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CONVERGENCE OF BS -ITERATION PROCEDURE IN UNIFORMLY CONVEX BANACH SPACES AND COMPARISON OF ITS RATE OF CONVERGENCE

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Abstract. In this paper, we introduce a new iteration procedure namely $BS-$iteration procedure which generalizes modified Vatan two step iteration procedure and modified $AK-$iteration procedure. We prove the strong convergence of these iteration procedures to a fixed point of an asymptotically nonexpansive map T defined on a nonempty closed convex and bounded subset of a uniformly convex Banach space X under the assumption that T is completely continuous. Also, we prove the weak convergence of these iteration procedures when X satisfies the Opial condition. We provide examples in support of our results and show that the modified $AK-$iteration procedure converges strongly to a fixed point but the modified Picard iteration procedure fails to converge. Further, we prove that the $BS-$iteration procedure converges faster to a fixed point of a Zamfirescu operator than a modified Mann iteration procedure in Banach spaces.

1. Introduction

In 1936, the concept of uniform convexity of a normed linear space was introduced by Clarkson [**6**]. In 1972, Goebel and Kirk [**7**] introduced the class of asymptotically nonexpansive maps and proved that every asymptotically nonexpansive selfmap defined on a nonempty closed convex and bounded subset of a uniformly convex Banach space has a fixed point.

In 1972, Zamfirescu [**18**] proved the following theorem which is a generalization of Banach contraction principle.

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THEOREM 1.1. ([**18**]) *Let X be a complete metric space and $T : X \to X$ be a Zamfirescu operator, i.e., there exist $0 < \alpha < 1$ and $0 < \beta, \gamma < \frac{1}{2}$ such that for all $x, y \in X$ atleast any one of the following hold.*

(i) $d(Tx, Ty) \leqslant \alpha d(x, y)$,

(ii) $d(Tx, Ty) \leqslant \beta[d(x, Tx) + d(y, Ty)]$,

(iii) $d(Tx, Ty) \leqslant \gamma[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point in X.

In 1989, Schu [**11**] introduced a modified Mann iteration procedure as follows: Let K be a nonempty convex subset of a normed linear space X and $T : K \to K$ be a map,

$$x_1 \in K,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n \tag{1.1}$$

for $n = 1, 2, ...$ where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$. Schu [**11**] studied strong convergence of this iteration procedure for asymptotically nonexpansive maps in Hilbert spaces.

If $\alpha_n = 1$ for all n in (1.1) then the iteration procedure (1.1) becomes

$$x_1 \in K,\ x_{n+1} = T^n x_n$$

and we call it 'modified Picard iteration procedure'.

In 1994, Tan and Xu [**14**] introduced a modified Ishikawa iteration procedure as follows:

$$x_1 \in K,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n((1 - \beta_n)x_n + \beta_n T^n x_n) \tag{1.2}$$

for $n = 1, 2, ...$ where $\{\alpha\}_{n=1}^{\infty}$ and $\{\beta\}_{n=1}^{\infty}$ are real sequences in $[0, 1]$. Also, Tan and Xu [**14**] proved that this iteration procedure converges weakly to a fixed point of an asymptotically nonexpansive selfmap in the setting of uniformly convex Banach spaces by using Opial condition.

An extensive research work is going on the topic 'the approximation of fixed points by iterative methods for asymptotically nonexpansive maps', for example, we refer [**2, 4, 5, 10, 12, 16**] and the related references there in. Inspired and motivated by this work, we modify the following iterations and study the weak and the strong convergence of these iteration procedures.

In 2015, Karakaya, Bouzara, Doğan and Atalan [**8**] introduced Vatan two step iteration process as follows.

$$\begin{cases} x_1 \in K \\ y_n = T((1 - \beta_n)x_n + \beta_n T x_n) \\ x_{n+1} = T((1 - \alpha_n)y_n + \alpha_n T y_n) \end{cases} \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$.

In 2016, Ullah and Arshad [**9**] introduced AK-iteration procedure

$$\begin{cases} x_1 \in K \\ z_n = T((1 - \beta_n)x_n + \beta_n T x_n) \\ y_n = T((1 - \alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = T y_n \end{cases} \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

Recently, Anthony and Mary [**2**] proved that the modified Mann iteration procedure (1.1) with $\alpha_n = \frac{n}{n+1}$ converges faster to a fixed point of Zamfirescu operator than the modified Mann iteration procedure (1.1) with $\epsilon \leqslant \alpha_n \leqslant (1-\epsilon)$ for $n = 1, 2, ...$ in the setting of uniformly convex Banach spaces.

In Section 2, we define new iteration procedures namely $BS-$iteration procedure, a modified Vatan two step iteration procedure and a modified $AK-$iteration procedure. Also, we present basic concepts that are required to develop the paper. In Section 3, we prove that these iteration procedures with appropriate conditions on control sequences converge strongly to a fixed point of an asymptotically nonexpansive map T defined on a closed, convex and bounded subset of a uniformly convex Banach space X under the assumption that T is completely continuous on X. Also, we prove that these iteation procedures converge weakly when X satisfies Opial condition. In Section 4, we provide examples in support of our results and show that the modified $AK-$iteration procedure converges to a fixed point but the modified Picard iteration procedure fails to converge. In Section 5, we prove that $BS-$iteration procedure converges faster to a fixed point of a Zamfirescu operator than the modified Mann iteration procedure [**2**] in Banach spaces.

2. Preliminaries

Let X be a normed linear space, K be a nonempty convex subset of X. Let $T: K \to K$ be a selfmap and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. We introduce '$BS-$iteration procedure' as follows:
For $x_1 \in K$, we define

$$\begin{aligned} z_n &= T^n((1-\beta_n)x_n + \beta_n T^n x_n) \\ y_n &= T^n((1-\alpha_n)z_n + \alpha_n T^n z_n) \\ x_{n+1} &= (1-\gamma_n)y_n + \gamma_n T^n y_n. \end{aligned} \tag{2.1}$$

If we choose $\gamma_n \equiv 0$ in (2.1) then we have the following iteration procedure:
For $x_1 \in K$,

$$\begin{aligned} y_n &= T^n((1-\beta_n)x_n + \beta_n T^n x_n) \\ x_{n+1} &= T^n((1-\alpha_n)y_n + \alpha_n T^n y_n). \end{aligned} \tag{2.2}$$

We call the iteration procedure (2.2), a modified Vatan two step iteration procedure. If $\gamma_n \equiv 1$ in (2.1) then we have

$$\begin{aligned} x_1 &\in K, \\ z_n &= T^n((1-\beta_n)x_n + \beta_n T^n x_n) \\ y_n &= T^n((1-\alpha_n)z_n + \alpha_n T^n z_n) \\ x_{n+1} &= T^n y_n, \end{aligned} \tag{2.3}$$

and we call it modified $AK-$iteration procedure.

Now, we recall the concepts and results that are required to develop this paper.

A map $T: K \to K$ is said to be "asymptotically nonexpansive" if there exists a sequence $\{k_n\}_{n=1}^{\infty}$ with $k_n \geqslant 1$ and $\lim\limits_{n\to\infty} k_n = 1$ such that $||T^n x - T^n y|| \leqslant k_n ||x-y||$ for all $x, y \in K$ and $n \geqslant 1$.

A map $T : K \to K$ is said to be completely continuous if for any sequence $\{x_n\}$ converges weakly to a point x_0 in K, the sequence $\{Tx_n\}$ converges to Tx_0.

Definition 2.1. ([**6**]) A uniformly convex Banach space X is a Banach space in which the following holds : for every $0 < \epsilon \leqslant 2$, there exists $\delta > 0$ such that for all $x, y \in X$ with $||x|| = ||y|| = 1$ and $||x - y|| \geqslant \epsilon$ implies that $||\frac{x+y}{2}|| < 1 - \delta$.

Definition 2.2. ([**17**]) A Banach space X is said to satisfy Opial condition if $x_n \to x$ weakly implies that $\liminf ||x_n - x|| < \liminf ||x_n - y||$ for all $y \neq x$.

Lemma 2.1. ([**15**]) *Let $p > 1$ and $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and $||\lambda x + (1 - \lambda)y||^p \leqslant \lambda||x||^p + (1 - \lambda)||y||^p - w_p(\lambda)g(||x - y||)$ for all $x, y \in B_r$ where $0 \leqslant \lambda \leqslant 1$, $B_r = \{x \in X : ||x|| \leqslant r\}$ and $w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.*

Lemma 2.2. ([**13**]) *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers such that the inequality $a_{n+1} \leqslant (1 + \delta_n)a_n + b_n$ for $n = 1, 2, 3...$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$ then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.3. ([**5**]) *Let K be a nonempty closed and convex subset of a uniformly convex Banach space and $T : K \to K$ be an asymptotically nonexpansive map. Then $I - T$ is demiclosed with respect to 0. That is, $x_n \to x$ weakly and $||x_n - Tx_n|| \to 0$ implies that $x \in F(T)$ where $F(T)$ is the set of all fixed points of T.*

Lemma 2.4. ([**1**]) *Let X be a reflexive Banach space that satisfies the Opial condition, K be a nonempty closed convex subset of X, $T : K \to X$ be a map such that $F(T) \neq \emptyset$ and $I - T$ be demiclosed at 0. Let $\{x_n\}$ be any sequence such that $\lim_{n\to\infty} ||x_n - p||$ exists for all $p \in F(T)$ and $||x_n - Tx_n|| \to 0$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Lemma 2.5. ([**12**]) *Let K be a nonempty closed subset of a uniformly convex Banach space X. Let $T : K \to K$ be a completely continuous map with $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K such that (a) for any $p \in F(T)$, $\lim_{n\to\infty} ||x_n - p||$ exists and, (b) $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.*

Definition 2.3. ([**3**]) Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be two sequences such that $||x_n - p|| \leqslant a_n$, $||u_n - p|| \leqslant b_n$ for $n = 1, 2, ...$ where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two positive real sequences such that $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Then we say that $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{u_n\}_{n=1}^{\infty}$ to p.

Lemma 2.6. ([**16**]) *If $\{k_n\}_{n=1}^{\infty}$ is a sequence in $[1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ then for any real number $p > 0$, both the series $\sum_{n=1}^{\infty} (k_n - 1)$ and $\sum_{n=1}^{\infty} (k_n^p - 1)$ converge or diverge together.*

3. Main results

PROPOSITION 3.1. *Let K be a nonempty closed, bounded and convex subset of a uniformly convex Banach space X and $T : K \to K$ be an asymptotically nonexpansive map with the sequence $\{k_n\}_{n=1}^{\infty}$ that satisfies $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. For any $x_1 \in K$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated by the $BS-$iteration procedure (2.1). Then*

(i) *for any fixed point p of T, $\lim_{n\to\infty} ||x_n - p||$ exists.*
Further, if $0 < \liminf \beta_n \leqslant \limsup \beta_n < 1$ then the following hold.

(ii) $\lim_{n\to\infty} ||x_n - T^n x_n|| = 0$

(iii) $\lim_{n\to\infty} ||T^n x_n - z_n|| = 0$

(iv) $\lim_{n\to\infty} ||x_n - z_n|| = 0$

(v) $\lim_{n\to\infty} ||T^n x_n - T^n z_n|| = 0$

(vi) $\lim_{n\to\infty} ||T^n z_n - z_n|| = 0$

(vii) $\lim_{n\to\infty} ||T^n z_n - y_n|| = 0$

(viii) $\lim_{n\to\infty} ||x_n - y_n|| = 0$

(ix) $\lim_{n\to\infty} ||y_n - z_n|| = 0$

(x) $\lim_{n\to\infty} ||T^n y_n - y_n|| = 0$, and

(xi) $\lim_{n\to\infty} ||x_n - Tx_n|| = 0.$

PROOF. Let p be a fixed point of T. Since K is bounded, there is a real number $r > 0$ such that $\sup\{||x - p|| : x \in K\} < r$. By Lemma 2.1, there is a continuous strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that $||\lambda x + (1 - \lambda)y||^2 \leqslant \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||)$ for x, y in B_r. We consider

$$\begin{aligned}
||z_n - p||^2 &= ||T^n((1-\beta_n)x_n + \beta_n T^n x_n) - p||^2 \\
&\leqslant k_n^2||(1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p)||^2 \\
&\leqslant k_n^2[(1-\beta_n)||x_n - p||^2 + \beta_n||T^n x_n - p||^2 - \beta_n(1-\beta_n)g(||x_n - T^n x_n||)] \\
&\leqslant k_n^2[(1-\beta_n)||x_n - p||^2 + \beta_n k_n^2||x_n - p||^2 - \beta_n(1-\beta_n)g(||x_n - T^n x_n||)] \\
&= k_n^2[1 + \beta_n(k_n^2 - 1)]||x_n - p||^2 - \beta_n(1-\beta_n)k_n^2 g(||x_n - T^n x_n||)
\end{aligned}$$

$$\leqslant k_n^4||x_n - p||^2 - \beta_n(1-\beta_n)k_n^2 g(||x_n - T^n x_n||) \tag{3.1}$$

and

$$\begin{aligned}
||y_n - p||^2 &= ||T^n((1-\alpha_n)z_n + \alpha_n T^n z_n) - p||^2 \\
&\leqslant k_n^2||(1-\alpha_n)z_n + \alpha_n T^n z_n - p||^2 \\
&\leqslant k_n^2[(1-\alpha_n)||z_n - p||^2 + \alpha_n||T^n z_n - p||^2 - \alpha_n(1-\alpha_n)g(||z_n - T^n z_n||)] \\
&\leqslant k_n^2[(1-\alpha_n)||z_n - p||^2 + \alpha_n k_n^2||z_n - p||^2 - \alpha_n(1-\alpha_n)g(||z_n - T^n z_n||)] \\
&\leqslant k_n^2[1 + \alpha_n(k_n^2 - 1)]||z_n - p||^2 - \alpha_n(1-\alpha_n)k_n^2 g(||z_n - T^n z_n||)
\end{aligned}$$

$$\leqslant k_n^4||z_n - p||^2 - \alpha_n(1-\alpha_n)k_n^2 g(||z_n - T^n z_n||)$$

$$\leqslant k_n^8||x_n - p||^2 - \beta_n(1-\beta_n)k_n^6 g(||x_n - T^n x_n||) - \alpha_n(1-\alpha_n) k_n^2 g(||z_n - T^n z_n||), \text{ (from (3.1))}. \tag{3.2}$$

Now we consider

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\gamma_n)y_n + \gamma_n T^n y_n - p||^2 \\
&\leqslant (1-\gamma_n)||y_n - p||^2 + \gamma_n||T^n y_n - p||^2 - \gamma_n(1-\gamma_n)g(||y_n - T^n y_n||) \\
&\leqslant (1-\gamma_n)||y_n - p||^2 + \gamma_n k_n^2||y_n - p||^2 - \gamma_n(1-\gamma_n)g(||y_n - T^n y_n||) \\
&= (1+\gamma_n(k_n^2 - 1))||y_n - p||^2 - \gamma_n(1-\gamma_n)g(||y_n - T^n y_n||)
\end{aligned}$$

$$\leqslant k_n^2||y_n - p||^2 - \gamma_n(1-\gamma_n)g(||y_n - T^n y_n||). \tag{3.3}$$

We write $A_n = \beta_n(1-\beta_n)k_n^8 g(||x_n - T^n x_n||)$, $B_n = \alpha_n(1-\alpha_n)k_n^6 g(||z_n - T^n z_n||)$ and $C_n = \gamma_n(1-\gamma_n)k_n^2 g(||y_n - T^n y_n||)$. Then from (3.2) and (3.3) it follows that

$$||x_{n+1} - p||^2 \leqslant ||x_n - p||^2 + (k_n^{10} - 1)||x_n - p||^2 - A_n - B_n - C_n$$

$$\leqslant ||x_n - p||^2 + r^2(k_n^{10} - 1) - A_n - B_n - C_n \tag{3.4}$$

$$\leqslant ||x_n - p||^2 + r^2(k_n^{10} - 1). \tag{3.5}$$

By Lemma 2.6, $\sum\limits_{n=1}^{\infty}(k_n^{10} - 1) < \infty$ and hence by Lemma 2.2, $\lim\limits_{n\to\infty}||x_n - p||$ exists for every fixed point p of T. This proves (i).

Now we assume that $0 < \liminf \beta_n \leqslant \limsup \beta_n < 1$ so that there exist $\eta_1, \eta_2 \in (0,1)$ and a positive integer n_0 such that $\eta_1 \leqslant \beta_n \leqslant \eta_2$ for all $n \geqslant n_0$. From the inequality (3.4), it follows that

$$\beta_n(1-\beta_n)k_n^8 g(||x_n - T^n x_n||) = A_n \leqslant (||x_n - p||^2 - ||x_{n+1} - p||^2) + r^2(k_n^{10} - 1).$$

Therefore

$$\eta_1(1-\eta_2)g(||x_n - T^n x_n||) \leqslant (||x_n - p||^2 - ||x_{n+1} - p||^2) + r^2(k_n^{10} - 1)$$

so that

$$g(||x_n - T^n x_n||) \leqslant \frac{||x_n - p||^2 - ||x_{n+1} - p||^2 + r^2(k_n^{10} - 1)}{\eta_1(1-\eta_2)}$$

for all $n \geqslant n_0$ and hence $\lim\limits_{n\to\infty} g(||x_n - T^n x_n||) = 0$.

Now we prove that $\lim\limits_{n\to\infty}||x_n - T^n x_n|| = 0$. Otherwise, there exist $\epsilon > 0$ and a strictly increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers such that $||x_{n_k} - T^{n_k}x_{n_k}|| \geqslant \epsilon$ for $k = 1, 2, 3...$. Since g is strictly increasing, we have $g(||x_{n_k} - T^{n_k}x_{n_k}||) \geqslant g(\epsilon)$ for $k = 1, 2, ...$ Therefore $\lim\limits_{k\to\infty} g(||x_{n_k} - T^{n_k}x_{n_k}||) \geqslant g(\epsilon) > 0$, a contradiction. Hence (ii) holds.

Since $z_n = T^n((1-\beta_n)x_n + \beta_n T^n x_n)$, we have $||T^n x_n - z_n|| \leqslant k_n\beta_n||x_n - T^n x_n||$ for $n = 1, 2, ...$ so that $\lim\limits_{n\to\infty}||T^n x_n - z_n|| = 0$. This proves (iii).

By the triangle inequality, we have $||x_n - z_n|| \leqslant ||x_n - T^n x_n|| + ||T^n x_n - z_n||$ for $n = 1, 2, ...$ so that (iv) follows from (ii)and (iii).

Since T is asymptotically nonexpansive, (v) follows from (iv).

By the triangle inequality, we have $||T^n z_n - z_n|| \leqslant ||T^n z_n - T^n x_n|| + ||T^n x_n - z_n||$ for $n = 1, 2, ...$ and hence (vi) follows from (iii) and (v).

Since $y_n = T^n((1-\alpha_n)z_n + \alpha_n T^n z_n)$, we have $||T^n z_n - y_n|| \leqslant k_n ||z_n - T^n z_n||$ for $n = 1, 2, ...$ so that (vii) follows from (vi).

Again by the triangle inequality, we have

$$||x_n - y_n|| \leqslant ||x_n - T^n x_n|| + ||T^n x_n - T^n z_n|| + ||T^n z_n - y_n|| \text{ for } n = 1, 2, ...$$

Hence (viii) follows from (ii), (v) and (vii).

It is easy to see that (ix) follows from (iv) and (viii).

We consider

$$\begin{aligned} ||T^n y_n - y_n|| &= ||T^n y_n - T^n((1-\alpha_n)z_n + \alpha_n T^n z_n)|| \\ &\leqslant k_n[(1-\alpha_n)||y_n - z_n|| + \alpha_n ||y_n - T^n z_n||]. \end{aligned}$$

Hence (x) follows from (vii) and (ix).

We consider

$$\begin{aligned} ||x_{n+1} - T^n x_{n+1}|| &\leqslant ||x_{n+1} - y_n|| + ||y_n - T^n y_n|| + ||T^n y_n - T^n x_{n+1}|| \\ &\leqslant (k_n + 1)||y_n - x_{n+1}|| + ||y_n - T^n y_n|| \\ &= ((k_n + 1)\gamma_n + 1)||T^n y_n - y_n|| \\ &\leqslant (k_n + 2)||T^n y_n - y_n|| \text{ for } n = 1, 2, 3... . \end{aligned}$$

Therefore from (x) it follows that

$$\lim_{n\to\infty} ||x_{n+1} - T^n x_{n+1}|| = 0. \tag{3.6}$$

Now we consider

$$\begin{aligned} ||Tx_n - x_n|| &\leqslant ||x_n - T^n x_n|| + ||T^n x_n - Tx_n|| \\ &\leqslant ||x_n - T^n x_n|| + k_1 ||x_n - T^{n-1} x_n|| \text{ for } n = 2, 3... . \end{aligned}$$

Hence (xi) follows from (ii) and (3.6). □

Remark 3.1. The conclusion of Proposition 3.1 remains true for modified Vatan two step iteration procedure (2.2) and modified $AK-$iteration procedure (2.3).

Theorem 3.1. *Let K be a nonempty closed, convex and bounded subset of a uniformly convex Banach space X. Let T be a completely continuous asymptotically nonexpansive map of K into itself with $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in* [0,1] *such that $0 < \liminf \beta_n \leqslant \limsup \beta_n < 1$. Then the iterative sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ generated by the $BS-$iteration procedure* (2.1) *converges strongly to a fixed point of T.*

Proof. Follows from Lemma 2.5 and Proposition 3.1. □

Theorem 3.2. *Let K be a nonempty closed, convex and bounded subset of a uniformly convex Banach space X that satisfies Opial condition. Let T be an asymptotically nonexpansive map of K into itself with $k_n \geqslant 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in* [0,1] *such that $0 < \liminf \beta_n \leqslant \limsup \beta_n < 1$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by the $BS-$iteration procedure* (2.1) *converges weakly to a fixed point of T.*

Proof. By Proposition 3.1, $\lim_{n\to\infty} ||x_n - p||$ exists for any fixed point p of T and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Also, by Lemma 2.3, $I - T$ is demiclosed at 0.
Since every uniformly convex Banach space is reflexive, the conclusion follows from Lemma 2.4. □

Remark 3.2. Since modified Vatan two step iteration procedure and modified $AK-$iteration procedure are special cases of $BS-$iteration procedure it follows that these iteration procedures converge strongly (weakly) to a fixed point under the hypotheses of Theorem 3.1 (Theorem 3.2).

Remark 3.3. By Lemma 2.6, the main results of this paper still valid if we replace $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ by $\sum_{n=1}^{\infty}(k_n^p - 1) < \infty$ where $p > 0$ is real.

4. Examples

In the following, we consider the example given in [**2**] in support of Theorem 3.1.

Example 4.1. Let $X = \mathbb{R}^2$ with the Euclidean norm so that X is a uniformly convex Banach space. Let $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant \frac{81}{100}\}$. We define $T : K \to K$ by $T(x_1, x_2) = (x_1^2, \sin x_2)$ so that $T^n(x_1, x_2) = (x_1^{2^n}, \sin^{(n)} x_2)$ where $\sin^{(n)} x_2$ is the composition of sine function over n times at x_2. For any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in K,

$$||T^n x - T^n y|| = \sqrt{|x_1^{2^n} - y_1^{2^n}|^2 + |\sin^{(n)} x_2 - \sin^{(n)} y_2|^2}.$$

It is easy to see that $|\sin^{(n)} x_2 - \sin^{(n)} y_2| \leqslant |x_2 - y_2|$ for $n = 1, 2, \ldots$.

Now we consider

$$\begin{aligned}|x_1^{2^n} - y_1^{2^n}| &= |x_1 - y_1||x_1^{2^n-1} + x_1^{2^n-2}y_1 + \ldots + x_1 y_1^{2^n-2} + y_1^{2^n-1}| \\ &\leqslant |x_1 - y_1|(\tfrac{9}{10})^{2^n-1}2^n\end{aligned}$$

so that

$$||T^n x - T^n y|| \leqslant \sqrt{|x_1 - y_1|^2 2^{2n}(.81)^{2^n-1} + |x_2 - y_2|^2} \leqslant (1 + 2^n(\tfrac{9}{10})^{2^n-1})||x - y||$$

for $n = 1, 2, ..$. Therefore T is an asymptotically nonexpansive map with $k_n = (1 + 2^n(\frac{9}{10})^{2^n-1})$ for $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty}(k_n - 1) = \sum_{n=1}^{\infty} 2^n(\frac{9}{10})^{2^n-1} < \infty$.

Now we show that T is completely continuous. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in K such that $u_n \to u_0$ weakly, where $u_n = (u_{n1}, u_{n2})$ for $n = 1, 2, \ldots$ and $u_0 = (u_{01}, u_{02}) \in K$. Therefore for any $y = (y_1, y_2) \in K$, $\lim_{n\to\infty}\langle u_n, y\rangle = \langle u_0, y\rangle$ where $\langle\cdot,\cdot\rangle$ represents the inner product. That is, $\lim_{n\to\infty}\sum_{i=1}^{2}(u_{ni} - u_{0i})y_i = 0$.If we take $y_1 = \frac{1}{2}$ and $y_2 = 0$ then $\lim_{n\to\infty} u_{n1} = u_{01}$ and, $y_1 = 0$ and $y_2 = \frac{1}{2}$ implies that $\lim_{n\to\infty} u_{n2} = u_{02}$ so that $\lim_{n\to\infty} u_n = u_0$. By using continuty of T, it follows that T is completely continuous. Let $x_1 \in K$ be arbitrary and $\alpha_n = \frac{n}{n+1}$, $\beta_n = \frac{2^n-1}{2^{n+1}}$ and $\gamma_n = 1$ for $n = 1, 2, \ldots$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by the

$BS-$iteration procedure (2.1). Let x_{n1} and x_{n2} be the first and second component of x_n respectively so that $x_n = (x_{n1}, x_{n2})$ for $n = 1, 2, \ldots$.
It is easy to see that

$$z_n = ((\tfrac{x_{n1}+n(x_{n1})^{2^n}}{n+1})^{2^n}, \sin^{(n)}(\tfrac{x_{n2}+n\sin^{(n)} x_{n2}}{n+1})),\ y_n = ((\tfrac{(2^n+1)z_{n1}+(2^n-1)(z_{n1})^{2^n}}{2^{n+1}})^{2^n},$$
$$\sin^{(n)}(\tfrac{(2^n+1)z_{n2}+(2^n-1)\sin^{(n)} z_{n2}}{2^{n+1}})) \text{ and } x_{n+1} = (y_{n1}^{2^n}, \sin^{(n)} y_{n2}).$$

Let

$$z_{n1} = (\tfrac{x_{n1}+n(x_{n1})^{2^n}}{n+1})^{2^n},\ z_{n2} = \sin^{(n)}(\tfrac{x_{n2}+n\sin^{(n)} x_{n2}}{n+1}),$$
$$y_{n1} = (\tfrac{(2^n+1)z_{n1}+(2^n-1)(z_{n1})^{2^n}}{2^{n+1}})^{2^n} \text{ and } y_{n2} = \sin^{(n)}(\tfrac{(2^n+1)z_{n2}+(2^n-1)\sin^{(n)} z_{n2}}{2^{n+1}}).$$

It is easy to see that

$$|z_{n1}| \leqslant (\tfrac{|x_{n1}|+n|x_{n1}|^{2^n}}{n+1})^{2^n} \leqslant |x_{n1}|^{2^n},\ |y_{n1}| \leqslant (\tfrac{(2^n+1)|z_{n1}|+(2^n-1)|z_{n1}|^{2^n}}{2^{n+1}})^{2^n} \leqslant |z_{n1}|^{2^n}$$
$$|x_{n+11}| = |y_{n1}|^{2^n} \leqslant |x_{n1}|^{2^{3n}} \text{ for } n = 1, 2, \ldots .$$

Hence, $|x_{n+11}| \leqslant |x_{11}|^{2^{\frac{3n(n+1)}{2}}}$ for $n = 1, 2, \ldots$ so that $\lim\limits_{n\to\infty} x_{n1} = 0$. Since $|\sin x| \leqslant |x|$ for every $x \in \mathbb{R}$ and $\sin|x| = |\sin x|$ for x sufficiently close to 0, we have

$$\begin{aligned}
|z_{n2}| &= |\sin^{(n)}(\tfrac{x_{n2}+n\sin^{(n)} x_{n2}}{n+1})| \\
&\leqslant |\sin^{(n-1)}(\tfrac{x_{n2}+n\sin^{(n)} x_{n2}}{n+1})| \\
&\ \vdots \\
&\leqslant |\sin(\tfrac{x_{n2}+n\sin^{(n)} x_{n2}}{n+1})| \\
&= \sin(\tfrac{|x_{n2}+n\sin^{(n)} x_{n2}|}{n+1}) \\
&\leqslant \sin|x_{n2}|.
\end{aligned}$$

Similarly, $|y_{n2}| \leqslant |z_{n2}|$ and $|x_{n+12}| = |\sin^{(n)} y_{n2}| \leqslant |y_{n2}| \leqslant |z_{n2}| \leqslant \sin|x_{n2}| \leqslant |x_{n2}|$. Therefore the sequence $\{x_{n2}\}$ is a decreasing sequence of non-negative real numbers so that $\lim\limits_{n\to\infty} x_{n2}$ exists, say $l \geqslant 0$. Hence $l \leqslant \sin l \leqslant l$, that is, $l = 0$. Therefore $\lim\limits_{n\to\infty} x_{n2} = 0$. Hence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point $0 = (0, 0)$ of T. ■

In the following, we show that the modified $AK-$iteration procedure (2.3) converges strongly to a fixed point of T whereas the modified Picard iteration procedure, i.e., $x_{n+1} = T^n x_n$ for $n = 1, 2, \ldots$, fails to converge under the hypotheses of Theorem 3.1.

Example 4.2. We consider the uniformly convex Banach space $\mathbb{R}$ and we define a map $T : [0, 1] \to [0, 1]$ by $Tx = 1 - x$ so that T is an asymptotically nonexpansive map with $k_n = 1$. Also, we observe that T is completely continuous and $T^n x = x$ if n is even and $T^n x = 1 - x$ if n is odd.

Let $x_1 \in [0, 1]$ and $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by the modified $AK-$iteration procedure (2.3). First we show that for any sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in $[0, 1]$, we have $x_n = x_{n+1}$ for n is even.

We consider

$$z_n = T^n((1-\beta_n)x_n + \beta_n T^n x_n) = T^n((1-\beta_n)x_n + \beta_n x_n) = T^n x_n = x_n.$$

We consider

$$y_n = T^n((1-\alpha_n)z_n + \alpha_n T^n z_n) = T^n((1-\alpha_n)z_n + \alpha_n z_n) = T^n z_n = z_n = x_n$$

so that $x_{n+1} = T^n y_n = T^n x_n = x_n$. Hence the limit of the sequence $\{x_n\}_{n=1}^{\infty}$ (if exists) is equal to the limit of the subsequence $\{x_{2n}\}_{n=1}^{\infty}$.

Now, we let $\beta_n = \frac{(2^n-1)}{3\times 2^n}$ for $n = 1, 2, 3...$ so that $\liminf \beta_n = \limsup \beta_n = \frac{1}{3}$. We consider

$$\begin{aligned} z_{2n+1} &= T^{2n+1}((1-\beta_{2n+1})x_{2n+1} + \beta_{2n+1}T^{2n+1}x_{2n+1}) \\ &= T^{2n+1}((1-\tfrac{2^{2n+1}-1}{3\times 2^{2n+1}})x_{2n+1} + \tfrac{2^{2n+1}-1}{3\times 2^{2n+1}}(1-x_{2n+1})) \\ &= T^{2n+1}(\tfrac{2^{2n+1}-1}{3\times 2^{2n+1}} + (\tfrac{2^{2n+1}+2}{3\times 2^{2n+1}})x_{2n+1}) \\ &= \tfrac{2\times 2^{2n+1}+1}{3\times 2^{2n+1}} - \tfrac{2^{2n+1}+2}{3\times 2^{2n+1}}x_{2n+1}. \end{aligned}$$

Therefore $z_{2n+1} - \frac{1}{2} = -\frac{2^{2n+1}+2}{3\times 2^{2n+1}}(x_{2n+1} - \frac{1}{2})$ and

$$|z_{2n+1} - \frac{1}{2}| = \frac{2^{2n+1}+2}{3\times 2^{2n+1}}|x_{2n+1} - \frac{1}{2}| \leqslant \frac{2}{3}|x_{2n+1} - \frac{1}{2}| = \frac{2}{3}|x_{2n} - \frac{1}{2}|.$$

On continuing this process, we have

$$|z_{2n+1} - \frac{1}{2}| \leqslant (\frac{2}{3})^n|x_2 - \frac{1}{2}| \text{ for } n = 1, 2... . \tag{4.1}$$

In the following, we show that the modified $AK-$ iteration procedure (2.3) converges to a fixed point and its convergence is independent of the choice of the sequence $\{\alpha_n\}$.

Case(i): Let $\alpha_n = \frac{1}{n}$ for $n = 1, 2, 3...$ so that $\sum_{n=1}^{\infty} \alpha_n = \infty$. We consider

$$\begin{aligned} y_{2n+1} &= T^{2n+1}((1-\alpha_{2n+1})z_{2n+1} + \alpha_{2n+1}T^{2n+1}z_{2n+1}) \\ &= T^{2n+1}((1-\tfrac{1}{2n+1})z_{2n+1} + \tfrac{1}{2n+1}(1-z_{2n+1})) \\ &= \tfrac{2n}{2n+1} + \tfrac{1-2n}{2n+1}z_{2n+1} \text{ so that } x_{2n+2} = \tfrac{1}{2n+1} + \tfrac{2n-1}{2n+1}z_{2n+1}. \end{aligned}$$

Therefore
$|x_{2n+2} - \frac{1}{2}| = \frac{2n-1}{2n+1}|z_{2n+1} - \frac{1}{2}| \leqslant |z_{2n+1} - \frac{1}{2}|$. By (4.1), it follows that $|x_{2n+2} - \frac{1}{2}| \leqslant (\frac{2}{3})^n|x_2 - \frac{1}{2}|$ for $n = 1, 2, ...$ so that $\lim_{n\to\infty} x_{2n} = \frac{1}{2}$ and hence $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Case(ii): Let $\alpha_n = \frac{1}{n^2}$ for $n = 1, 2, 3...$ so that $\sum_{n=1}^{\infty} \alpha_n < \infty$. We consider

$$\begin{aligned} y_{2n+1} &= T^{2n+1}((1-\alpha_{2n+1})z_{2n+1} + \alpha_{2n+1}T^{2n+1}z_{2n+1}) \\ &= T^{2n+1}((1-\tfrac{1}{(2n+1)^2})z_{2n+1} + \tfrac{1}{(2n+1)^2}(1-z_{2n+1}) \\ &= T^{2n+1}(\tfrac{1}{4n^2+4n+1} + \tfrac{4n^2+4n-1}{4n^2+4n+1}z_{2n+1}) \\ &= \tfrac{4n^2+4n}{4n^2+4n+1} - \tfrac{4n^2+4n-1}{4n^2+4n+1}z_{2n+1}. \end{aligned}$$

Therefore $x_{2n+2} = T^{2n+1}y_{2n+1} = \frac{1}{4n^2+4n+1} + \frac{4n^2+4n-1}{4n^2+4n+1}z_{2n+1}$ so that

$$|x_{2n+2} - \frac{1}{2}| = \frac{4n^2+4n-1}{4n^2+4n+1}|z_{2n+1} - \frac{1}{2}| \leqslant |z_{2n+1} - \frac{1}{2}|$$

for $n = 1, 2, ...$ and by proceeding as in *case* (i) we have $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Case(iii): Let $\alpha_n = \frac{n+1}{n+2}$ for $n = 1, 2, 3...$ so that $\sum_{n=1}^{\infty} \alpha_n = \infty$. We consider

$$\begin{aligned} y_{2n+1} &= T^{2n+1}((1-\alpha_{2n+1})z_{2n+1} + \alpha_{2n+1}T^{2n+1}z_{2n+1}) \\ &= T^{2n+1}(\tfrac{2n+2}{2n+3} - \tfrac{2n+1}{2n+3}z_{2n+1}) \\ &= \tfrac{1}{2n+3} + \tfrac{2n+1}{2n+3}z_{2n+1}. \end{aligned}$$

Therefore

$$\begin{aligned} |x_{2n+2} - \frac{1}{2}| = |T^{2n+1}y_{2n+1} - \frac{1}{2}| &= |\frac{2n+2}{2n+3} - \frac{2n+1}{2n+3}z_{2n+1} - \frac{1}{2}| \\ &= | - \frac{2n+1}{2n+3}(z_{2n+1} - \frac{1}{2})| \leqslant |z_{2n+1} - \frac{1}{2}| \end{aligned}$$

for $n = 1, 2, ...$ It is easy to see that the iterative sequence $\{x_n\}_{n=1}^{\infty}$ converges to $\frac{1}{2}$.

Here we observe that the modified Picard iteration procedure $x_{n+1} = T^n x_n$ for $n = 1, 2, 3...$ does not converge. For example, it is easy to see that for any $x_1 \in [0, 1]$ with $x_1 \neq \frac{1}{2}$, we have $x_n = \begin{cases} x_1 & \text{if } n \equiv 0, 1 \pmod 4 \\ 1 - x_1 & \text{if } n \equiv 2, 3 \pmod 4 \end{cases}$ so that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge.

5. Comparison of rate of convergence

In the following, we prove that the $BS-$iteration procedure converges faster to a fixed point of Zamfirescu operator than the modified Mann iteration procedure (1.1) that was considered by Anthony and Mary [**2**] with $\alpha_n = \frac{n}{n+1}$ for $n = 1, 2, ...$

THEOREM 5.1. *Let X be a Banach space, K be a nonempty closed convex and bounded subset of X. Let $T : K \to K$ be a Zamfirescu operator. Let $x_1 \in K$ and $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence generated by $BS-$iteration procedure* (2.1). *Let $u_1 \in K$ and $u_{n+1} = (1 - \frac{1}{n+1})T^n u_n + \frac{1}{n+1}u_n$ for $n = 1, 2, ...$. Then both the sequences $\{x_n\}$ and $\{u_n\}$ converge to the unique fixed point p of T and $\{x_n\}$ converges faster than $\{u_n\}$ to p.*

PROOF. Since T is a Zamfirescu operator, we write

$$||Tx - Ty|| \leqslant \delta||x - y|| + 2\delta||x - Tx||$$

for all $x, y \in K$, where $0 < \delta < 1$. By Theorem 1.1, T has a unique fixed point p in K.
It is easy to see that $||T^n x - p|| \leqslant \delta^n||x - p||$ for all $x \in K$ and $n = 1, 2, ...$. We consider

$$\begin{aligned} ||z_n - p|| &= ||T^n((1 - \beta_n)x_n + \beta_n T^n x_n) - p|| \\ &\leqslant \delta^n||(1 - \beta_n)x_n + \beta_n T^n x_n - p|| \\ &\leqslant \delta^n[(1 - \beta_n)||x_n - p|| + \beta_n||T^n x_n - p||] \\ &\leqslant \delta^n(1 - \beta_n + \beta_n\delta^n)||x_n - p|| \ . \end{aligned} \tag{5.1}$$

We consider

$$||y_n - p|| = ||T^n((1 - \alpha_n)z_n + \alpha_n T^n z_n) - p||$$

$$\leqslant \delta^n[||(1-\alpha_n)z_n + \alpha_n T^n z_n - p||]$$
$$\leqslant \delta^n[(1-\alpha_n)||z_n - p|| + \alpha_n||T^n z_n - p||]$$
$$\leqslant \delta^n(1-\alpha_n+\alpha_n\delta^n)||z_n - p|| \ . \tag{5.2}$$

Again we consider

$$||x_{n+1} - p|| = ||(1-\gamma_n)y_n + \gamma_n T^n y_n - p||$$
$$\leqslant (1-\gamma_n)||y_n - p|| + \gamma_n||T^n y_n - p||$$
$$\leqslant (1-\gamma_n+\gamma_n\delta^n)||y_n - p|| \ . \tag{5.3}$$

By (5.1), (5.2) and (5.3), we have

$$||x_{n+1}-p|| \leqslant \delta^{2n}(1-\alpha_n+\alpha_n\delta^n)(1-\beta_n+\beta_n\delta^n)(1-\gamma_n+\gamma_n\delta^n) \text{ for } n = 1, 2, \dots \ . \tag{5.4}$$

By using the inequality (5.4), it is easy to see that

$$||x_{n+1}-p|| \leqslant \delta^{n(n+1)} \prod_{k=1}^{n}(1-\alpha_k+\alpha_k\delta^k)(1-\beta_k+\beta_k\delta^k)(1-\gamma_k+\gamma_k\delta^k)||x_1-p|| \ . \tag{5.5}$$

Now we consider

$$||u_{n+1} - p|| = ||(1 - \tfrac{1}{n+1})T^n u_n + \tfrac{1}{n+1}u_n - p||$$
$$\leqslant \tfrac{n\delta^n+1}{n+1}||u_n - p|| \text{ for } n = 1, 2, \dots \ .$$

By repeated application of this inequality, we have

$$||u_{n+1} - p|| \leqslant \prod_{k=1}^{n} \frac{k\delta^k + 1}{k+1}||u_1 - p||. \tag{5.6}$$

Let $a_n = \delta^{n(n+1)} \prod\limits_{k=1}^{n} (1 - \alpha_k + \alpha_k\delta^k)(1 - \beta_k + \beta_k\delta^k)(1 - \gamma_k + \gamma_k\delta^k)||x_1 - p||$,

$b_n = \prod\limits_{k=1}^{n} \frac{k\delta^k+1}{k+1}||u_1 - p||$ so that

$$||x_{n+1} - p|| \leqslant a_n \tag{5.7}$$

and

$$||u_{n+1} - p|| \leqslant b_n \ . \tag{5.8}$$

Since $\frac{a_{n+1}}{a_n} = \delta^{2n+2}(1-\alpha_{n+1}+\alpha_{n+1}\delta^{n+1})(1-\beta_{n+1}+\beta_{n+1}\delta^{n+1})(1-\gamma_{n+1}+\gamma_{n+1}\delta^{n+1})$ for $n = 1, 2, \dots$, we have $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$ so that $\sum\limits_{n=1}^{\infty} a_n < \infty$ and hence $\lim\limits_{n\to\infty} a_n = 0$. By (5.7), the sequence $\{x_n\}_{n=1}^{\infty}$ converges to p.

Similarly, we prove that $\lim\limits_{n\to\infty} b_n = 0$ so that the sequence $\{u_n\}_{n=1}^{\infty}$ converges to p. Now, we let $c_n = \frac{a_n}{b_n}$ so that

$$\frac{c_{n+1}}{c_n} = \frac{\delta^{2n+2}(1-\alpha_{n+1}+\alpha_{n+1}\delta^{n+1})(1-\beta_{n+1}+\beta_{n+1}\delta^{n+1})(1-\gamma_{n+1}+\gamma_{n+1}\delta^{n+1})}{\frac{(n+1)\delta^{n+1}+1}{n+2}}$$
$$\leqslant \frac{\delta^{2n+2}(1-\alpha_{n+1}+\alpha_{n+1}\delta^{n+1})(1-\beta_{n+1}+\beta_{n+1}\delta^{n+1})(1-\gamma_{n+1}+\gamma_{n+1}\delta^{n+1})}{\delta^{n+1}}$$
$$\leqslant \delta^{n+1}(1 - \alpha_{n+1} + \alpha_{n+1}\delta^{n+1})(1 - \beta_{n+1} + \beta_{n+1}\delta^{n+1})(1 - \gamma_{n+1} + \gamma_{n+1}\delta^{n+1}).$$

Therefore $\lim\limits_{n\to\infty} \frac{c_{n+1}}{c_n} = 0 < 1$ so that $\sum\limits_{n=1}^{\infty} c_n < \infty$ and hence $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$.

Therefore $\{x_n\}$ converges to p faster than $\{u_n\}$ and the conclusion of the theorem holds. $\square$

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