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MAGIC AND ANTIMAGIC LABELING OF m COPIES OF FLOWER SNARK AND RELATED GRAPHS

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Abstract. An antimagic labeling of a graph with q edges and p vertices is a bijection from the set of edges to the integers $1, 2, \ldots, q$ such that all p vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an *antimagic labeling*. For odd $n \geqslant 5$, the *flower snark* $F_n = (V, E)$ is a simple undirected cubic graph with $4n$ vertices and $6n$ edges, where $V = \{b_i : 0 \leqslant i \leqslant n-1\} \cup \{c_i : 0 \leqslant i \leqslant 2n-1\} \cup \{a_i : 0 \leqslant i \leqslant n-1\}$ and $E = \{b_i b_{(i+1)(mod\ n)} : 0 \leqslant i \leqslant n-1\} \cup \{c_i c_{(i+1)(mod\ 2n)} : 0 \leqslant i \leqslant 2n-1\} \cup \{a_i b_i, a_i c_i, a_i c_{n+i} : 0 \leqslant i \leqslant n-1\}$. For $n = 3$ or even $n \geqslant 4$, F_n is called the related graph of flower snark.In this paper, we show that (a, d)-antimagic and super (a, d)-vertex antimagic total labeling of mF_n and related graphs.

1. Introduction

Let $G = (V, E)$ be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $p = |V(G)|$, $q = |E(G)|$ be the number of vertices and edges of G respectively.

The *labeling* (or valuation) of a graph is any mapping that maps some set of graph elements to a set of numbers (usually positive or non negative integers). If the domain is the edge set then it is called *edge labeling*. If the domain is the vertex set then it is called *vertex labeling*. If the domain consists of both edge set and vertex set then it is called *total labeling*. The most complete recent survey of graph labeling is in [**11**].

In [**13**] Hartsfield et al. introduced the concepts of an antimagic graphs. An *antimagic labeling* of a graph with q edges and p vertices is a bijection from the set

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of edges to the integers $1, 2, \ldots, q$ such that all p vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called *antimagic* if it has an antimagic labeling.

Bodendiek et al. [**4**] defined the concept of an (a, d)-antimagic graph as a special case of an antimagic graph. They showed [**5**] that the theory of linear Diophantine equations and other concepts of number theory can be applied to determine the set of all connected (a, d)-antimagic graphs.

In both magic and antimagic labelings, we consider the sum of all labels associated with a graph element. This will be called the *weight* of the element. A connected graph $G = (V, E)$ is said to be (a, d)-*antimagic* if there exists positive integers a, d and a bijection $f : E \to \{1, 2, 3, \ldots, q\}$ such that $W = \{w(v)|v\epsilon V\} = \{a, a + d, a + 2d, \ldots, a + (n - 1)d\}$ is the set of weights. The map f is called a (a, d)-*antimagic labeling* of G.

An (a, d)-*vertex antimagic total* (in short, (a, d)-VAT) labeling of G is a bijection $f : V \cup E \to \{1, 2, 3, \ldots, p + q\}$ with the set of weights of all vertices in G is $\{a, a+d, a+2d, \ldots, a+(n-1)d\}$, for some integers $a > 0$ and $d \geqslant 0$. If $d = 0$, then we call f a vertex magic total labeling (in short, VMT labeling). A *vertex magic total labeling* is an assignment of the integers from $\{1, 2, \ldots, p + q\}$ to the vertices and edges of G so that at each vertex, the vertex label and the labels on the edges incident at that vertex, add to a fixed contant C. This constant C is called *magic constant*.

MacDougall et al. [**18**] have proved that the generalized Petersen graphs $P(n, k)$ are vertex-magic total when n is even and $k \leqslant (n/2) - 1$. They conjecture that all $P(n, k)$ are vertex-magic total when $k \leqslant (n - 1)/2$. If all vertices receive p smallest labels, then (a, d)-vertex antimagic total labeling is called a *super (a,d)-vertex antimagic total labeling* (in short, super (a, d)-VAT).

Bača et al [**1**] introduce the notions of the vertex antimagic total labeling (VATL) and (a, d)-vertex antimagic total labeling ((a, d)-VATL), and conjecture that all regular graphs are (a, d)-VATL. In [**2**], Baca et al. study an antimagic labeling of generalized Petersen graphs. And also in [**3**] Baca et al. prove that a vertex-magic total labeling for the generalized Petersen graphs $P(m, n)$ for all $n > 3$, $1 \leqslant m \left\lfloor \frac{n-1}{2} \right\rfloor$.

J. A. MacDougall et al. [**16**] introduced vertex magic total labelings of graphs. MacDougall et al. [**17**] show: C_n has super vertex-magic total labeling if and only if n is odd, and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjecture that no tree has a super vertex-magic total labeling and that K_{4n} has a super vertex-magic labeling when $n > 1$. In [**12**], Gomez proves the conjecture: If $n \equiv 0 \mod 4$, $n > 4$, then K_n has a super vertex-magic total labeling.

The flower snark graph, defined by Isaacs [**14**], is certainly one of the most famous cubic graphs that theorists have come across. The well-known Four Color Theorem is equivalent to the statement that no snark is planar, including flower snark. Fiorini proved that flower snark is hypo-Hamiltonian. Yue et al. [**21**] have proved flower snark and related graphs are super vertex magic total. More details on flower snark can be found in [**6**],[**7**], and [**19**].

For odd $n \geqslant 5$, the *flower snark* $F_n = (V, E)$ is a simple undirected cubic graph with $4n$ vertices and $6n$ edges, where $V = \{b_i : 0 \leqslant i \leqslant n-1\} \cup \{c_i : 0 \leqslant i \leqslant 2n-1\} \cup \{a_i : 0 \leqslant i \leqslant n-1\}$ and $E = \{b_i b_{(i+1)(mod\ n)} : 0 \leqslant i \leqslant n-1\} \cup \{c_i c_{(i+1)(mod\ 2n)} : 0 \leqslant i \leqslant 2n-1\} \cup \{a_i b_i, a_i c_i, a_i c_{n+i} : 0 \leqslant i \leqslant n-1\}$. For $n = 3$ or even $n \geqslant 4$, F_n is called the related graph of a *flower snark*. Figure 1 shows the general form of the flower snark F_5.

Cranston [**9**] proved that for $k > 2$, every k-regular bipartite graph is antimagic. For non-bipartite regular graphs, Liang and Zhu [**15**] proved that every cubic graph is antimagic. That result was generalized by Cranston, Liang and Zhu [**10**], who proved that odd degree regular graphs are antimagic. Hartsield and Ringel [**13**] proved that every 2-regular graph is anti-magic. Chang, Liang, Pan, and Zhu [**8**] proved that every even degree regular graph is antimagic. The graphs consider in this work are antimagic because they are 3-regular(or cubic) graphs.

Since flower snark is a 3-regular graph, according to the definition of super vertex-magic total labeling, flower snark F_n is super vertex-magic only if the magic constant C is $23n + 2$.

In this paper we will investigate the existence of (a, d)-antimagic and super (a, d)-VAT labeling for a union of m copies of Flower snark and related graphs.

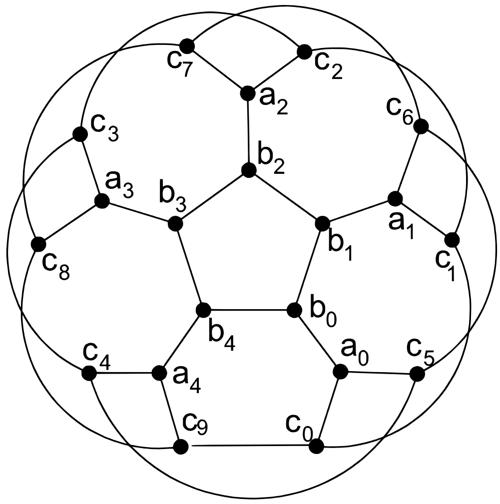

FIGURE 1. Flower snark F_5

2. Necessary Conditions for Antimagic Labeling

Assume that F_n is (a, d)-antimagic on $|V(F_n)| = 4n$ vertices and $|E(F_n)| = 6n$ edges. Let $f : E(F_n) \to \{1, 2, 3, \ldots, 6n\}$ be an edge labeling and $W = \{w(v)|v\epsilon V(F_n)\} = \{a, a+d, a+2d, \ldots, a+(4n-1)d\}$ be the set of weights. Thus

$$\sum_{e \in E(F_n)} f(e) = \frac{6n(6n+1)}{2}$$

$$\sum_{v \in V(F_n)} w(v) = 4na + 2nd(4n-1).$$

Clearly, the following equations (2.1), (2.2) hold

$$2 \sum_{e \in E(F_n)} f(e) = \sum_{v \in V(F_n)} w(v) \tag{2.1}$$

$$6n(6n+1) = 4na + 2nd(4n-1). \tag{2.2}$$

From the linear Diophantine equation (2.2) we have

$$d = \frac{3(6n+1) - 2a}{4n-1}. \tag{2.3}$$

The minimal value of a weight assigned to a vertex of degree three is $a = 6$. Thus we get the upper bound on the value d, as $0 < d < \frac{9}{2}$. This implies that,

(1) If d is odd, we get exactly two different solutions $(a, d) = (7n + 2, 1)$ or $(a, d) = (3n + 3, 3)$ respectively.
(2) If d is even, the associated value of a is not an integer. So we will not get the desired labeling when $d = 2$ or 4.
(3) Id $d \geqslant 6$ we will get the desired labeling of the graph, since the upper bound of d is $\frac{9}{2}$.

2.1. Basic Properties of Super (a, d)-VAT labeling. Suppose that graph G has a super (a, d)-VAT labeling. If δ is the smallest degree in G then the minimum possible vertex-weight is

$$1 + (p+1) + (p+2) + \cdots + (p+\delta) \geqslant 1 + p\delta + \frac{\delta(\delta+1)}{2} \tag{2.4}$$

Let Δ be the largest degree of G then

$$d \leqslant 1 + \frac{\Delta(2p + 2q - \Delta + 1) - \delta(2p + \delta + 1)}{2(p-1)} \tag{2.5}$$

The minimal value of a when we apply $\delta = \Delta = 3$ is 10 and the upper bound on the value d, as $d < \frac{11}{2}$. This implies that,

(1) If d is even, we get exactly three different solutions $(a, d) = (23n + 2, 0)$ or $(a, d) = (19n + 3, 2)$ or $(a, d) = (15n + 4, 4)$ respectively.
(2) If d is odd, we will get the value of a in decimal. So we will not get the desired labeling when $d = 1$, 3 or 5.

2.2. Basic Counting for Super Vertex Magic Labeling. Set $M = p + q$ and let S_p be the sum of the vertex labels and S_q the sum of the edge labels. Since the labels are the numbers $1, 2, \ldots, M$ we have as the sum of all labels

$$S_p + S_q = \sum_{1}^{M} i = \binom{M+1}{2}$$

At each vertex v_i we have

$$f(v_i) + \sum f(v_i u) = k. \tag{2.6}$$

We sum this over all p vertices v_i. This adds each vertex label once and each edge label twice, so that

$$S_p + 2S_q = pk \tag{2.7}$$

$$S_q + \binom{M+1}{2} = pk \tag{2.8}$$

The edge labels are all distinct (as are all the vertex labels). The edges could conceivably receive the q smallest labels or, at the other extreme, the q largest labels, or anything between. Consequently we have

$$\sum_{1}^{e} i \leqslant S_q \leqslant \sum_{p+1}^{M} i \tag{2.9}$$

A similar result holds for S_p. Combining (2.8) and (2.9), we get

$$\binom{M+1}{2} + \binom{q+1}{2} \leqslant pk \leqslant 2\binom{M+1}{2} - \binom{p+1}{2} \tag{2.10}$$

which will give the range of feasible values for k.

It is clear from (2.6) that when k is given and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. Surprisingly, however, the vertex labels do not completely determine the labeling. Having assigned the vertex labels to a graph, it may be possible to assign the edge labels to the graph in several different ways.

In following theorems, we prove that mF_n and related graphs are $(7nm+2,1)$-antimagic, super $(23nm+2,0)$-VAT, and super $(19nm+3,2)$-VAT graphs.

3. Antimagic Labeling of Flower Snark and Related Graphs

THEOREM 3.1. *If $n \geqslant 3$, then mF_n and related graphs are $(7nm+2,1)$-antimagic.*

PROOF. **Case 1:** When n is odd.

The labeling f^j of the jth copy of F_n, is given by

$$f^j(b_i b_{(i+1)(mod\ n)}) = 4nm - mi - j + 1,\ 0 \leqslant i \leqslant n-1.$$

$$f^j(c_i c_{(i+1)(mod\ 2n)}) = \begin{cases} 2nm + mi + j; & 0 \leqslant i \leqslant n-1, for\ all\ i\ even \\ 2nm - mi - m + j; & 1 \leqslant i \leqslant n-2,\ for\ all\ i\ odd \\ mi + j; & n \leqslant i \leqslant 2n-1,\ for\ all\ i\ odd \\ 4nm - mi - m + j; & n+1 \leqslant i \leqslant 2n-2,\ for\ all\ i\ even \end{cases}$$

$$f^j(a_i b_i) = mi + j,\ 0 \leqslant i \leqslant n-1$$

$$f^j(a_i c_i) = 6nm - mi - j + 1,\ 0 \leqslant i \leqslant n-1$$

$$f^j(a_i c_{n+i}) = 5nm - mi - j + 1,\ 0 \leqslant i \leqslant n-1$$

Hence it is clear that all edge labels $\{1, 2, \ldots, 6nm\}$ are distinct.

Then we can easily see that the set of weights are,

$$W^j(b_i) = \begin{cases} 7nm + m + 2 - j; & i = 0 \\ 8nm - mi - j + m + 2; & 1 \leqslant i \leqslant n-1 \end{cases}$$

$W^j(a_i) = 11nm - mi - j + 2,\ 0 \leqslant i \leqslant n-1$

$$W^j(c_i) = \begin{cases} 10nm - m + j + 1; & i = 0 \\ 10nm - mi + j + 1 - 2m; & 1 \leqslant i \leqslant n-2,\ for\ all\ i\ odd \\ 10nm - mi + j + 1; & n+2 \leqslant i \leqslant 2n-1,\ for\ all\ i\ odd \\ 10nm - mi + j + 1; & 2 \leqslant i \leqslant n-1,\ for\ all\ i\ even \\ 10nm - mi + j + 1 - 2m; & n+1 \leqslant i \leqslant 2n-2,\ for\ all\ i\ even \\ 9nm - m + j + 1; & i = n \end{cases}$$

Hence it is clear that vertex weights of mF_n and related graphs constitute the set of consecutive integers $\{7nm+2, 7nm+3, \ldots, (11nm+1)\}$ when n is odd.

Case 2: When n is even.
The labeling f^j of the jth copy of F_n, is given by

$$f^j(b_i b_{(i+1) mod\ n}) = 4nm - mi + 1 - j, \quad 0 \leqslant i \leqslant n-1.$$

$$f^j(c_i c_{(i+1) mod\ 2n}) = \begin{cases} 2nm - mi + j - m; & 0 \leqslant i \leqslant n-2,\ for\ all\ i\ even \\ 2nm + mi + j - m; & 1 \leqslant i \leqslant n-1,\ for\ all\ i\ odd \\ nm + mi + m + j; & n \leqslant i \leqslant 2n-2,\ for\ all\ i\ even \\ 3nm - mi - 3m + j; & n+1 \leqslant i \leqslant 2n-3,\ for\ all\ i\ odd \\ 2nm + j - 2m; & i = 2n-1 \end{cases}$$

$$f^j(a_i b_i) = mi + j, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_i c_i) = 6nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_i c_{n+i}) = 5nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

Hence it is clear that all edge labels $\{1, 2, \ldots, 6nm\}$ are distinct.
Then we can easily see that the set of weights of vertices are,

$$W^j(b_i) = \begin{cases} 7nm + m + 2 - j; & i = 0 \\ 8nm - mi - j + m + 2; & 1 \leqslant i \leqslant n-1 \end{cases}$$

$W^j(a_i) = 11nm - mi - j + 2;\ \ 0 \leqslant i \leqslant n-1$

$$W^j(c_i) = \begin{cases} 10nm - mi + j - 3m + 1; & 0 \leqslant i \leqslant n-2,\ for\ all\ i\ even \\ 10nm - mi + j - m + 1; & n+2 \leqslant i \leqslant 2n-2,\ for\ all\ i\ even \\ 10nm - mi + j - m + 1; & 1 \leqslant i \leqslant n-1,\ for\ all\ i\ odd \\ 10nm - mi + j - 3m + 1; & n+1 \leqslant i \leqslant 2n-3,\ for\ all\ i\ odd \\ 9nm + j + 1 - 2m; & i = 2n-1 \\ 10nm + j - m + 1; & i = n \end{cases}$$

Hence it is clear that vertex weights of mF_n and related graphs constitute the set of consecutive integers $\{7nm+2, 7nm+3, \ldots, (11nm+1)\}$ when n is even. Hence mF_n and related graphs are $(7nm+2, 1)$-antimagic graphs. $\square$

4. Super Vertex Magic Total Labeling of Flower Snark and Related Graphs

THEOREM 4.1. *If $n \geqslant 3$, then mF_n and related graphs are super $(23nm+2, 0)$-vertex antimagic total graphs. Otherwise, it can be stated, if $n \geqslant 3$, then mF_n and related graphs are super vertex magic total graphs with magic constant $23nm+2$.*

PROOF. **Case 1:** When n is odd.
The labeling f^j of the jth copy of F_n, is given by

$$f^j(b_i b_{(i+1) mod\ n}) = 8nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1.$$

$$f^j(c_i c_{(i+1) mod\ 2n}) = \begin{cases} 6nm + mi + j; & 0 \leqslant i \leqslant n-1,\ for\ all\ i\ even \\ 6nm - mi + j - m; & 1 \leqslant i \leqslant n-2,\ for\ all\ i\ odd \\ 4nm + mi + j; & n \leqslant i \leqslant 2n-1,\ for\ all\ i\ odd \\ 8nm - mi + j - m; & n+1 \leqslant i \leqslant 2n-2,\ for\ all\ i\ even \end{cases}$$

$$f^j(a_i b_i) = 4nm + mi + j, \quad 0 \leqslant i \leqslant n-1$$

$$f^j(a_i c_i) = 10nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

$$f^j(a_i c_{n+i}) = 9nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

Hence it is clear that all edge labels $\{4nm+1, 4nm+2, \ldots, 10mn\}$ are distinct. Recalling equation (2.10) we get the magic sum as,

$$\binom{M+1}{2} + \binom{q+1}{2} \leqslant pk \leqslant 2\binom{M+1}{2} - \binom{p+1}{2}$$

$$\binom{10nm+1}{2} + \binom{6nm+1}{2} \leqslant pk \leqslant 2\binom{10nm+1}{2} - \binom{4nm+1}{2}$$

$$(10nm+1)(5nm)+(6nm+1)(3nm) \leqslant 4nmk \leqslant (10nm+1)(10nm)-(4nm+1)(2nm)$$

$$\frac{(50n^2m^2+5nm)}{4nm} + \frac{18n^2m^2+3nm}{4nm} \leqslant k \leqslant \frac{(100n^2m^2+10nm)}{4nm} - \frac{8n^2m^2+2nm}{4nm}$$

$$\frac{(68n^2m^2+8nm)}{4nm} \leqslant k \leqslant \frac{(92n^2m^2+8nm)}{4nm}$$

$$(17nm+2) \leqslant k \leqslant (23nm+2)$$

Here, we get the magic constant as $k = 23nm+2$. Then labelling of vertices are given by,

$$g_f^j(v) = k - \Sigma f^j(uv)$$

$$W^j = \{g_f^j(v)/v \in V(G)\}$$

Hence we get distinct labels for the vertices starting from $\{1, 2, \ldots, 8nm\}$. According to the definition of super vertex-magic labeling, we thus conclude that mF_n and related graphs are super vertex-magic total graphs with magic sum $23nm + 2$ when n is odd.

Case 2: When n is even.
The labeling f^j of the jth copy of F_n, is given by

$$f^j(b_i b_{(i+1) mod\ n}) = 8nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1.$$

$$f^j(c_i c_{(i+1) mod\ 2n}) = \begin{cases} 6nm - mi + j - m; & 0 \leqslant i \leqslant n-2, \ for\ all\ i\ even \\ 6nm + mi + j - m; & 1 \leqslant i \leqslant n-1, \ for\ all\ i\ odd \\ 5nm + mi + j + m; & n \leqslant i \leqslant 2n-2, \ for\ all\ i\ even \\ 7nm - mi + j - 3m; & n+1 \leqslant i \leqslant 2n-3, \ for\ all\ i\ odd \\ 6nm + j - 2m; & i = 2n-1 \end{cases}$$

$$f^j(a_i b_i) = 4nm + mi + j, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_i c_i) = 10nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_i c_{n+i}) = 9nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

Hence it is clear that all edge labels $\{4nm + 1, 4nm + 2, \ldots, 10mn\}$ are distinct.
Then labeling of vertices are given by,

$$g_f^j(v) = k - \Sigma f^j(uv)$$

$$W^j = \{g_f^j(v)/v \in V(G)\}$$

Hence we get distinct labels for the vertices starting from $\{1, 2, \ldots, 8nm\}$. According to the definition of super vertex-magic labeling, we thus conclude that mF_n and related graphs are super vertex-magic total graphs with magic sum $23nm + 2$ when n is even. □

5. Super Vertex Antimagic Total Labeling of Flower Snark and Related Graphs

THEOREM 5.1. *If $n \geqslant 3$, then mF_n and related graphs are super $(19nm+3, 2)$-vertex antimagic total graphs.*

PROOF. **Case 1:** When n is odd.
The labeling f^j of the jth copy of F_n, is given by

$$f^j(b_i b_{(i+1) mod\ n}) = 8nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1.$$

$$f^j(c_i c_{(i+1) mod\ 2n}) = \begin{cases} 6nm + mi + j; & 0 \leqslant i \leqslant n-1, \ for\ all\ i\ even \\ 6nm - mi + j - m; & 1 \leqslant i \leqslant n-2, \ for\ all\ i\ odd \\ 4nm + mi + j; & n \leqslant i \leqslant 2n-1, \ for\ all\ i\ odd \\ 8nm - mi + j - m; & n+1 \leqslant i \leqslant 2n-2, \ for\ all\ i\ even \end{cases}$$

$$f^j(a_ib_i) = 4nm + mi + j, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_ic_i) = 10nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_ic_{n+i}) = 9nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

Hence it is clear that all edge labels $\{4nm+1, 4nm+2, \ldots, 10mn\}$ are distinct. Then labelling of vertices are given by,

$$f^j(b_i) = \begin{cases} m+1-j; & i = 0 \\ nm - mi - j + m + 1; & 1 \leqslant i \leqslant n-1 \end{cases}$$

$$f^j(a_i) = 4nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

$$f^j(c_i) = \begin{cases} 3nm + j - m; & i = 0 \\ 3nm - mi + j - 2m; & 1 \leqslant i \leqslant n-2, \ for\ all\ i\ odd \\ 3nm - mi + j; & n+2 \leqslant i \leqslant 2n-1, \ for\ all\ i\ odd \\ 3nm - mi + j; & 2 \leqslant i \leqslant n-1, \ for\ all\ i\ even \\ 3nm - mi + j - 2m; & n+1 \leqslant i \leqslant 2n-2, \ for\ all\ i\ even \\ 2nm + j - m; & i = n \end{cases}$$

Then we can easily see that the set of weights of vertices are,

$$W^j(b_i) = \begin{cases} 19nm - 2j + 2m + 3; & i = 0 \\ 21nm - 2mi - 2j + 2m + 3; & 1 \leqslant i \leqslant n-1 \end{cases}$$

$$W^j(a_i) = 27nm - 2mi - 2j + 3, \quad 0 \leqslant i \leqslant n-1$$

$$W^j(c_i) = \begin{cases} 25nm + 2j - 2m + 1; & i = 0 \\ 25nm - 2mi + 2j - 4m + 1; & 1 \leqslant i \leqslant n-2, \ for\ all\ i\ odd \\ 25nm - 2mi + 2j + 1; & n+2 \leqslant i \leqslant 2n-1, \ for\ all\ i\ odd \\ 25nm - 2mi + 2j + 1; & 2 \leqslant i \leqslant n-1, \ for\ all\ i\ even \\ 25nm - 2mi + 2j - 4m + 1; & n+1 \leqslant i \leqslant 2n-2, \ for\ all\ i\ even \\ 23nm + 2j - 2m + 1; & i = n \end{cases}$$

Thus the weights of vertices of mF_n constitute the set of integers $\{19nm + 3, 19nm + 5, \ldots, (27nm + 1)\}$ when n is odd.

Case 2: When n is even.
The labeling f^j of the jth copy of F_n, is given by

$$f^j(b_ib_{(i+1)mod\ n}) = 8nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1.$$

$$f^j(c_ic_{(i+1)mod\ 2n}) = \begin{cases} 6nm - mi + j - m; & 0 \leqslant i \leqslant n-2, \ for\ all\ i\ even \\ 6nm + mi + j - m; & 1 \leqslant i \leqslant n-1, \ for\ all\ i\ odd \\ 5nm + mi + j + m; & n \leqslant i \leqslant 2n-2, \ for\ all\ i\ even \\ 7nm - mi + j - 3m; & n+1 \leqslant i \leqslant 2n-3, \ for\ all\ i\ odd \\ 6nm + j - 2m; & i = 2n-1 \end{cases}$$

$$f^j(a_ib_i) = 4nm + mi + j, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_ic_i) = 10nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$
$$f^j(a_ic_{n+i}) = 9nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

Hence it is clear that all edge labels $\{4nm+1, 4nm+2, \ldots, 10mn\}$ are distinct. Then labeling of vertices are given by,

$$f^j(b_i) = \begin{cases} m - j + 1; & i = 0 \\ nm - mi - j + m + 1; & 1 \leqslant i \leqslant n-1 \end{cases}$$

$$f^j(a_i) = 4nm - mi - j + 1, \quad 0 \leqslant i \leqslant n-1$$

$$f^j(c_i) = \begin{cases} 3nm - mi + j - 3m; & 0 \leqslant i \leqslant n-2, \ for\ all\ i\ even \\ 3nm - mi + j - m; & n+2 \leqslant i \leqslant 2n-2, \ for\ all\ i\ even \\ 3nm - mi + j - m; & 1 \leqslant i \leqslant n-1, \ for\ all\ i\ odd \\ 3nm - mi + j - 3m; & n+1 \leqslant i \leqslant 2n-3, \ for\ all\ i\ odd \\ 2nm + j - 2m; & i = 2n-1 \\ 3nm + j - m; & i = n \end{cases}$$

Then we can easily see that the set of weights of vertices are,

$$W^j(b_i) = \begin{cases} 19nm - 2j + 2m + 3; & i = 0 \\ 21nm - 2mi - 2j + 2m + 3; & 1 \leqslant i \leqslant n-1 \end{cases}$$

$$W^j(a_i) = 27nm - 2mi - 2j + 3; \quad 0 \leqslant i \leqslant n-1$$

$$W^j(c_i) = \begin{cases} 25nm - 2mi + 2j - 6m + 1; & 0 \leqslant i \leqslant n-2, \ for\ all\ i\ even \\ 25nm - 2mi + 2j - 2m + 1; & n+2 \leqslant i \leqslant 2n-2, \ for\ all\ i\ even \\ 25nm - 2mi + 2j - 2m + 1; & 1 \leqslant i \leqslant n-1, \ for\ all\ i\ odd \\ 25nm - 2mi + 2j - 6m + 1; & n+1 \leqslant i \leqslant 2n-3, \ for\ all\ i\ odd \\ 23nm + 2j - 4m + 1; & i = 2n-1 \\ 25nm + 2j - 2m + 1; & i = n \end{cases}$$

Thus the weights of vertices of mF_n and related graphs constitute the set of integers $\{19nm+3, 19nm+5, \ldots, 27nm+1\}$ when n is even. Hence mF_n and related graphs are $(19nm+3, 2)$-super vertex antimagic total graphs. $\square$

5.1. Open Problem. In this paper we have shown the existence of a $(7nM+2, 1)$-antimagic, super $(23nM+2, 0)$-VAT and super $(19nM+3, 2)$-VAT labeling of flower snark and related graphs. Now we put forward the following two conjectures.

Conjecture 1. For $n \geqslant 3$, then MF_n and related graphs are $(3n+3, 3)$-antimagic.

Conjecture 2. For $n \geqslant 3$, then MF_n and related graphs are super $(15n+4, 4)$-VAT.

Illustrative example of the labeling given in the proof of Theorem 3.1 case:1 is displayed in the figure

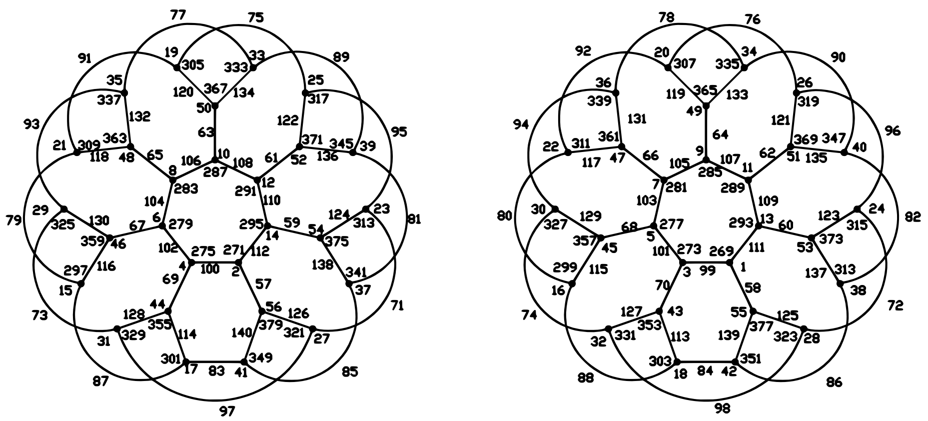

FIGURE 2. $(100, 1)$**-Antimagic labeling of** F_7

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