+ : Q \to P$ is a residual of $f : P \to Q$ and $g^+ : R \to Q$ is a residual of $g : Q \to R$, then we prove that $f^+ \circ g^+$ is a residual of $g \circ f$.

2. Preliminaries

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. ([**1**]) A relation L on A satisfying the property reflexive, anti symmetric and transitive is called a partial order relation on A.
" $\leqslant$ " is generally used for partial orders. If " $\leqslant$ " is a partial order on A, then we call $(A, \leqslant)$ as a partially ordered set (Poset).

Definition 2.2. ([**1**]). A poset $(L, \leqslant)$ is called a lattice if every subset of L with exactly two elements has supremum and infimum in L. $(L, \leqslant)$ is a lattice $a, b \in L$ if and only if $\{a, b\}$ has supremum and infimum in L. If $(L, \vee, \wedge)$ be any lattice. Then

(i) non empty set H of L is called a sublattice of L if $a \wedge b \in H$ and $a \vee b \in H$ for all $a, b \in H$;

(ii) A sublattice H of L is said to be convex if

$$(\forall a, b \in H)(\forall c \in L)(a \leqslant b \ \wedge \ a \leqslant c \leqslant b \implies c \in H).$$

Definition 2.3. ([1]) If in a poset $(P, \leqslant)$ for every $a, b \in P$, either $a \leqslant b$ or $b \leqslant a$ holds, then $(P, \leqslant)$ is called a chain or simply ordered set.

Every chain is a lattice but not vice versa.

Definition 2.4. ([**1**]) An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following axioms:
(1a) $x \vee x = x$ and (1b) $x \wedge x = x$;
(2a) $x \vee y = y \vee x$ and (2b) $x \wedge y = y \wedge x$;
(3a) $(x \vee y) \vee z = x \vee (y \vee z)$ and (3b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
(4a)) $(x \vee y) \wedge y = y$ and (4b) $(x \wedge y) \vee y = y$.
In any lattice $(L, \vee, \wedge)$ the following are equivalent:
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and
$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.5. ([**1**]) A Lattice $(L, \vee, \wedge)$ satisfying any of the above four equations is called a distributive lattice.

Definition 2.6. ([**1**]) If $(L, \vee, \wedge)$ is a lattice then an element $0 \in L$ is called a zero element or least element of L if $0 \wedge a = 0 \ \forall \ a \in L$ and an element 1 of L is called 1 element or greatest element if $a \vee 1 = 1 \ \forall \ a \in L$. If L has 0 and 1 then L is called a bounded lattice.

DEFINITION 2.7. ([**1**]) A bounded lattice L is called complemented if to each $a \in L$ there exists $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$. A lattice L is said to be relatively complemented if for any $x, y \in L$ such that $x \leqslant y$, the bounded lattice $[x, y] = \{z \in L \mid x \leqslant z \leqslant y\}$ is a complemented lattice.

It is well known that a lattice L is distributive if and only if relative compliments of any element in any interval $[x, y], x \leqslant y$ are unique.

DEFINITION 2.8. ([**1**]) A bounded distributive and complemented lattice is called a Boolean algebra.

DEFINITION 2.9. ([**1**]) A sublattice I of L is called an ideal of L if $i \in I, a \in L$ imply $a \wedge i \in I$.

(1) An ideal I of L is said to be proper if $I \neq L$
(2) An ideal I of L is said to be prime if
    (1) $a \wedge b \in I$, $a, b \in L \Rightarrow$ either $a \in I$ or $b \in I$ and
    (2) $I \neq L$.
(3) An ideal M of L is called maximal if
    (i) $M \neq L$
    (ii) If U is an ideal of L such that $M \subseteq U \subseteq L \Rightarrow$ either $M = U$ or $U = L$.

In the following, we give the definition of an ADL:

DEFINITION 2.10. ([**1**]) An almost distributive lattice (in further ADL) is an algebra $(L, \vee, \wedge)$ of type $(2, 2)$ satisfying
(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$,
for all $a, b, c \in L$.

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

EXAMPLE 2.1. ([**1**]) Let X be a non-empty set. Let fix $x_0 \in X$. For any $x, y \in L$, define

$$x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \quad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a discrete ADL.

For any $a, b \in L$, we say that a is less than or equals to b and write $a \leqslant b$, if $a \wedge b = a$. Then "$\leqslant$" is a partial ordering on L.

The following hold in any ADL L. Here onwards by L we mean an ADL $(L, \vee, \wedge, 0)$.

THEOREM 2.1 ([**1**]). *For any $a, b \in L$, we have*
(1) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b$, $a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$
(4) $a \wedge b = a \Longleftrightarrow a \vee b = b$ *and* $a \wedge b = b \Longleftrightarrow a \vee b = a$
(5) $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *whenever* $a \leqslant b$
(6) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(7) $\wedge$ *is associative in* L
(8) $a \wedge b \wedge c = b \wedge a \wedge c$
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \wedge b = 0 \Longleftrightarrow b \wedge a = 0$
(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert L into a distributive lattice.

THEOREM 2.2 ([**1**]). *Let $(L, \vee, \wedge, 0)$ be an ADL with* 0.
Then the following are equivalent:
(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice*
(2) $a \vee b = b \vee a$ *for all* $a, b \in L$
(3) $a \wedge b = b \wedge a$ *for all* $a, b \in L$
(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ *for all* $a, b, c \in L$.

PROPOSITION 2.1 ([**1**]). *Let $(L, \vee, \wedge, 0)$ be an ADL. Then for any $a, b, c \in L$ with $a \leqslant b$, we have*
(1) $a \wedge c \leqslant b \wedge c$
(2) $c \wedge a \leqslant c \wedge b$
(3) $c \vee a \leqslant c \vee b$.

DEFINITION 2.11. ([**3**]). An element $m \in L$ is called maximal if it is maximal as in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a$ implies $m = a$.

THEOREM 2.3 ([**1**]). *Let L be an ADL and $m \in L$.*
Then the following are equivalent:
(1) m *is maximal with respect to* $\leqslant$
(2) $m \vee a = m$ *for all* $a \in L$
(3) $m \wedge a = a$ *for all* $a \in L$.

LEMMA 2.1 ([**1**]). *Let L be an ADL with a maximal element m and $x, y \in L$. If $x \wedge y = y$ and $y \wedge x = x$ then x is maximal if and only if y is maximal.Also the following conditions are equivalent:*
(i) $x \wedge y = y$ *and* $y \wedge x = x$
(ii) $x \wedge m = y \wedge m$.

3. Residuated Mapping's on ADL's

In this section, we give the definition of a residuated map on an ADL and that of its residual. Certain equivalent conditions for an isotone map to become a residuated map are discussed. If P and Q are two ADL's, then we prove that a map $f : P \to Q$ is residuated if and only if $f^{-1}(I)$ is a principal initial segment in P, for any principal initial segment I in Q and deduced that $f : P \to Q$ is residuated if and only if there exists an isotone $g : Q \to P$ such that $g(y) \wedge x = x$ if and only if $y \wedge f(x) = f(x)$, for any $x \in P, y \in Q$. If $f^{+} : Q \to P$ is a residual of $f : P \to Q$ and $g^{+} : R \to Q$ is a residual of $g : Q \to R$, then we prove that $f^{+}og^{+}$ is a residual of gof.

Definition 3.1. Let P and Q be two Almost Distributive Lattices(ADL's). A mapping $f : P \to Q$ is said to be an isotone if for any $a, b \in P$,

$$a \wedge b = b \implies f(a) \wedge f(b) = f(b).$$

Definition 3.2. Let P and Q be two ADL's. An isotone $f : P \to Q$ is called residuated if there exists an isotone map $g : Q \to P$ such that $g(f(x)) \wedge x = x$, for any $x \in P$ and $y \wedge f(g(y)) = f(g(y))$, for any $y \in Q$. In this case, g is called a residual of f.

Example 3.1. Let $P = Z$, the set of all integers and let $Q = R$, the set of all real numbers together with the usual orderings. Define $f : P \to Q$ by $f(n) = n$, for any $n \in Z$. Then f is residuated and its residual $g : Q \to P$ is given by $g(y) = [y]$, the largest integer less than or equal to y.

From here onwards, P, Q and R denote ADL's.

Definition 3.3. A subset I of an ADL P is called an initial segment in P if for any $x, y \in P, y \wedge x = x$ and $y \in I \implies x \in I$.

Definition 3.4. For any $x \in P$, the set $(x] = \{a \in P \mid x \wedge a = a\}$ is an initial segment in P and is called the principal initial segment (PIS) generated by $\mathbf{x}$.

Theorem 3.1. *Let P, Q be two ADL's. A map $f : P \to Q$ is an isotone map if and only if $f^{-1}(I)$ is an initial segment in P, for each initial segment I in Q.*

Proof. Let P and Q be two ADL's. Assume that $f : P \to Q$ is an isotone map and $a, b \in P$. Let I be an initial segment in Q.
Suppose $b \wedge a = a$ and $b \in f^{-1}(I)$. Then $f(b) \wedge f(a) = f(a)$ (since f is isotone). As $f(b) \in I, f(b) \wedge f(a) = f(a)$ and I is an initial segment in Q, we get $f(a) \in I$. Then $a \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is an initial segment in P, for each initial segment I in Q.

On the other hand, assume $f^{-1}(I)$ is an initial segment in P, for each initial segment I in Q. Let $a, b \in P$ and $b \wedge a = a$ in P and $I = (f(b)]$. Then I is an initial segment in Q and $f(b) \in I$. Thus $b \in f^{-1}(I)$ and $f^{-1}(I)$ is an initial segment in P. So, $a \in f^{-1}(I)$ and $f(a) \in I = (f(b)]$. Finally, $f(b) \wedge f(a) = f(a)$. Therefore, f is an isotone map. $\square$

Theorem 3.2. *Let $f : P \to Q$ and $g : Q \to P$ be two maps. Then the following conditions are equivalent*

(i) *for any $x \in P$, $y \in Q$, $g(y) \wedge x = x \Leftrightarrow y \wedge f(x) = f(x)$.*

(ii) $f^{-1}((y]) = (g(y)]$, *for any $y \in Q$.*

PROOF. $(i) \Longrightarrow (ii)$ Assume (i). Let $y \in Q$. Then $t \in f^{-1}((y])$. Thus $y \wedge f(t) = f(t)$ and $g(y) \wedge t = t$ (By (i)). So $t \in (g(y)]$ holds. Therefore, $f^{-1}((y]) = (g(y)]$, for any $y \in Q$.

$(ii) \Longrightarrow (i)$ Suppose $f^{-1}((y]) = (g(y)]$ for any $y \in Q$. Let $x \in P, y \in Q$. Then $g(y) \wedge x = x$. Thus $x \in (g(y)]$ and $x \in f^{-1}((y])$ (By (ii). So, $f(x) \in (y]$ Finally $y \wedge f(x) = f(x)$. □

THEOREM 3.3. *Let P, Q be two ADL's with a maximal element m. An isotone map $f : P \to Q$ is residuated if and only if $f^{-1}(I)$ is a principal initial segment(PIS) in P, for any principal initial segment(PIS) I in Q.*

PROOF. Suppose $f : P \to Q$ is an isotone residuated map. Then there exists an isotone map $g : Q \to P$ such that

$$g(f(x)) \wedge x = x, \text{ for all } x \in P \text{ and } y \wedge f(g(y)) = f(g(y)), \text{ for all } y \in Q.$$

Let $I = (y]$ be a PIS in Q and take $x = g(y)$. Let $t \in f^{-1}(I)$. Then $f(t) \in I = (y]$ and $y \wedge f(t) = f(t)$. Thus $g(y) \wedge g(f(t)) = g(f(t))$ (Since g is isotone) and $g(y) \wedge g(f(t)) \wedge t = g(f(t)) \wedge t$. Further, we have $g(y) \wedge t = t$ (Since $g(f(t)) \wedge t = t$, for any $t \in P$) and $t \in (g(y)]$. Therefore, $f^{-1}(I) \subseteq (g(y)] \longrightarrow (1)$.

Now, let $t \in (g(y)]$. Thus $g(y) \wedge t = t$ and $f(g(y)) \wedge f(t) = f(t)$ (Since f is isotone). Now, $y \wedge f(t) = y \wedge f(g(y)) \wedge f(t) = f(g(y)) \wedge f(t) = f(t)$. Thus $f(t) \in (y] = I$ and hence $t \in f^{-1}(I)$. Therefore, $(g(y)] \subseteq f^{-1}(I) \longrightarrow (2)$.

From (1) and (2) we get that $f^{-1}(I) = (g(y)]$ Therefore, $f^{-1}(I)$ is a PIS in P.

On the other hand, assume the given condition. Suppose $f : P \to Q$ is an isotone map. Define $g : Q \to P$ as follows. Let $y \in Q$. Then $(y]$ is a PIS in Q and $f^{-1}((y])$ is a PIS in P. Thus $f^{-1}((y]) = (x]$, for some $x \in P$. Define $g(y) = x \wedge m$, where $f^{-1}((y]) = (x] = (x \wedge m]$. Let $y_1, y_2 \in Q$ be such that $y_1 = y_2$. Thus $(y_1] = (y_2]$ and $f^{-1}((y_1]) = f^{-1}((y_2])$. So, further, we have $(x_1] = (x_2]$ and $x_1 \wedge m = x_2 \wedge m$. Then $g(y_1) = g(y_2)$. Therefore, g is well-defined. Since $f^{-1}((y]) = (x] = (x \wedge m] = (g(y)]$, for any $y \in Q$, by above Theorem 3.2, we get that

$$g(y) \wedge x = x \Leftrightarrow y \wedge f(x) = f(x), \text{ for any} x \in P, y \in Q\ ,\ \longrightarrow, (3)$$

Then by taking $y = f(x)$, we have $g(f(x)) \wedge x = x$ and by taking $x = g(y)$, we have $y \wedge f(g(y)) = f(g(y))$, for any $x \in P, y \in Q$.

Let $x_1, x_2 \in P$. Now, $x_2 \wedge x_1 = x_1$ implies $g(f(x_2)) \wedge x_2 \wedge x_1 = g(f(x_2)) \wedge x_1$ and $x_2 \wedge x_1 = g(f(x_2)) \wedge x_1$ (Since $g(f(x_2)) \wedge x_2 = x_2$). Thus $g(f(x_2)) \wedge x_1 = x_1$ and $f(x_2) \wedge f(x_1) = f(x_1)$ in Q (By (3)).

Let $y_1, y_2 \in Q$. Now, $y_2 \wedge y_1 = y_1$ implies $y_2 \wedge y_1 \wedge f(g(y_1)) = y_1 \wedge f(g(y_1))$ and $y_2 \wedge f(g(y_1)) = f(g(y_1))$ (Since $y_1 \wedge f(g(y_1)) = f(g(y_1))$). Thus $g(y_2) \wedge g(y_1) = g(y_1)$ in P (By (3)). Therefore, f and g are isotone maps.

Thus f is residuated. □

COROLLARY 3.1. *An isotone map $f : P \to Q$ is residuated if and only if there exists a map $g : Q \to P$ such that $g(y) \wedge x = x \Leftrightarrow y \wedge f(x) = f(x)$, for any $x \in P, y \in Q$ and in this case g is a residual of f.*

PROOF. Suppose $f : P \to Q$ is an isotone residuated map. Then there exists an isotone map $g : Q \to P$ such that $g(f(x)) \wedge x = x$, for any $x \in P$ and $y \wedge f(g(y)) = f(g(y))$, for any $y \in Q$.

Suppose $g(y) \wedge x = x$. Then $f(g(y)) \wedge f(x) = f(x)$ (Since f is isotone) and $y \wedge f(g(y)) \wedge f(x) = y \wedge f(x)$. Thus $f(g(y)) \wedge f(x) = y \wedge f(x)$ and $f(x) = y \wedge f(x)$.

On the other hand, assume $y \wedge f(x) = f(x)$. Then $g(y) \wedge g(f(x)) = g(f(x))$ (Since g is isotone) and $g(y) \wedge g(f(x)) \wedge x = g(f(x)) \wedge x$. Thus $g(y) \wedge x = x$.

Converse follows from Theorem 3.2 and Theorem 3.3. □

DEFINITION 3.5. Let P, Q be two ADL's with maximal elements. Two maps $g, h : P \to Q$ are said to be equivalent, if, $g(x) \wedge h(x) = h(x)$ and $h(x) \wedge g(x) = g(x)$ for any $x \in P$.

In the following result, we prove that the residual of a residuated map is unique upto equivalence.

COROLLARY 3.2. *Let P, Q be two ADL's with maximal elements. Suppose an isotone map $f : P \to Q$ is residuated. If there exist two maps $g : Q \to P$ and $h : Q \to P$ such that*

(i) *$g(y) \wedge x = x \Leftrightarrow y \wedge f(x) = f(x)$ and*

(ii) *$h(y) \wedge x = x \Leftrightarrow y \wedge f(x) = f(x)$ for any $x \in P, y \in Q$,*

then g, h are equivalent.

PROOF. Let $x \in P, y \in Q$. By the given hypothesis, we get that

$$g(y) \wedge x = x \Leftrightarrow h(y) \wedge x = x \longrightarrow (1).$$

Since $g(y) \wedge g(y) = g(y)$, by (1), we get that $h(y) \wedge g(y) = g(y)$. Since $h(y) \wedge h(y) = h(y)$, by (1), we get that $g(y) \wedge h(y) = h(y)$. Hence g, h are equivalent. □

THEOREM 3.4. *Let $f : P \to Q$ and $g : Q \to R$ be two isotone residuated maps. If $f^+ : Q \to P$ is a residual of f and $g^+ : R \to Q$ is a residual of g, then $f^+ \circ g^+$ is a residual of $g \circ f$.*

PROOF. Suppose $f^+ : Q \to P$ is a residual of f and $g^+ : R \to Q$ is a residual of g. Then, by Corollary 3.1, we have $f^+(y) \wedge x = x \Leftrightarrow y \wedge f(x) = f(x)$ and $g^+(z) \wedge y = y \Leftrightarrow z \wedge g(y) = g(y)$ for all $x \in P, y \in Q, z \in R$. Now, for any $x \in P$ and $z \in R$, $f^+(g^+(z)) \wedge x = x$ implies that $g^+(z) \wedge f(x) = f(x)$ and $z \wedge g(f(x)) = g(f(x))$.

Now, $z \wedge g(f(x)) = g(f(x))$ implies $g^+(z) \wedge f(x) = f(x)$ and $f^+(g^+(z)) \wedge x = x$. By Corollary 3.1, we get that $g \circ f : P \to R$ is an isotone residuated map. Thus $f^+ \circ g^+$ is a residual of $g \circ f$. □

COROLLARY 3.3. *Let $f : P \to Q$ be an isotone residuated map and $f^+ : Q \to P$ be a residual of f. Then*

(1) $(fof^+of)(x) \wedge f(x) = f(x)$

(2) $f(x) \wedge (fof^+of)(x) = (fof^+of)(x)$, *for all* $x \in P$

(3) $f^+(y) \wedge (f^+ofof^+)(y) = (f^+ofof^+)(y)$

(4) $(f^+ofof^+)(y) \wedge f^+(y) = f^+(y)$, *for all* $y \in Q$.

Proof. Since f^+ is a residual of f, then, by definition, we have $f^+(f(x)) \wedge x = x$, for all $x \in P$ and $y \wedge f(f^+(y)) = f(f^+(y))$, for all $y \in Q$. So that $f(f^+(f(x))) \wedge f(x) = f(x)$.

Also, we have $y \wedge f(f^+(y)) = f(f^+(y))$.

Replacing y by $f(x)$, we get that $f(x) \wedge f(f^+(f(x))) = f(f^+(f(x)))$.

Again, we have $y \wedge f(f^+(y)) = f(f^+(y))$, so that $f^+(y) \wedge (f^+(f(f^+(y)))) = f^+(f(f^+(y)))$.

Finally, we have $f^+(f(x)) \wedge x = x$.

Replacing x by $f^+(y)$, we get that $(f^+(f(f^+(y)))) \wedge f^+(y) = f^+(y)$. $\square$

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