ON GENERALIZED FUZZY GENERALIZED FUZZY BI-IDEALS OF TERNARY SEMIGROUPS

Gnanasigamani Mohanraj and M. Vela

Abstract. We introduce the notion of $S$-fuzzy generalized bi-ideal. We introduce ternary $S$ and ternary $T$-products of fuzzy sets of ternary semigroup. We find interrelationship between ternary $S$-product and ternary $T$-product. We redefine $S$-fuzzy generalized bi-ideal by using ternary $S$-product and ternary $T$-product of ternary semigroup. We introduce the notion of $S$-union of fuzzy sets. We establish that $S$-union of $S$-fuzzy bi-ideal is again a $S$-fuzzy bi-ideal.

1. Introduction


2. Preliminaries

A non-empty set $R$ is called a ternary semigroup if there exists a mapping $R \times R \times R \to R$ denoted by juxtaposition that satisfies: $(abc)de = a(bcd)e = ab(cde)$ for all $a, b, c, d, e \in R$. A non-empty set $B$ of $R$ is called generalized bi-ideal if $BRBRB \subseteq B$. The generalized bi-ideal $B$ of $R$ is called bi-ideal if $BBB \subseteq B$. A mapping $\mu : X \to [0, 1]$ is called a fuzzy set of $X$. The fuzzy set $\mu$ of $R$ is called generalized fuzzy bi-ideal if $\mu(xwyvz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in R$.

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R. The generalized fuzzy bi-ideal $\mu$ of $R$ is called a fuzzy bi-ideal if $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in R$. The fuzzy set $\mu$ of $R$ is called generalized anti fuzzy bi-ideals if $\mu(xwyz) \leq \max\{\mu(x), \mu(y), \mu(z)\}$, for all $x, y, z, u, v \in R$. The generalized anti fuzzy bi-ideals $\mu$ of $R$ is called anti fuzzy bi-ideal if $\mu(xyz) \leq \max\{\mu(x), \mu(y), \mu(z)\}$, for all $x, y, z \in R$.

### 3. S-fuzzy bi-ideals

**Definition 3.1.** The binary operation $S$ on $[0,1]$ is called a $S$-norm on $[0,1]$ if satisfies the following conditions:

1. $(S1)$ $S(x,0) = S(0,x) = x$ (boundary condition)
2. $(S2)$ $S(x,y) = S(y,x)$ (commutativity)
3. $(S3)$ $S(S(x,y), z) = S(x, S(y,z))$ (associativity)
4. $(S4)$ If $x^* \leq x$ and $y^* \leq y$ then $S(x^*, y^*) \leq S(x,y)$ (monotonicity)

for all $x, y, z, x^*, y^* \in [0,1]$.

**Definition 3.2.** (6) The binary operation $T$ on $[0,1]$ is called a triangular norm [T-norm] on $[0,1]$ which satisfies $S2$ to $S4$ and $T(x,1) = T(1,x) = x$.

**Theorem 3.1.** (3) The function $S : [0,1] \times [0,1] \rightarrow [0,1]$ is a $S$-norm (T-conorm) if and only if there exist a $T$-norm (S-conorm) such that $S(x,y)=1-T(1-x,1-y)(1)$ for all $x, y \in [0,1]$.

**Remark 3.1.** (1) By above Theorem 3.1, for each $S$-norm $S$, there exists $T$-norm satisfying Equation (1) and that $T$-norm is a called $S$-conorm.

(2) For each $T$-norm $T$, Theorem 3.1, there exists $S$-norm $S$ satisfying $T(x,y) = 1-S(1-x,1-y)$ and that $S$-norm $S$ is called $T$-conorm.

(3) Various $S$-norms and corresponding $S$-conorms are tabulated as follows

<table>
<thead>
<tr>
<th>$S$-norm</th>
<th>$T$-norm (S-conorm)</th>
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</thead>
<tbody>
<tr>
<td>$S_M(x,y) = \max{x,y}$</td>
<td>$T_M(x,y) = \min{x,y}$</td>
</tr>
<tr>
<td>$S_P(x,y) = x + y - x \cdot y$</td>
<td>$T_P(x,y) = x \cdot y$</td>
</tr>
<tr>
<td>$S_L(x,y) = \min{x+y,1}$</td>
<td>$T_L(x,y) = \max{x+y-1,0}$</td>
</tr>
<tr>
<td></td>
<td>$S_D(x,y) = \begin{cases} 1 \quad &amp; \text{if } x,y \in [0,1) \ \max{x,y} &amp; \text{otherwise,} \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$T_D(x,y) = \begin{cases} 0 \quad &amp; \text{if } x,y \in [0,1) \ \min{x,y} &amp; \text{otherwise,} \end{cases}$</td>
</tr>
<tr>
<td>Hamacher class $S$-norm for $\lambda \in [0,\infty]$</td>
<td>Hamacher class $T$-norm for $\lambda \in [0,\infty]$</td>
</tr>
<tr>
<td>$(S'_\lambda)(x,y) = \begin{cases} S_D(x,y) \quad &amp; \text{if } \lambda = 0 \ 1 \quad &amp; \text{if } x = y = 1 \ \frac{x+y-xy-(1-\lambda)xy}{1-(\lambda)x+y} &amp; \text{otherwise.} \end{cases}$</td>
<td>$(T'_\lambda)(x,y) = \begin{cases} T_D(x,y) \quad &amp; \text{if } \lambda = \infty \ 0 \quad &amp; \text{if } \lambda = x = y = 0 \ \frac{xy}{x+(1-\lambda)(x+y-xy)} &amp; \text{otherwise.} \end{cases}$</td>
</tr>
</tbody>
</table>

**Theorem 3.2.** (3) Every $T$-norm $[0,1]$ satisfies the inequality as follows

$T_D(x,y) \leq T(x,y) \leq T_M(x,y)$, for all $x, y \in [0,1]$.
Theorem 3.3. Every $S$-norm $S$ satisfies the inequality

\[ S_D(x, y) \leq S(x, y) \leq S_M(x, y), \quad \text{for all } x, y \in [0, 1]. \]

Proof. By Theorem 3.2, we have $T_M(x, y) \geq T(x, y) \geq T_D(x, y)$, for all $x, y \in [0, 1]$. By Theorem 3.1, $S_M(x, y) \leq S(x, y) \leq S_D(x, y)$, for all $x, y \in [0, 1]$. \qed

Hereafter, $R$ denotes a ternary semigroup and $S$ denotes $S$-norm on $[0, 1]$, whereas $T$ denotes a corresponding $S$-conorm on $[0, 1]$ unless otherwise specified.

Definition 3.3. The fuzzy set $\mu$ of $R$ is called generalized $S$-fuzzy bi-ideal if $\mu(xwyz) \leq S(\mu(x), S(\mu(y), \mu(z)))$, for all $x, y, z, w, v \in R$.

Definition 3.4. The generalized $S$-fuzzy bi-ideal $\mu$ of $R$ is called a $S$-fuzzy bi-ideal of $R$ if $\mu(xyz) \leq S(\mu(x), S(\mu(y), \mu(z)))$ for all $x, y, z, v \in R$.

Definition 3.5. ([7]) The fuzzy set $\mu$ of $R$ is called generalized $T$-fuzzy bi-ideal if $\mu(xwyz) \geq T(\mu(x), T(\mu(y), \mu(z)))$, for all $x, y, z, w, v \in R$.

Definition 3.6. ([7]) The generalized $T$-fuzzy bi-ideal $\mu$ of $R$ is called $T$-fuzzy bi-ideal of $R$ if $\mu(xyz) \geq T(\mu(x), T(\mu(y), \mu(z)))$, for all $x, y, z \in R$.

Here, we redefine $S$-fuzzy bi-ideals by using ternary $S$-products and ternary $T$-products.

Definition 3.7. The ternary $S$-product and the ternary $T$-product of the fuzzy sets $\lambda, \mu, \sigma$ of $R$ denoted by $\lambda \circ_S \mu \circ_S \sigma$ and $\lambda \cdot_T \mu \cdot_T \sigma$ are defined as follows:

\[
(\lambda \circ_S \mu \circ_S \sigma)(x) = \begin{cases} 
\inf_{x=abc} S(\lambda(a), S(\mu(b), \sigma(c))) & \text{if } x = abc \\
1 & \text{otherwise}
\end{cases}
\]

\[
(\lambda \cdot_T \mu \cdot_T \sigma)(x) = \begin{cases} 
\sup_{x=abc} T(\lambda(a), T(\mu(b), \sigma(c))) & \text{if } x = abc \\
0 & \text{otherwise}
\end{cases}
\]

Remark 3.2. (1) By taking $S$-norm as $S_M$-norm, then the ternary $S$-product becomes ternary “$\circ$” product

\[
(\lambda \circ \mu \circ \sigma)(x) = \begin{cases} 
\inf_{x=abc} \{\max\{\lambda(a), \mu(b), \sigma(c)\}\} & \text{if } x = abc \\
1 & \text{otherwise}
\end{cases}
\]

(2) By taking $T$-norm as $T_M$-norm, then the ternary $T$-product becomes ternary “$.$” product

\[
(\lambda \cdot \mu \cdot \sigma)(x) = \begin{cases} 
\sup_{x=abc} \{\min\{\lambda(a), \mu(b), \sigma(c)\}\} & \text{if } x = abc \\
0 & \text{otherwise}
\end{cases}
\]

Definition 3.8. The fuzzy sets 0 and 1 of $R$ are defined as follows:

\[
0(x) = 0, 1(x) = 1 \quad \text{for all } x \in R
\]
Theorem 3.4. The fuzzy set $\mu$ is a $S$-fuzzy generalized bi-ideal of $R$ if and only if $\mu \subseteq \mu \circ_S \mu \circ_S \mu$.

Proof. For a $S$-fuzzy generalized bi-ideal $\mu$ of $R$ and if $x$ cannot be expressible as $x = awbvc$, then $(\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu)(x) = 1 \geq \mu(x)$. Now,

$$(\mu \circ_S 0 \circ_S \mu) \circ_S 0 \circ_S \mu(x) = \inf_{x=awb} S((\mu \circ_S 0 \circ_S \mu)(u), S(0(v), \mu(c)))$$

$$= \inf_{x=awb} S(\inf_{u=awb} S(\mu(a), S(0(w), \mu(b))), \mu(c))$$

$$= \inf_{x=awb} S(\mu(a), S(\mu(b), \mu(c)))$$

Now

$$\mu(awbvc) \leq S(\mu(a), S(\mu(b), \mu(c)))$$

implies $\mu(x) \leq \inf_{x=awb} S(\mu(a), S(\mu(b), \mu(c)))$. Thus

$$\mu(x) \leq \inf_{x=awb} S(\mu(a), S(\mu(b), \mu(c))) \leq \inf_{x=awb} S(\mu(a), S(\mu(b), \mu(c)))$$

$$= (\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu)(x)$$

Conversely,

$$\mu(xwyvz) \leq ((\mu \circ_S 0 \circ_S \mu) \circ_S 0 \circ_S \mu)(xwyvz) \leq S(\mu(x), S(\mu(y), \mu(z)))$$

Hence $\mu$ is a $S$-fuzzy generalized bi-ideal of $R$.

Theorem 3.5. The fuzzy set $\mu$ is a $S$-fuzzy-bi-ideal of $R$ if and only if

(i) $\mu \subseteq \mu \circ_S \mu \circ_S \mu$.

(ii) $\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu$

Proof. By Theorem 3.4, $\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu$, when $\mu$ is a $S$-fuzzy bi-ideal. If $x$ cannot be expressible as $x = abc$, then $(\mu \circ_S 0 \circ_S \mu)(x) = 1 \geq \mu(x)$. Now $\mu(abc) \leq S(\mu(a), S(\mu(b), \mu(c)))$. Then,

$$\mu(x) \leq \inf_{x=abc} S(\mu(a), S(\mu(b), \mu(c))) = (\mu \circ_S 0 \circ_S \mu)(x).$$

Conversely, By Theorem 3.4, $\mu$ is a $S$-fuzzy generalized bi-ideal

$$\mu(abc) \leq (\mu \circ_S \mu \circ_S \mu)(abc) \leq S(\mu(0), (\mu(b), \mu(c)))$$

Hence $\mu$ is a $S$ fuzzy-bi-ideal of $R$.

Theorem 3.6. If $S$ is a $S$-norm and $T$ is it $S$-conorm (T-norm), then

(i) $1 - (\lambda \circ_S \mu \circ_S \sigma) = (1 - \lambda) \cdot_T (1 - \mu) \cdot_T (1 - \sigma)$

(ii) $1 - (\lambda \cdot_T \mu \cdot_T \sigma) = (1 - \lambda) \circ_S (1 - \mu) \circ_S (1 - \sigma)$,

for any fuzzy set $\lambda, \mu$ and $\sigma$ of $R$. 
Proof. For the fuzzy sets $\lambda, \mu$ and $\sigma$ of $R$,
\[
(\lambda \circ_S \mu \circ_S \sigma)(x) = \inf_{x=abc} S(\lambda(a), S(\mu(b), \sigma(c)))
\]
\[
= \inf_{x=abc} 1 - T(1 - \lambda(a), 1 - S(\mu(b), \sigma(c)))
\]
\[
= \inf_{x=abc} 1 - T(1 - \lambda(a), 1 - (1 - T(1 - \mu(b), 1 - \sigma(c))))
\]
\[
= \inf_{x=abc} 1 - T(1 - \lambda(a), T(1 - \mu(b), 1 - \sigma(c)))
\]
\[
= \inf_{x=abc} 1 - T((1 - \lambda)(a), T((1 - \mu)(b), (1 - \sigma)(c)))
\]
\[
= 1 - \sup_{x=abc} T((1 - \lambda)(a), T((1 - \mu)(b), (1 - \sigma)(c))).
\]

Then,
\[
(1 - (\lambda \circ_S \mu \circ_S \sigma))(x) = \sup_{x=abc} T((1 - \lambda)(a), T((1 - \mu)(b), (1 - \sigma)(c))).
\]

Therefore
\[
1 - (\lambda \circ_S \mu \circ_S \sigma) = (1 - \lambda) \cdot_T (1 - \mu) \cdot_T (1 - \sigma).
\]

Now,
\[
(\lambda \cdot_T \mu \cdot_T \sigma)(x) = \sup_{x=abc} T(\lambda(a), T(\mu(b), \sigma(c)))
\]
\[
= \sup_{x=abc} 1 - S(1 - \lambda(a), 1 - T(\mu(b), \sigma(c)))
\]
\[
= \sup_{x=abc} 1 - S(1 - \lambda(a), 1 - (1 - S(1 - \mu(b), 1 - \sigma(c))))
\]
\[
= \sup_{x=abc} 1 - S(1 - \lambda(a), S(1 - \mu(b), 1 - \sigma(c)))
\]
\[
= \sup_{x=abc} 1 - S((1 - \lambda)(a), S((1 - \mu)(b), (1 - \sigma)(c)))
\]
\[
= 1 - \inf_{x=abc} S((1 - \lambda)(a), S((1 - \mu)(b), (1 - \sigma)(c)))
\]

Then,
\[
1 - (\lambda \cdot_T \mu \cdot_T \sigma)(x) = \inf_{x=abc} S((1 - \lambda)(a), S((1 - \mu)(b), (1 - \sigma)(c)))
\]

Therefore
\[
1 - (\lambda \cdot_T \mu \cdot_T \sigma) = (1 - \lambda) \circ_S (1 - \mu) \circ_S (1 - \sigma).
\]

**Theorem 3.7.** The fuzzy set $\mu$ is a $S$-fuzzy generalized bi-ideal of $R$ if and only if there exists $S$-conorm $T$ such that $(1 - \mu) \cdot_T 1 \cdot_T (1 - \mu) \cdot_T 1 \subseteq 1 - \mu$.

**Proof.** For a $S$-fuzzy generalized bi-ideal $\mu$ of $R$ and by Theorem 3.4, $\mu \subseteq \mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu$. Then
\[
1 - (\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu) \subseteq 1 - \mu.
\]

By Theorem 3.6, we have
\[
(1 - \mu) \cdot_T (1 - 0) \cdot_T (1 - \mu) \cdot_T (1 - 0) \cdot_T (1 - \mu) \subseteq 1 - \mu.
\]

Thus
\[
(1 - \mu) \cdot_T 1 \cdot_T (1 - \mu) \cdot_T 1 \cdot_T (1 - \mu) \subseteq 1 - \mu.
\]

Conversely, by Theorem 3.6, we have
\[
1 - (\mu \circ_S 0 \circ_S \mu \circ_S 0 \circ_S \mu) = (1 - \mu) \cdot_T 1 \cdot_T (1 - \mu) \cdot_T 1 \cdot_T (1 - \mu) \subseteq 1 - \mu.
\]
Thus
\[ \mu \subseteq \mu \circ S 0 \circ S \mu \circ S 0 \circ S \mu. \]
By Theorem 3.4, \( \mu \) is a \( S \)-fuzzy generalized bi-ideal of \( R \).

**Theorem 3.8.** The fuzzy set \( \mu \) is a \( S \)-fuzzy bi-ideal of \( R \) if and only if there exists \( S \)-conorm \( T \) such that
\begin{align*}
(1 - \mu) \cdot T (1 - \mu) \cdot T (1 - \mu) & \subseteq (1 - \mu), \\
(1 - \mu) \cdot T 1 \cdot T 1 \cdot T (1 - \mu) & \subseteq 1 - \mu
\end{align*}

**Proof.** For a \( S \)-fuzzy bi-ideal \( \mu \) of \( R \) and by Theorem 3.7, there exist \( S \)-conorm \( T \),
\[ (1 - \mu) \cdot T (1 - \mu) \cdot T (1 - \mu) \subseteq 1 - \mu. \]
By Theorem 3.5, \( \mu \subseteq \mu \circ S \mu \circ S \mu \) implies \( 1 - (\mu \circ S \mu \circ S \mu) \subseteq 1 - \mu \). By Theorem 3.6, there exist \( S \)-conorm \( T \),
\[ (1 - \mu) \cdot T (1 - \mu) \cdot T (1 - \mu) = 1 - (\mu \circ S \mu \circ S \mu) \subseteq 1 - \mu. \]
Conversely, by Theorem 3.6, we have that
\[ 1 - (\mu \circ S \mu \circ S \mu) \subseteq (1 - \mu) \cdot T (1 - \mu) \cdot T (1 - \mu) \subseteq 1 - \mu \]
implies
\[ \mu \subseteq \mu \circ S \mu \circ S \mu. \]
Similarly, by Theorem 3.5
\[ \mu \subseteq \mu \circ S 0 \circ S \mu \circ S 0 \circ S \mu. \]
Thus \( \mu \) is a \( S \)-fuzzy bi-ideal of \( R \).

**Theorem 3.9.** The fuzzy set \( \mu(x) = \begin{cases} t & \text{if } x \in B \\ s & \text{otherwise} \end{cases} \)
for \( 0 \leq t \leq s \leq 1 \) is a \( S \)-fuzzy generalized bi-ideal of \( R \) for all \( S \)-norms \( S \) if and only if \( B \) is a generalized bi-ideal of \( R \).

**Proof.** Let \( B \) be a generalized bi-ideal of \( R \) and let \( \mu \) be the fuzzy set defined as above for \( 0 \leq t \leq s \leq 1 \). If \( t = s \), then \( \mu \) is constant. Thus \( \mu \) is \( S \)-fuzzy generalized bi-ideal. Otherwise if \( xuyz \in B \), then for all \( u, v \in R \) holds
\[ \mu(xuyvz) = \begin{cases} t & \text{if } x \in B \\ s & \text{otherwise} \end{cases} \]
If \( xuyvz \notin B \), then either \( x \notin B \) or \( y \notin B \) or \( z \notin B \). Now,
\[ \mu(xuyvz) = s = S_M(\mu(x), S_M(\mu(y), \mu(z))) \]
and by Theorem 3.3,
\[ \mu(xuyvz) \leq S_M(\mu(x), S_M(\mu(y), \mu(z))) \leq S(\mu(x), S(\mu(y), \mu(z))) \]
for any \( S \)-norms and for \( x, y, z \in B \). \( u, v \in R \). Therefore \( \mu \) is a \( S \)-fuzzy generalized bi-ideal, for all \( S \)-norms \( S \).

Conversely, for \( u, v \in R \) and \( x, y, z \in B \),
\[ t = S_M(\mu(x), S_M(\mu(y), \mu(z))) \geq \mu(xuyvz), \text{ implies } xuyvz \in B \]. Thus \( B \) is a generalized bi-ideal of \( R \).
**Theorem 3** is an anti fuzzy generalized bi-ideal of \( R \) for \( 0 \leq x, y, z, w, v \leq 1 \). Therefore \( S \) is a bi-ideal of \( R \). Consequently, by Theorem 3, \( S \) is a fuzzy generalized bi-ideal of \( R \). Therefore \( S \) is a bi-ideal of \( R \).

**Proof.** By taking \( S \)-norm as \( S_M \) in Theorem 3.9, we get the result. \( \square \)

**Theorem 3.10.** The fuzzy set
\[
\mu(x) = \begin{cases} 
  t & \text{if } x \in B, \\
  s & \text{otherwise}
\end{cases}
\]
for \( 0 \leq t \leq s \leq 1 \) is a \( S \)-fuzzy bi-ideal of \( R \) for all \( S \)-norms \( S \) if and only if \( B \) is a bi-ideal of \( R \).

**Proof.** Let \( B \) be a bi-ideal of \( R \) and let \( \mu \) be the fuzzy set defined as above for \( 0 \leq t \leq s \leq 1 \). If \( t = s \), then \( \mu \) is constant. Thus \( \mu \) is \( S \)-fuzzy bi-ideal. By Theorem 3.9, \( \mu \) is a \( S \)-fuzzy generalized bi-ideal of \( R \). If \( x, y, z \in B \), then \( xyz \in B \) implies
\[
\mu(xyz) = t \leq S_M(\mu(x), S_M(\mu(y), \mu(z))) \leq S(\mu(x), S(\mu(y), \mu(z)))
\]
If \( xyz \notin B \), then \( x \notin B \) or \( y \notin B \) or \( z \notin B \). Thus
\[
\mu(xyz) = s = S_M(\mu(x), S_M(\mu(y), \mu(z))) \leq S(\mu(x), S(\mu(y), \mu(z))).
\]
Therefore \( \mu \) is a \( S \)-fuzzy bi-ideal for all \( S \)-norms.

Conversely, by Theorem 3.9, \( B \) is generalized bi-ideal. If \( x, y, z \in B \), then \( t = S_M(\mu(x), S_M(\mu(y), \mu(z))) \geq \mu(xyz) \) implies \( xyz \in B \). Thus \( B \) is a bi-ideal of \( R \). \( \square \)

**4. \( T \)-fuzzy bi-ideals**

**Theorem 4.1.** The fuzzy set \( \mu \) is a \( S \)-fuzzy generalized bi-ideal of \( R \) if and only if there exists \( S \)-conorm \( T \) such that \( 1 - \mu \) is a \( T \)-fuzzy generalized bi-ideal of \( R \).

**Proof.** If \( \mu \) is a \( S \)-fuzzy generalized bi-ideal of \( R \), then by Theorem 3.1, there exists \( S \)-conorm \( T \) such that \( S(x, y) = 1 - T(1 - x, 1 - y) \) for all \( x, y \in [0, 1] \). For \( x, y, z, w, v \in R \),
\[
\mu(xwyz) \leq S(\mu(x), S(\mu(y), \mu(z)))
\]
\[
= 1 - T(1 - \mu(x), 1 - S(\mu(y), \mu(z)))
\]
\[
= 1 - T(1 - \mu(x), 1 - (1 - T(1 - \mu(y), 1 - \mu(z))))
\]
\[
= 1 - T(1 - \mu(x), T(1 - \mu(y), 1 - \mu(z)))
\]
\[
= 1 - T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z)))
\]
Therefore
\[
-\mu(xwyz) \geq -1 + T((1 - \mu(x), T((1 - \mu)(y), (1 - \mu)(z))))
\]
Then, \((1 - \mu)(xwyvz) \geq T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z)))\) and \(1 - \mu\) is a \(T\)-fuzzy generalized bi-ideal of \(R\).

Conversely,
\[
(1 - \mu)(xwyvz) \geq T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z)))
\]
\[
= T(1 - \mu(x), T(1 - \mu(y), 1 - \mu(z)))
\]
\[
= 1 - S(1 - (1 - \mu(x), 1 - (1 - \mu(y), 1 - \mu(z))))
\]
\[
= 1 - S(\mu(x), 1 - (1 - S(1 - (1 - \mu(y), 1 - (1 - \mu(z)))))
\]
\[
= 1 - S(\mu(x), S(\mu(y), \mu(z)))
\]

Thus
\[-1 + \mu(xwyvz) \leq -1 + S(\mu(x), S(\mu(y), \mu(z))).\]

Then, \((xwyvz) \leq S(\mu(x), S(\mu(y), \mu(z)))\). Therefore, \(\mu\) is a \(S\)-fuzzy generalized bi-ideal of \(R\).

\textbf{Theorem 4.2.} The fuzzy set \(\mu\) is a \(S\)-fuzzy bi-ideal of \(R\) if and only if there exists \(S\)-conorm \(T\) such that \(1 - \mu\) is a \(T\)-fuzzy bi-ideal of \(R\).

\textbf{Proof.} If \(\mu\) is a \(S\)-fuzzy bi-ideal of \(R\), then by Theorem 4.1, \(1 - \mu\) is a \(T\)-fuzzy generalized bi-ideal. For \(x, y, z \in R\) and by Theorem 3.1,
\[
\mu(xyz) \leq S(\mu(x), S(\mu(y), \mu(z)))
\]
\[
= 1 - T(1 - \mu(x), 1 - S(\mu(y), \mu(z)))
\]
\[
= 1 - T(1 - \mu(x), 1 - (1 - T(1 - \mu(y), 1 - \mu(z))))
\]
\[
= 1 - T(1 - \mu(x), T(1 - \mu(y), 1 - \mu(z)))
\]
\[
= 1 - T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z))))
\]

Then \((1 - \mu)(xyz) \geq T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z))))\). Thus \(\mu\) is a \(T\)-fuzzy bi-ideal of \(R\).

Conversely, by Theorem 4.1, \((xwyvz) \leq S(\mu(x), S(\mu(y), \mu(z)))\). Now,
\[
(1 - \mu)(xyz) \geq T((1 - \mu)(x), T((1 - \mu)(y), (1 - \mu)(z))))
\]
\[
= T(1 - \mu(x), T(1 - \mu(y), 1 - \mu(z)))
\]
\[
= 1 - S(1 - (1 - \mu(x), 1 - T(1 - \mu(y), 1 - \mu(z))))
\]
\[
= 1 - S(\mu(x), 1 - (1 - S(1 - (1 - \mu(y), 1 - (1 - \mu(z))))
\]
\[
= 1 - S(\mu(x), S(\mu(y), \mu(z)))
\]

Then
\[
\mu(xyz) \leq S(\mu(x), S(\mu(y), \mu(z)))
\]

Therefore \(\mu\) is a \(S\)-fuzzy bi-ideal of \(R\).

\textbf{Theorem 4.3.} The fuzzy set \(\mu\) is a \(T\)-fuzzy generalized bi-ideal of \(R\) if and only if \(\mu \cdot T \geq T \cdot \mu \leq \mu\)
Thus, Theorem 3.5 holds only if there exists $T$-conorm $S$ such that $1 - \mu$ is a $S$-fuzzy generalized bi-ideal. Now, by Theorem 3.7, we have

$$(1 - (1 - \mu)) \ast_T (1 - (1 - \mu)) \ast_T (1 - (1 - \mu)) \subseteq 1 - (1 - \mu).$$

Thus,

$$\mu \ast_T 1 \ast_T \mu \ast_T 1 \ast_T \mu \subseteq \mu.$$

Conversely,

$$\mu(xwyvz) \geq ((\mu \ast_T 1 \ast_T \mu) \ast_T 1 \ast_T \mu)(xwyvz)$$
$$= T((\mu \ast_T 1 \ast_T \mu)(xwy), T(1(v), \mu(z)))$$
$$= T((\mu \ast_T 1 \ast_T \mu)(xwy), \mu(z))$$
$$\geq T(T(\mu(x), T(1(w), \mu(y))), \mu(z))$$
$$= T(T(\mu(x), \mu(y)), \mu(z))$$
$$= T(\mu(x), T(\mu(y), \mu(z))), \text{ for all } x, y, z, w, v \in R$$

Thus $\mu$ is a $T$-fuzzy generalized bi-ideal. \hfill \Box

**Theorem 4.4.** The fuzzy set $\mu$ is a $T$-fuzzy bi-ideal of $R$ if and only if

(i) $\mu \ast_T 1 \ast_T \mu \subseteq \mu$
(ii) $\mu \ast_T 1 \ast_T \mu \ast_T 1 \ast_T \mu \subseteq \mu$.

**Proof.** Let $\mu$ be $T$-fuzzy bi-ideal of $R$. By Theorem 4.3,

$$\mu \ast_T 1 \ast_T \mu \ast_T 1 \ast_T \mu \subseteq \mu.$$  

By Theorem 4.2, $1 - \mu$ is a $S$-fuzzy bi-ideal, for $T$-conorm $S$ by Theorem 3.6, we have

$$(1 - \mu) \subseteq (1 - \mu) \circ_S (1 - \mu) \circ_S (1 - \mu)$$

and

$$(1 - \mu) \subseteq 1 - [(1 - (1 - \mu)) \ast_T (1 - (1 - \mu)) \ast_T (1 - (1 - \mu))].$$

Then, $\mu \ast_T 1 \ast_T \mu \subseteq \mu$.

Conversely, By Theorem 4.3, $\mu$ is a $T$-fuzzy generalized bi-ideal.

$$\mu(abc) \geq (\mu \ast_T 1 \ast_T \mu)(abc) \geq T(\mu(a), T(\mu(b), \mu(c))), \text{ for } a, b, c \in R$$

Thus $\mu$ is a $T$-fuzzy bi-ideal. \hfill \Box

**Theorem 4.5.** The fuzzy set $\mu$ is a $T$-fuzzy generalized bi-ideal of $R$ if and only if $1 - \mu \subseteq (1 - \mu) \circ_S 0 \circ_S (1 - \mu) \circ_S (1 - \mu)$, for $T$-conorm $S$.

**Proof.** Let $\mu$ be a $T$-fuzzy generalized bi-ideal of $R$. By Theorem 4.1, there exists $T$-conorm $S$ such that $1 - \mu$ is a $S$-fuzzy generalized bi-ideal of $R$. By Theorem 3.5, holds

$$1 - \mu \subseteq (1 - \mu) \circ_S 0 \circ_S (1 - \mu) \circ_S 0 \circ_S (1 - \mu).$$
Conversely, by Theorem 3.6 we have
\[
1 - \mu \subseteq (1 - \mu) \circ_S 0 \circ_S (1 - \mu) \circ_S 0 \circ_S (1 - \mu)
= 1 - (\mu \cdot_T (1 - 0) \cdot_T \mu \cdot_T (1 - 0) \cdot_T \mu)
= 1 - (\mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu)
\]
Thus \( \mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu \subseteq \mu \). Finally, By Theorem 4.3, we have \( \mu \) is a \( T \)-fuzzy generalized bi-ideal of \( R \).

**Theorem 4.6.** The fuzzy set \( \mu \) is a \( T \)-fuzzy bi-ideal of \( R \) if and only if
(i) \((1 - \mu) \subseteq (1 - \mu) \circ_S (1 - \mu) \circ_S (1 - \mu)\)
(ii) \(1 - \mu \subseteq (1 - \mu) \circ_S 0 \circ_S (1 - \mu) \circ_S 0 \circ_S (1 - \mu)\), for \( T \)-conorm \( S \).

**Proof.** Let \( \mu \) be a \( T \)-fuzzy bi-ideal of \( R \). By Theorem 4.1, \( 1 - \mu \) is a \( S \)-fuzzy bi-ideal of \( R \). By Theorem 3.5, we have
\[
1 - \mu \subseteq (1 - \mu) \circ_S (1 - \mu) \circ_S (1 - \mu),
1 - \mu \subseteq (1 - \mu) \circ_S 0 \circ_S (1 - \mu) \circ_S 0 \circ_S (1 - \mu)
\]
Conversely, by Theorem 3.6,
\[
1 - \mu \subseteq (1 - \mu) \circ_S (1 - \mu) \circ_S (1 - \mu) = 1 - (\mu \cdot_T 1 \cdot_T \mu).
\]
Thus \( \mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu \subseteq \mu \).
Similarly,
\[
\mu \cdot_T 1 \cdot_T \mu \cdot_T 1 \cdot_T \mu \subseteq \mu.
\]
By Theorem 4.4, we have \( \mu \) is a \( T \)-fuzzy bi-ideals of \( R \).

**Theorem 4.7.** The fuzzy set \( \mu \) defined by
\[
\mu(x) = \begin{cases} 
s & \text{if } x \in B \\
t & \text{otherwise} \end{cases}
\]
for \( 0 \leq t \leq s \leq 1 \) is a \( T \)-fuzzy bi-ideals of \( R \) for all \( T \)-norms if and only if \( B \) is a bi-ideal of \( R \).

**Proof.** If \( B \) is a bi-ideal of \( R \). Now, \( t \leq s \) implies \( 1 - s \leq 1 - t \). Then
\[
(1 - \mu)(x) = \begin{cases} 
1 - s & \text{if } x \in B \\
1 - t & \text{otherwise} \end{cases}
\]
By Theorem 3.9, \( 1 - \mu \) is a \( S \)-fuzzy bi-ideal for all \( S \)-norms. By Theorem 4.1, \( \mu \) is a \( T \)-fuzzy bi-ideal of all \( T \)-norms.
Conversely, \( \mu \) is a \( T_M \)-fuzzy bi-ideal of \( R \). Let \( x, y, z \in B \). Then \( \mu(xyz) \geq T_M(\mu(x), T_M(\mu(y), \mu(z))) = s \) implies \( xyz \in B \). For \( u, v \in R \), we have
\[
\mu(xuvz) \geq T_M(\mu(x), T_M(\mu(y), \mu(z))) = s.
\]
Then \( xuvz \in B \), for all \( x, y, z \in B \) and for all \( u, v \in R \). Thus \( B \) is a bi-ideal of \( R \).
5. S-Union and T-intersection

**Definition 5.1.** For the fuzzy sets $\mu$ and $\lambda$ of $R$ and $S$-norm $S$, the $S$-union of $\mu$ and $\lambda$ denoted by $(S(\mu, \lambda))(x)$ is defined as follows: $(S(\mu, \lambda))(x) = S(\mu(x), \lambda(x))$ for all $x \in R$.

**Theorem 5.1.** If $\mu$ and $\lambda$ are $S$-fuzzy generalized bi-ideals of $R$, then $S(\mu, \lambda)$ is a $S$-fuzzy generalized bi-ideal of $R$.

**Proof.** For $S$-fuzzy generalized bi-ideals $\mu$ and $\lambda$ of $R$,

$$(S(\mu, \lambda))(xwyvz) = S(\mu(xwyvz), \lambda(xwyvz))$$

$$\leq S\left(S\left[\mu(x), S\left(\mu(y), \mu(z)\right)\right], S\left[\lambda(x), S\left(\lambda(y), \lambda(z)\right)\right]\right)$$

$$= S\left(\mu(x), S\left[\lambda(x), S\left(\lambda(y), \lambda(z)\right)\right], S\left(\mu(y), \mu(z)\right)\right)$$

$$= S\left(S\left[\mu(x), S\left(\lambda(x), S\left(\lambda(y), \lambda(z)\right)\right)\right], S\left(\mu(y), \mu(z)\right)\right)$$

$$= S\left(S\left[S\left(\mu(x), \lambda(x)\right), S\left(\lambda(y), \lambda(z)\right)\right], S\left(\mu(y), \mu(z)\right)\right)$$

Therefore $(S(\mu, \lambda))(xwyvz) \leq S((S(\mu, \lambda))(x), S((S(\mu, \lambda))(y), (S(\mu, \lambda))(z)))$, for all $x, w, y, v, z \in R$. Hence $S(\mu, \lambda)$ is a $S$-fuzzy generalized bi-ideal of $R$.

**Corollary 5.1.** Union of any two anti fuzzy generalized bi-ideals of a ternary semigroup $R$ is an anti fuzzy generalized bi-ideal of $R$.

**Proof.** By taking $S$-norm as $S_{M}$-norm in Theorem 5.1, we get the result.
THEOREM 5.2. If $\mu$ and $\lambda$ are $S$-fuzzy bi-ideals of $R$, then $S(\mu, \lambda)$ is a $S$-fuzzy bi-ideal of $R$.

**Proof.** If $\mu$ and $\lambda$ are $S$-fuzzy bi-ideals of $R$, by Theorem 5.1, $S(\mu, \lambda)$ is a $S$-fuzzy generalized fuzzy bi-ideals of $R$.

$$(S(\mu, \lambda))(xyz) = S\left(\mu(xyz), \lambda(xyz)\right)$$

$$\leq S\left(S\left[\mu(x), S\left(\mu(y), \mu(z)\right)\right], S\left[\lambda(x), S\left(\lambda(y), \lambda(z)\right)\right]\right)$$

$$= S\left(S\left(S\left(\mu(x), \mu(y)\right), S\left(\mu(z), \mu(y)\right)\right), S\left(S\left(\lambda(x), \lambda(y)\right), S\left(\lambda(y), \lambda(z)\right)\right)\right)$$

$$= S\left(S\left(S\left(\mu(x), \lambda(x)\right), S\left(\lambda(y), \lambda(z)\right), S\left(\lambda(y), \lambda(z)\right)\right), S\left(\mu(y), \mu(z)\right)\right)$$

$$= S\left(S\left(\mu(x), \lambda(x)\right), S\left(\lambda(y), \lambda(z)\right), S\left(\mu(y), \mu(z)\right)\right)$$

Thus $S(\mu, \lambda)(xyz) \leq S(S(\mu, \lambda)(x), S((S(\mu, \lambda)(y), (S(\mu, \lambda)(z)))$, for all $x, y, z \in R$. Therefore $S(\mu, \lambda)$ is a $S$-fuzzy bi-ideals of $R$.

**Corollary 5.2.** (9) If $\mu$ and $\lambda$ are anti fuzzy bi-ideal of a semigroup $R$, then $\mu \cup \lambda$ is an antifuzzy bi-ideal of $R$.

**Proof.** By taking $S$-norm as $S_M$-norm in Theorem 5.1, we get the result.

**References**


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