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ON GENERALIZED FUZZY GENERALIZED FUZZY BI-IDEALS OF TERNARY SEMIGROUPS

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Abstract. We introduce the notion of S-fuzzy generalized bi-ideal. We introduce ternary S and ternary T-products of fuzzy sets of ternary semigroup. We find interrelationship between ternary S-product and ternary T-product. We redefine S-fuzzy generalized bi-ideal by using ternary S-product and ternary T-product of ternary semigroup. We introduce the notion of S-union of fuzzy sets. We establish that S-union of S-fuzzy bi-ideal is again a S-fuzzy bi-ideal.

1. Introduction

Biswas [**1**] was first introduced subgroups and anti fuzzy subgroups. Madad Khan and Tauseef Asif [**9**]introduced the notion of Characterizations of semigroups by their anti fuzzy ideals. Muhammad Shabir [**8**]introduced the notion of characterizations of ternary semigroups by their anti fuzzy ideals. Prince Willians [**10**] introduced the notion of S-fuzzy left h-ideal of hemirings. Mohanraj and Vela [**6**] introduced the notions T-fuzzy lateral ideal of ternary semigroups. We [**7**] discussed TL-bi-ideals of ternary semigroups. A systematic study concerning the properties and related matters of T-norms and S-norms have been made by Klement [**3**].

2. Preliminaries

A non-empty set R is called a ternary semigroup if there exists a mapping $R\times R\times R \to R$ denoted by juxtaposition that satisfies : $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in R$. A non-empty set B of R is called generalized bi-ideal if $BRBRB \subseteq B$. The generalized bi-ideal B of R is called bi-ideal if $BBB \subseteq B$. A mapping $\mu : X \to [0, 1]$ is called a fuzzy set of X. The fuzzy set μ of R is called generalized fuzzy bi-ideal if $\mu(xwyvz) \geqslant min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in$

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R. The generalized fuzzy bi-ideal μ of R is called a fuzzy bi-ideal if $\mu(xyz) \geqslant min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in R$. The fuzzy set μ of R is called generalized anti fuzzy bi-ideals if $\mu(xwyvz) \leqslant max\{\mu(x), \mu(y), \mu(z)\}$, for all $x, y, z, u, v \in R$. The generalized anti fuzzy bi-ideals μ of R is called anti fuzzy bi-ideal if $\mu(xyz) \leqslant max\{\mu(x), \mu(y), \mu(z)\}$,for all $x, y, z \in R$.

3. S-fuzzy bi-ideals

Definition 3.1. The binary operation S on $[0, 1]$ is called a S-norm on $[0, 1]$ if satisfies the following conditions:

($S1$) $S(x, 0) = S(0, x) = x$ (boundary condition)
($S2$) $S(x, y) = S(y, x)$ (commutativity)
($S3$) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)
($S4$) If $x^* \leqslant x$ and $y^* \leqslant y$ then $S(x^*, y^*) \leqslant S(x, y)$ (monotonicity)
for all $x, y, z, x^*, y^* \in [0, 1]$.

Definition 3.2. ([**6**]) The binary operation T on $[0, 1]$ is called a triangular norm[T-norm] on $[0, 1]$ which satisfies $S2$ to $S4$ and $T(x, 1) = T(1, x) = x$

Theorem 3.1. ([**3**]) *The function $S : [0, 1] \times [0, 1] \to [0, 1]$ is a S-norm(T-conorm) if and only if there exist a T-norm(S-conorm) such that S(x,y)=1-T(1-x,1-y)-(1) for all $x, y \in [0, 1]$.*

Remark 3.1. (1) By above Theorem 3.1, for each S-norm S, there exists T-norm satisfying Equation (1) and that T-norm is a called S-conorm.

(2) For each T-norm T, Theorem 3.1, there exists S-norm S satisfying $T(x, y) = 1 - S(1 - x, 1 - y)$ and that S-norm S is called T-conorm.

(3) Various S-norms and corresponding S-conorms are tabulated as follows

S-norm	T-norm(S-conorm)
$S_M(x, y) = max\{x, y\}$,	$T_M(x, y) = min\{x, y\}$,
$S_P(x, y) = x + y - x \cdot y$,	$T_P(x, y) = x \cdot y$,
$S_L(x, y) = min\{x + y, 1\}$,	$T_L(x, y) = max\{x + y - 1, 0\}$,
$S_D(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1) \\ max\{x, y\} & \text{is otherwise,} \end{cases}$	$T_D(x, y) = \begin{cases} 0 & \text{if } x, y \in [0, 1) \\ min\{x, y\} & \text{is otherwise,} \end{cases}$
Hamacher class S-norm for$\lambda \in [0, \infty]$	Hamacher class T-norm for $\lambda \in [0, \infty]$
$(S^H_\lambda)(x, y) = \begin{cases} S_D(x, y) & \text{if } \lambda = 0 \\ 1 & \text{if } x = y = 1 \\ \frac{x+y-xy-(1-\lambda)xy}{1-(1-\lambda)xy} & \text{otherwise.} \end{cases}$	$(T^H_\lambda)(x, y) = \begin{cases} T_D(x, y) & \text{if } \lambda = \infty \\ 0 & \text{if } \lambda = x = y = 0 \\ \frac{xy}{\lambda+(1-\lambda)(x+y-xy)} & \text{otherwise.} \end{cases}$

Theorem 3.2. ([**3**]) *Every T-norm $[0, 1]$ satisfies the inequality as follows*

$$T_D(x, y) \leqslant T(x, y) \leqslant T_M(x, y), \text{ for all } x, y \in [0, 1].$$

THEOREM 3.3. *Every S-norm S satisfies the inequality*

$$S_D(x,y) \leqslant S(x,y) \leqslant S_M(x,y), \quad \textit{for all } x,y \in [0,1].$$

PROOF. By Theorem 3.2, we have $T_M(x,y) \geqslant T(x,y) \geqslant T_D(x,y)$, for all $x,y \in [0,1]$, $1 - T_M(x,y) \leqslant 1 - T(x,y) \leqslant 1 - T_D(x,y)$. By Theorem 3.1, $S_M(x,y) \leqslant S(x,y) \leqslant S_D(x,y)$, for all $x,y \in [0,1]$ □

Hereafter, R denotes a ternary semigroup and S denotes S-norm on $[0,1]$, whereas T denotes a corresponding S-conorm on $[0,1]$ unless otherwise specified,

DEFINITION 3.3. The fuzzy set μ of R is called generalized S-fuzzy bi-ideal if $\mu(xwyvz) \leqslant S(\mu(x), S(\mu(y), \mu(z)))$, for all $x,y,z,w,v \in R$.

DEFINITION 3.4. The generalized S-fuzzy bi-ideal μ of R is called a S-fuzzy bi-ideal of R if $\mu(xyz) \leqslant S(\mu(x), S(\mu(y), \mu(z)))$ for all $x,y,z,\in R$.

DEFINITION 3.5. ([**7**]) The fuzzy set μ of R is called generalized T-fuzzy bi-ideal if $\mu(xwyvz) \geqslant T(\mu(x), T(\mu(y), \mu(z)))$, for all $x,y,z,w,v \in R$.

DEFINITION 3.6. ([**7**]) The generalized T-fuzzy bi-ideal μ of R is called T-fuzzy bi-ideal of R if $\mu(xyz) \geqslant T(\mu(x), T(\mu(y), \mu(z)))$, for all $x,y,z \in R$.

Here, we redefine S-fuzzy bi-ideals by using ternary S-products and ternary T-products.

DEFINITION 3.7. The ternary S-product and the ternary T-product of the fuzzy sets λ, μ and σ of R denoted by $\lambda \circ_S \mu \circ_S \sigma$ and $\lambda \cdot_T \mu \cdot_T \sigma$ are defined as follows:

$$(\lambda \circ_S \mu \circ_S \sigma)(x) = \begin{cases} \inf\limits_{x=abc} S(\lambda(a), S(\mu(b), \sigma(c))) & \text{if } x = abc \\ 1 & \text{otherwise} \end{cases}$$

$$(\lambda \cdot_T \mu \cdot_T \sigma)(x) = \begin{cases} \sup\limits_{x=abc} T(\lambda(a), T(\mu(b), \sigma(c))) & \text{if } x = abc \\ 0 & \text{otherwise} \end{cases}$$

REMARK 3.2. (1) By taking S-norm as S_M-norm, then the ternary S-product becomes ternary "$\circ$" product

$$(\lambda \circ \mu \circ \sigma)(x) = \begin{cases} \inf\limits_{x=abc} \{max\{\lambda(a), \mu(b), \sigma(c))\}\} & \text{if } x = abc \\ 1 & \text{otherwise} \end{cases}$$

(2) By taking T-norm as T_M-norm, then the ternary T-product becomes ternary "$\cdot$" product

$$(\lambda \cdot \mu \cdot \sigma)(x) = \begin{cases} \sup\limits_{x=abc} \{min\{\lambda(a), \mu(b), \sigma(c)\}\} & \text{if } x = abc \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION 3.8. The fuzzy sets 0 and 1 of R are defined as follows

$$0(x) = 0 \ , 1(x) = 1 \text{ for all } x \in R$$

THEOREM 3.4. *The fuzzy set μ is a S-fuzzy generalized bi-ideal of R if and only if $\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu$.*

PROOF. For a S-fuzzy generalized -bi-ideal μ of R and if x cannot be expressible as $x = awbvc$, then $(\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu)(x) = 1 \geqslant \mu(x)$. Now,

$$\begin{aligned}((\mu \circ_S 0 \circ_S \mu) \circ_S 0 \circ_S \mu)(x) &= \inf_{x=uvc} S((\mu \circ_S 0 \circ_S \mu)(u), S(0(v), \mu(c))) \\ &= \inf_{x=uvc} S(\inf_{u=awb} S(\mu(a), S(0(w), \mu(b))), \mu(c)) \\ &= \inf_{x=uvc} \inf_{u=awb} S(\mu(a), S(\mu(b), \mu(c)))\end{aligned}$$

Now

$$\mu(awbvc) \leqslant S(\mu(a), S(\mu(b), \mu(c)))$$

implies $\mu(x) \leqslant \inf_{x=awbvc} S(\mu(a), S(\mu(b), \mu(c)))$ If $x = uvc$ and $u = awb$, then $x = (awb)vc$. Thus

$$\begin{aligned}\mu(x) &\leqslant \inf_{x=uvc} \inf_{u=awb} S(\mu(a), S(\mu(b), \mu(c))) \leqslant \inf_{x=awbvc} S(\mu(a), S(\mu(b), \mu(c))) \\ &= (\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu)(x)\end{aligned}$$

Conversely,

$$\mu(xwyvz) \quad \leqslant \quad ((\mu \circ_S 0 \circ_S \mu) \circ_S 0 \circ_s \mu)(xwyvz) \leqslant S(\mu(x), S(\mu(y), \mu(z)))$$

Hence μ is a S-fuzzy generalized bi-ideal of R. □

THEOREM 3.5. *The fuzzy set μ is a S-fuzzy-bi-ideal of R if and only if*
(i) $\mu \subseteq \mu \circ_S \mu \circ_S \mu$.
(ii) $\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu$

PROOF. By Theorem 3.4,$\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu$, when μ is a S-fuzzy bi-ideal If x cannot be expressible as $x = abc$, then $(\mu \circ_S \mu \circ_S \mu)(x) = 1 \geqslant \mu(x)$.
Now $\mu(abc) \leqslant S(\mu(a), S(\mu(b), \mu(c)))$. Then,

$$\mu(x) \leqslant \inf_{x=abc} S(\mu(a), S(\mu(b), \mu(c))) = (\mu \circ_S \mu \circ_S \mu)(x).$$

Conversely, By Theorem 3.4, μ is a S-fuzzy generalized bi-ideal

$$\mu(abc) \quad \leqslant \quad (\mu \circ_S \mu \circ_S \mu)(abc) \leqslant S(\mu(a), S(\mu(b), \mu(c)))$$

Hence μ is a S fuzzy-bi-ideal of R. □

THEOREM 3.6. *If S-is a S-norm and T is it S-conorm(T-norm), then*
(i) $1 - (\lambda \circ_S \mu \circ_S \sigma) = (1 - \lambda) \cdot_T (1 - \mu) \cdot_T (1 - \sigma)$
(ii) $1 - (\lambda \cdot_T \mu \cdot_T \sigma) = (1 - \lambda) \circ_S (1 - \mu) \circ_S (1 - \sigma)$,
for any fuzzy set λ, μ and σ of R.

PROOF. For the fuzzy sets λ, μ and σ of R,

$$\begin{aligned}(\lambda \circ_S \mu \circ_S \sigma)(x) &= \inf_{x=abc} S(\lambda(a), S(\mu(b), \sigma(c))) \\ &= \inf_{x=abc} 1 - T(1-\lambda(a), 1 - S(\mu(b), \sigma(c))) \\ &= \inf_{x=abc} 1 - T(1-\lambda(a), 1-(1-T(1-\mu(b), 1-\sigma(c)))) \\ &= \inf_{x=abc} 1 - T(1-\lambda(a), T(1-\mu(b), 1-\sigma(c))) \\ &= \inf_{x=abc} 1 - T((1-\lambda)(a), T((1-\mu)(b), (1-\sigma)(c))) \\ &= 1 - \sup_{x=abc} T((1-\lambda)(a), T((1-\mu)(b), (1-\sigma)(c))).\end{aligned}$$

Then,

$$(1 - (\lambda \circ_S \mu \circ_S \sigma))(x) = \sup_{x=abc} T((1-\lambda)(a), T((1-\mu)(b), (1-\sigma)(c))).$$

Therefore

$$1 - (\lambda \circ_S \mu \circ_S \sigma) = (1-\lambda) \cdot_T (1-\mu) \cdot_T (1-\sigma).$$

Now,

$$\begin{aligned}(\lambda \cdot_T \mu \cdot_T \sigma)(x) &= \sup_{x=abc} T(\lambda(a), T(\mu(b), \sigma(c))) \\ &= \sup_{x=abc} 1 - S(1-\lambda(a), 1 - T(\mu(b), \sigma(c))) \\ &= \sup_{x=abc} 1 - S(1-\lambda(a), 1-(1-S(1-\mu(b), 1-\sigma(c)))) \\ &= \sup_{x=abc} 1 - S(1-\lambda(a), S(1-\mu(b), 1-\sigma(c))) \\ &= \sup_{x=abc} 1 - S((1-\lambda)(a), S((1-\mu)(b), (1-\sigma)(c))) \\ &= 1 - \inf_{x=abc} S((1-\lambda)(a), S((1-\mu)(b), (1-\sigma)(c)))\end{aligned}$$

Then, $1 - (\lambda \cdot_T \mu \cdot_T \sigma)(x) = \inf_{x=abc} S((1-\lambda)(a), S((1-\mu)(b), (1-\sigma)(c)))$
Therefore $1 - (\lambda \cdot_T \mu \cdot_T \sigma) = (1-\lambda) \circ_S (1-\mu) \circ_S (1-\sigma)$ □

THEOREM 3.7. *The fuzzy set μ is a S-fuzzy generalized bi-ideal of R if and only if there exists S-conorm T such that $(1-\mu) \cdot_T 1 \cdot_T (1-\mu) \cdot_T 1 \cdot_T (1-\mu) \subseteq 1-\mu$.*

PROOF. For a S-fuzzy generalized bi-ideal μ of R and by Theorem 3.4, $\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu$. Then

$$1 - (\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu) \subseteq 1 - \mu.$$

By Theorem 3.6, we have

$$(1-\mu) \cdot_T (1-0) \cdot_T (1-\mu) \cdot_T (1-0) \cdot_T (1-\mu) \subseteq 1-\mu.$$

Thus

$$(1-\mu) \cdot_T 1 \cdot_T (1-\mu) \cdot_T 1 \cdot_T (1-\mu) \subseteq 1-\mu.$$

Conversely, by Theorem 3.6, we have

$$1 - (\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu) = (1-\mu) \cdot_T 1 \cdot_T (1-\mu) \cdot_T 1 \cdot_T (1-\mu) \subseteq 1-\mu.$$

Thus

$$\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu.$$

By Theorem 3.4, μ is a S-fuzzy generalized bi-ideals of R. □

THEOREM 3.8. *The fuzzy set μ is a S-fuzzy bi-ideal of R if and only if there exists S-conorm T such that*

(i) $(1-\mu) \cdot_T (1-\mu) \cdot_T (1-\mu) \subseteq (1-\mu)$.

(ii) $(1-\mu) \cdot_T 1 \cdot_T (1-\mu) \cdot_T 1 \cdot_T (1-\mu) \subseteq 1-\mu$

PROOF. For a S-fuzzy bi-ideal μ of R and by Theorem 3.7, there exist S-conorm T,

$$(1-\mu) \cdot_T (1-\mu) \cdot_T (1-\mu) \subseteq 1-\mu.$$

By Theorem 3.5, $\mu \subseteq \mu \circ_S \mu \circ_S \mu$ implies $1-(\mu \circ_S \mu \circ_S \mu) \subseteq 1-\mu$. By Theorem 3.6, there exist S-conorm T,

$$(1-\mu) \cdot_T (1-\mu) \cdot_T (1-\mu) = 1-(\mu \circ_S \mu \circ_S \mu) \subseteq 1-\mu.$$

Conversely, by Theorem 3.6, we have that

$$1-(\mu \circ_S \mu \circ_S \mu) = (1-\mu) \cdot_T (1-\mu) \cdot_T (1-\mu) \subseteq 1-\mu$$

implies

$$\mu \subseteq \mu \circ_S \mu \circ_S \mu.$$

Similarly, by Theorem 3.5

$$\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu.$$

Thus μ is a S-fuzzy bi-ideal of R. □

THEOREM 3.9. *The fuzzy set*

$$\mu(x) = \begin{cases} t & \text{if } x \in B \\ s & \text{otherwise} \end{cases}$$

for $0 \leqslant t \leqslant s \leqslant 1$ is a S-fuzzy generalized bi-ideal of R for all S-norms S if and only if B is a generalized bi-ideal of R.

PROOF. Let B be a generalized bi-ideal of R and let μ be the fuzzy set defined as above for $0 \leqslant t \leqslant s \leqslant 1$. If $t = s$, then μ is constant. Thus μ is S-fuzzy generalized bi-ideal. Otherwise if $xyz \in B$,then for all $u, v \in R$ holds

$$\mu(xuyvz) = t \leqslant S_M(\mu(x), S_M(\mu(y), \mu(z))).$$

If $xuyvz \notin B$, then either $x \notin B$ or $y \notin B$ or $z \notin B$. Now,

$$\mu(xuyvz) = s = S_M(\mu(x), S_M(\mu(y), \mu(z)))$$

and by Theorem 3.3,

$$\mu(xuyvz) \leqslant S_M(\mu(x), S_M(\mu(y), \mu(z))) \leqslant S(\mu(x), S(\mu(y), \mu(z)))$$

for any S-norms and for $x, y, z \in B$,$u, v \in R$. Therefore μ is a S-fuzzy generalized bi-ideals, for all S-norms S.

Conversely, for $u, v \in R$ and $x, y, z \in B$,
$t = S_M(\mu(x), S_M(\mu(y), \mu(z))) \geqslant \mu(xuyvz)$, implies $xuyvz \in B$. Thus B is a generalized bi-ideal of R. □

COROLLARY 3.1 ([**8**]). *A non-empty subset B of a ternary semigroup R is a generalized fuzzy bi-ideal of R if and only if the fuzzy subset μ of R defined by*

$$\mu(x) = \begin{cases} t & \text{if } x \in B, \\ s & \text{otherwise} \end{cases}$$

is an anti fuzzy generalized bi-ideal of R, where $t, s \in [0,1]$ $s \geqslant t$.

PROOF. By taking S-norm as S_M in Theorem 3.9, we get the result. □

THEOREM 3.10. *The fuzzy set*

$$\mu(x) = \begin{cases} t & \text{if } x \in B \\ s & \text{otherwise} \end{cases}$$

for $0 \leqslant t \leqslant s \leqslant 1$ is a S-fuzzy bi-ideal of R for all S-norms S if and only if B is a bi-ideal of R.

PROOF. Let B be a bi-ideal of R and let μ be the fuzzy set defined as above for $0 \leqslant t \leqslant s \leqslant 1$. If $t = s$, then μ is constant. Thus μ is S-fuzzy bi-ideal. By Theorem 3.9,μ is a S-fuzzy generalized bi-ideal of R.If $x, y, z \in B$,then $xyz \in B$ implies

$$\mu(xyz) = t \leqslant S_M(\mu(x), S_M(\mu(y), \mu(z))) \leqslant S(\mu(x), S(\mu(y), \mu(z)))$$

If $xyz \notin B$, then $x \notin B$ or $y \notin B$ or $z \notin B$. Thus

$$\mu(xyz) = s = S_M(\mu(x), S_M(\mu(y), \mu(z))) \leqslant S(\mu(x), S(\mu(y), \mu(z))).$$

Therefore μ is a S-fuzzy bi-ideal for all S-norms.

Conversely, by Theorem 3.9, B is generalized bi-ideal. If $x, y, z \in B$, then $t = S_M(\mu(x), S_M(\mu(y), \mu(z))) \geqslant \mu(xyz)$ implies $xyz \in B$.Thus B is a bi-ideal of R. □

4. T-fuzzy bi-ideals

THEOREM 4.1. *The fuzzy set μ is a S-fuzzy generalized bi-ideal of R if and only if there exists S-conorm T such that $1 - \mu$ is a T-fuzzy generalized bi-ideal of R.*

PROOF. If μ is a S-fuzzy generalized bi-ideal of R,then by Theorem 3.1, there exists S-conorm T such that $S(x, y) = 1 - T(1 - x, 1 - y)$ for all $x, y \in [0, 1]$. For $x, y, z, w, v \in R$.

$$\begin{aligned} \mu(xwyvz) &\leqslant S(\mu(x), S(\mu(y), \mu(z))) \\ &= 1 - T(1 - \mu(x), 1 - S(\mu(y), \mu(z))) \\ &= 1 - T(1 - \mu(x), 1 - (1 - T(1 - \mu(y), 1 - \mu(z)))) \\ &= 1 - T(1 - \mu(x), T(1 - \mu(y), 1 - \mu(z))) \\ &= 1 - T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z))) \end{aligned}$$

. Therefore

$$-\mu(xwyvz) \geqslant -1 + T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z)))).$$

Then, $(1-\mu)(xwyvz) \geqslant T((1-\mu)(x), T((1-\mu)(y),(1-\mu)(z))))$ and $1-\mu$ is a T-fuzzy generalized bi-ideal of R.

Conversely,

$$\begin{aligned}
(1-\mu)(xwyvz) &\geqslant T((1-\mu)(x), T((1-\mu)(y),(1-\mu)(z)))) \\
&= T(1-\mu(x), T(1-\mu(y), 1-\mu(z)))) \\
&= 1-S(1-(1-\mu(x), 1-T(1-\mu(y), 1-\mu(z)))) \\
&= 1-S(\mu(x), 1-(1-S(1-(1-\mu(y)), 1-(1-\mu(z))))) \\
&= 1-S(\mu(x), S(\mu(y), \mu(z)))
\end{aligned}$$

Thus

$$-1+\mu(xwyvz) \leqslant -1+S(\mu(x), S(\mu(y), \mu(z))).$$

Then, $\mu(xwyvz) \leqslant S(\mu(x)), S(\mu(y), \mu(z)))$. Therefore, μ is S-fuzzy generalized bi-ideals of R. □

THEOREM 4.2. *The fuzzy set μ is a S-fuzzy bi-ideal of R if and only if there exists S-conorm T such that $1-\mu$ is a T-fuzzy bi-ideal of R.*

PROOF. If μ is a S-fuzzy bi-ideal of R , then by Theorem 4.1, $1-\mu$ is a T-fuzzy generalized bi-ideal. For $x, y, z \in R$ and by Theorem 3.1,

$$\begin{aligned}
\mu(xyz) &\leqslant S(\mu(x), S(\mu(y), \mu(z))) \\
&= 1-T(1-\mu(x), 1-S(\mu(y), \mu(z))) \\
&= 1-T(1-\mu(x), 1-(1-T(1-\mu(y), 1-\mu(z)))) \\
&= 1-T(1-\mu(x), T(1-\mu(y), 1-\mu(z)))) \\
&= 1-T((1-\mu)(x), T((1-\mu)(y),(1-\mu)(z))))
\end{aligned}$$

Then $(1-\mu)(xyz) \geqslant T((1-\mu)(x), T((1-\mu)(y),(1-\mu)(z))))$. Thus μ is a T-fuzzy bi-ideals of R.

Conversely, by Theorem 4.1, $\mu(xwyvz) \leqslant S(\mu(x), S(\mu(y), \mu(z)))$. Now,

$$\begin{aligned}
(1-\mu)(xyz) &\geqslant T((1-\mu)(x), T((1-\mu)(y),(1-\mu)(z)))) \\
&= T(1-\mu(x), T(1-\mu(y), 1-\mu(z)))) \\
&= 1-S(1-(1-\mu(x), 1-T(1-\mu(y), 1-\mu(z)))) \\
&= 1-S(\mu(x), 1-(1-S(1-(1-\mu(y)), 1-(1-\mu(z))))) \\
&= 1-S(\mu(x), S(\mu(y), \mu(z)))
\end{aligned}$$

Then

$$\mu(xyz) \leqslant S(\mu(x)), S(\mu(y), \mu(z))).$$

Therefore μ is a S-fuzzy bi-ideal of R. □

THEOREM 4.3. *The fuzzy set μ is a T-fuzzy generalized bi-ideal of R if and only if $\mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu \subseteq \mu$*

PROOF. Let μ be T-fuzzy generalized bi-ideal of R. By Theorem 4.1, there exists T-conorm S such that $1-\mu$ is a S-fuzzy generalized bi-ideal. Now, by Theorem 3.7, we have

$$(1-(1-\mu))\cdot_T(1-0)\cdot_T(1-(1-\mu))\cdot_T(1-0)\cdot_T(1-(1-\mu))\subseteq 1-(1-\mu).$$

Thus,

$$\mu\cdot_T 1\cdot_T\mu\cdot_T 1\cdot_T\mu\subseteq\mu.$$

Conversely,

$$\begin{aligned}
\mu(xwyvz) &\geqslant ((\mu\cdot_T 1\cdot_T\mu)\cdot_T 1\cdot_T\mu)(xwyvz)\\
&= T((\mu\cdot_T 1\cdot_T\mu)(xwy),T(1(v),\mu(z)))\\
&= T((\mu\cdot_T 1\cdot_T\mu)(xwy),\mu(z))\\
&\geqslant T(T(\mu(x),T(1(w),\mu(y))),\mu(z))\\
&= T(T(\mu(x),\mu(y)),\mu(z))\\
&= T(\mu(x),T(\mu(y),\mu(z))),\ \textit{for all } x,y,z,w,v\in R
\end{aligned}$$

Thus μ is a T-fuzzy generalized bi-ideal. □

THEOREM 4.4. *The fuzzy set μ is a T-fuzzy bi-ideal of R if and only if*
(i) $\mu\cdot_T\mu\cdot_T\mu\subseteq\mu$
(ii) $\mu\cdot_T 1\cdot_T\mu\cdot_T 1\cdot_T\mu\subseteq\mu$.

PROOF. Let μ be T-fuzzy bi-ideal of R. By Theorem 4.3,

$$\mu\cdot_T 1\cdot_T\mu\cdot_T 1\cdot_T\mu\subseteq\mu.$$

By Theorem 4.2, $1-\mu$ is a S-fuzzy bi-ideal, for T-conorm S by Theorem 3.6, we have

$$(1-\mu)\subseteq(1-\mu)\circ_S(1-\mu)\circ_S(1-\mu)$$

and

$$(1-\mu)\subseteq 1-[(1-(1-\mu))\cdot_T(1-(1-\mu))\cdot_T(1-(1-\mu))]$$
$$1-\mu=1-(\mu\cdot_T\mu\cdot_T\mu).$$

Then, $\mu\cdot_T\mu\cdot_T\mu\subseteq\mu$.

Conversely, By Theorem 4.3, μ is a T-fuzzy generalized bi-ideal.

$$\mu(abc)\geqslant(\mu\cdot_T\mu\cdot_T\mu)(abc)\geqslant T(\mu(a),T(\mu(b),\mu(c))),\ for\ a,b,c\in R$$

Thus μ is a T-fuzzy bi-ideal. □

THEOREM 4.5. *The fuzzy set μ is a T-fuzzy generalized bi-ideal of R if and only if* $1-\mu\subseteq(1-\mu)\circ_S 0\circ_S(1-\mu)\circ_S 0\circ_S(1-\mu)$*, for T-conorm S.*

PROOF. Let μ be a T-fuzzy generalized bi-ideal of R. By Theorem 4.1, there exists T-conorm S such that $1-\mu$ is a S-fuzzy generalized bi-ideal of R. By Theorem 3.5, holds

$$1-\mu\subseteq(1-\mu)\circ_S 0\circ_S(1-\mu)\circ_S 0\circ_S(1-\mu).$$

Conversely, by Theorem 3.6 we have

$$\begin{aligned} 1-\mu &\subseteq (1-\mu)\circ_S 0 \circ_S (1-\mu)\circ_S 0\circ_S (1-\mu) \\ &= 1-(\mu \cdot_T (1-0)\cdot_T \mu \cdot_T (1-0)\cdot_T \mu) \\ &= 1-(\mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu) \end{aligned}$$

Thus $\mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu \subseteq \mu$. Finaly, By Theorem 4.3, we have μ is a T-fuzzy generalized bi-ideal of R. □

THEOREM 4.6. *The fuzzy set μ is a T-fuzzy bi-ideal of R if and only if*
(i) $(1-\mu) \subseteq (1-\mu)\circ_S (1-\mu)\circ_S (1-\mu)$
(ii) $1-\mu \subseteq (1-\mu)\circ_S 0\circ_S (1-\mu)\circ_S 0\circ_S (1-\mu)$, *for T-conorm S.*

PROOF. Let μ be a T-fuzzy bi-ideal of R. By Theorem 4.1, $1-\mu$ is a S-fuzzy bi-ideal of R. By Theorem 3.5, we have

$$1-\mu \subseteq (1-\mu)\circ_S (1-\mu)\circ_S (1-\mu),$$
$$1-\mu \subseteq (1-\mu)\circ_S 0\circ_S (1-\mu)\circ_S 0\circ_S (1-\mu)$$

Conversely, by Theorem 3.6,

$$1-\mu \subseteq (1-\mu)\circ_S (1-\mu)\circ_S (1-\mu) = 1-(\mu \cdot_T \mu \cdot_T \mu).$$

Thus $\mu \cdot_T \mu \cdot_T \mu \subseteq \mu$.
Similarly,

$$\mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu \subseteq \mu.$$

By Theorem 4.4, we have μ is a T-fuzzy bi-ideals of R. □

THEOREM 4.7. *The fuzzy set μ defined by*

$$\mu(x) = \begin{cases} s & \text{if } x \in B \\ t & \text{otherwise} \end{cases}$$

for $0 \leqslant t \leqslant s \leqslant 1$ is a T-fuzzy bi-ideals of R for all T-norms if and only if B is a bi-ideal of R.

PROOF. If B is a bi-ideal of R. Now, $t \leqslant s$ implies $1-s \leqslant 1-t$.Then

$$(1-\mu)(x) = \begin{cases} 1-s & \text{if } x \in B \\ 1-t & otherwise \end{cases}$$

By Theorem 3.9, $1-\mu$ is a S-fuzzy bi-ideal for all S-norms. By Theorem 4.1, μ is a T-fuzzy bi-ideal of all T-norms.

Conversely, μ is a T_M-fuzzy bi-ideal of R. Let $x, y, z \in B$. Then $\mu(xyz) \geqslant T_M(\mu(x), T_M(\mu(y), \mu(z))) = s$ implies $xyz \in B$. For $u, v \in R$, we have

$$\mu(xuyvz) \geqslant T_M(\mu(x), T_M(\mu(y), \mu(z))) = s.$$

Then $xuyvz \in B$, for all $x, y, z \in B$ and for all $u, v \in R$. Thus B is a bi-ideal of R. □

5. S-Union and T-intersection

Definition 5.1. For the fuzzy sets μ and λ of R and S-norm S, the S-union of μ and λ denoted by $S(\mu, \lambda)$ is defined as follows: $(S(\mu, \lambda))(x) = S(\mu(x), \lambda(x))$ for all $x \in R$.

Theorem 5.1. *If μ and λ are S-fuzzy generalized bi-ideals of R, then $S(\mu, \lambda)$ is a S- fuzzy generalized bi-ideal of R.*

Proof. For S-fuzzy generalized bi-ideals μ and λ of R,

$$\begin{aligned}
(S(\mu,\lambda))(xwyvz) &= S\Big(\mu(xwyvz), \lambda(xwyvz)\Big) \\
&\leqslant S\Big(S\Big[\mu(x), S\Big(\mu(y),\mu(z)\Big)\Big], S\Big[\lambda(x), S\Big(\lambda(y),\lambda(z)\Big)\Big]\Big) \\
&= S\Big(\mu(x), S\Big[S\Big(\mu(y),\mu(z)\Big), S\Big[\lambda(x), S\Big(\lambda(y),\lambda(z)\Big)\Big]\Big]\Big) \\
&= S\Big(\mu(x), S\Big[S\Big[\lambda(x), S\Big(\lambda(y),\lambda(z)\Big)\Big], S\Big(\mu(y),\mu(z)\Big)\Big]\Big) \\
&= S\Big(S\Big[\mu(x), S\Big[\lambda(x), S\Big(\lambda(y),\lambda(z)\Big)\Big]\Big], S\Big(\mu(y),\mu(z)\Big)\Big) \\
&= S\Big(S\Big[S\Big(\mu(x),\lambda(x)\Big), S\Big(\lambda(y),\lambda(z)\Big)\Big], S\Big(\mu(y),\mu(z)\Big)\Big) \\
&= S\Big(\Big(S\Big[\Big(S\Big(\mu,\lambda\Big)\Big)(x), S\Big(\lambda(y),\lambda(z)\Big)\Big], S\Big(\mu(y),\mu(z)\Big)\Big) \\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x), S\Big[S\Big(\mu(y),\mu(z)\Big), S\Big(\lambda(y),\lambda(z)\Big)\Big]\Big) \\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x), S\Big[\mu(y), S\Big(\mu(z), S\Big(\lambda(y),\lambda(z)\Big)\Big)\Big]\Big) \\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x), S\Big[\mu(y), S\Big(S\Big(\lambda(y),\lambda(z)\Big),\mu(z)\Big)\Big]\Big) \\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x), S\Big[\mu(y), S\Big(\lambda(y), S\Big(\lambda(z),\mu(z)\Big)\Big)\Big]\Big) \\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x), S\Big[S\Big(\mu(y),\lambda(y)\Big), S\Big(\mu(z),\lambda(z)\Big)\Big]\Big) \\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x), S\Big[\Big(S\Big(\mu,\lambda\Big)\Big)(y), \Big(S\Big(\mu,\lambda\Big)\Big)(z)\Big]\Big)
\end{aligned}$$

Therefore $(S(\mu,\lambda))(xwyvz) \leqslant S((S(\mu,\lambda))(x), S((S(\mu,\lambda))(y), (S(\mu,\lambda))(z)))$,
for all $x, w, y, v, z \in R$.Hence $S(\mu, \lambda)$ is a S-fuzzy generalized bi-ideal of R. □

Corollary 5.1. ([**8**]) *Union of any two anti fuzzy generalized bi-ideals of a ternary semigroup R is an anti fuzzy generalized bi-ideal of R.*

Proof. By taking S-norm as S_M-norm in Theorem 5.1, we get the result. □

THEOREM 5.2. *If μ and λ are S-fuzzy bi-ideals of R,then $S(\mu,\lambda)$ is a S-fuzzy bi-ideal of R.*

PROOF. If μ and λ are S-fuzzy bi-ideals of R.By Theorem 5.1,$S(\mu,\lambda)$ is a S-fuzzy generalized fuzzy bi-ideals of R.

$$\begin{aligned}
(S(\mu,\lambda))(xyz) &= S\Big(\mu(xyz),\lambda(xyz)\Big)\\
&\leqslant S\Big(S\Big[\mu(x),S\Big(\mu(y),\mu(z)\Big)\Big],S\Big[\lambda(x),S\Big(\lambda(y),\lambda(z)\Big)\Big]\Big)\\
&= S\Big(\mu(x),S\Big[S\Big(\mu(y),\mu(z)\Big),S\Big[\lambda(x),S\Big(\lambda(y),\lambda(z)\Big)\Big]\Big]\Big)\\
&= S\Big(\mu(x),S\Big[S\Big[\lambda(x),S\Big(\lambda(y),\lambda(z)\Big)\Big],S\Big(\mu(y),\mu(z)\Big)\Big]\Big)\\
&= S\Big(S\Big[\mu(x),S\Big[\lambda(x),S\Big(\lambda(y),\lambda(z)\Big)\Big],S\Big(\mu(y),\mu(z)\Big)\Big]\Big)\\
&= S\Big(S\Big[S\Big(\mu(x),\lambda(x)\Big),S\Big(\lambda(y),\lambda(z)\Big)\Big],S\Big(\mu(y),\mu(z)\Big)\Big)\\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x),S\Big[S\Big(\lambda(y),\lambda(z)\Big),S\Big(\mu(y),\mu(z)\Big)\Big]\Big)\\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x),S\Big[S\Big(\mu(y),\mu(z)\Big),S\Big(\lambda(y),\lambda(z)\Big)\Big]\Big)\\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x),S\Big[\mu(y),S\Big(\mu(z),S\Big(\lambda(y),\lambda(z)\Big)\Big)\Big]\Big)\\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x),S\Big[\mu(y),S\Big(S\Big(\lambda(y),\lambda(z)\Big),\mu(z)\Big)\Big]\Big)\\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x),S\Big[\mu(y),S\Big(\lambda(y),S\Big(\lambda(z),\mu(z)\Big)\Big)\Big]\Big)\\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x),S\Big[S\Big(\mu(y),\lambda(y)\Big),S\Big(\mu(z),\lambda(z)\Big)\Big]\Big)\\
&= S\Big(\Big(S\Big(\mu,\lambda\Big)\Big)(x),S\Big[\Big(S\Big(\mu,\lambda\Big)\Big)(y),\Big(S\Big(\mu,\lambda\Big)\Big)(z)\Big]\Big)
\end{aligned}$$

Thus$(S(\mu,\lambda))(xyz) \leqslant S(S(\mu,\lambda))(x),S((S(\mu,\lambda))(y),(S(\mu,\lambda))(z)))$, for all $x,y,z \in R$. Therefore $S(\mu,\lambda)$is a S-fuzzy bi-ideals of R. □

COROLLARY 5.2. ([**9**]) *If μ and λ are anti fuzzy bi-ideal of a semigroup R, then $\mu \cup \lambda$ is an antifuzzy bi-ideal of R.*

PROOF. By taking S-norm as S_M-norm in Theorem 5.1, we get the result. □

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