MAGIC AND ANTIMAGIC LABELING OF $m$ COPIES OF FLOWER SNARK AND RELATED GRAPHS

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Abstract. An antimagic labeling of a graph with $q$ edges and $p$ vertices is a bijection from the set of edges to the integers $1, 2, \ldots, q$ such that all $p$ vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an antimagic labeling. For odd $n \geq 5$, the flower snark $F_n = (V, E)$ is a simple undirected cubic graph with $4n$ vertices and $6n$ edges, where $V = \{b_i : 0 \leq i \leq n-1\} \cup \{c_i : 0 \leq i \leq 2n-1\}$ and $E = \{b_i b_{i+1} \text{ (mod } n): 0 \leq i \leq n-1\} \cup \{c_i c_{i+1} \text{ (mod } 2n): 0 \leq i \leq 2n-1\} \cup \{a_i b_{i+1} : 0 \leq i \leq n-1\} \cup \{a_i c_{i+1} : 0 \leq i \leq n-1\} \cup \{a_i a_{i+1} : 0 \leq i \leq n-1\}$. For $n = 3$ or even $n \geq 4$, $F_n$ is called the related graph of flower snark. In this paper, we show that $(a, d)$-antimagic and super $(a, d)$-vertex antimagic total labeling of $mF_n$ and related graphs.

1. Introduction

Let $G = (V, E)$ be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $p = |V(G)|$, $q = |E(G)|$ be the number of vertices and edges of $G$ respectively.

The labeling (or valuation) of a graph is any mapping that maps some set of graph elements to a set of numbers (usually positive or non negative integers). If the domain is the edge set then it is called edge labeling. If the domain is the vertex set then it is called vertex labeling. If the domain consists of both edge set and vertex set then it is called total labeling. The most complete recent survey of graph labeling is in [11].

In [13] Hartsfield et al. introduced the concepts of an antimagic graphs. An antimagic labeling of a graph with $q$ edges and $p$ vertices is a bijection from the set
of edges to the integers 1, 2, ..., q such that all p vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an antimagic labeling.

Bodendiek et al. [4] defined the concept of an \((a,d)\)-antimagic graph as a special case of an antimagic graph. They showed [5] that the theory of linear Diophantine equations and other concepts of number theory can be applied to determine the set of all connected \((a,d)\)-antimagic graphs.

In both magic and antimagic labelings, we consider the sum of all labels associated with a graph element. This will be called the weight of the element. A connected graph \(G = (V, E)\) is said to be \((a,d)\)-antimagic if there exists positive integers \(a, d\) and a bijection \(f : E \rightarrow \{1, 2, 3, \ldots, q\}\) such that \(W = \{w(v) | v \in V\} = \{a, a + d, a + 2d, \ldots, a + (n - 1)d\}\) is the set of weights. The map \(f\) is called a \((a,d)\)-antimagic labeling of \(G\).

An \((a,d)\)-vertex antimagic total (in short, \((a,d)\)-VAT) labeling of \(G\) is a bijection \(f : V \cup E \rightarrow \{1, 2, 3, \ldots, p + q\}\) with the set of weights of all vertices in \(G\) is \(\{a, a + d, a + 2d, \ldots, a + (n - 1)d\}\), for some integers \(a > 0\) and \(d \geq 0\). If \(d = 0\), then we call \(f\) a vertex magic total labeling (in short, VMT labeling). A vertex magic total labeling is an assignment of the integers from \(\{1, 2, 3, \ldots, p + q\}\) to the vertices and edges of \(G\) so that at each vertex, the vertex label and the labels on the edges incident at that vertex, add to a fixed constant \(C\). This constant \(C\) is called magic constant.

MacDougall et al. [18] have proved that the generalized Petersen graphs \(P(n,k)\) are vertex-magic total when \(n\) is even and \(k \leq (n/2) - 1\). They conjecture that all \(P(n,k)\) are vertex-magic total when \(k \leq (n - 1)/2\). If all vertices receive \(p\) smallest labels, then \((a,d)\)-vertex antimagic total labeling is called a super \((a,d)\)-vertex antimagic total labeling (in short, super \((a,d)\)-VAT).

Baca et al. [1] introduce the notions of the vertex antimagic total labeling (VATL) and \((a,d)\)-vertex antimagic total labeling \(((a,d)\)-VATL), and conjecture that all regular graphs are \((a,d)\)-VATL. In [2], Baca et al. study an antimagic labeling of generalized Petersen graphs. And also in [3] Baca et al. prove that a vertex-magic total labeling for the generalized Petersen graphs \(P(m,n)\) for all \(n > 3, 1 \leq m \leq \lfloor \frac{n+1}{2} \rfloor\).

J. A. MacDougall et al. [16] introduced vertex magic total labelings of graphs. MacDougall et al. [17] show: \(C_n\) has super vertex-magic total labeling if and only if \(n\) is odd, and no wheel, ladder, fan, friendship graph, complete bipartite graph or graph with a vertex of degree 1 has a super vertex-magic total labeling. They conjecture that no tree has a super vertex-magic total labeling and that \(K_n\) has a super vertex-magic labeling when \(n > 1\). In [12], Gomez proves the conjecture: If \(n \equiv 0 \mod 4, n > 4\), then \(K_n\) has a super vertex-magic total labeling.

The flower snark graph, defined by Isaacs [14], is certainly one of the most famous cubic graphs that theorists have come across. The well-known Four Color Theorem is equivalent to the statement that no snark is planar, including flower snark. Fiorini proved that flower snark is hypo-Hamiltonian. Yue et al. [21] have proved flower snark and related graphs are super vertex magic total. More details on flower snark can be found in [6],[7], and [19].
For odd $n \geq 5$, the flower snark $F_n = (V, E)$ is a simple undirected cubic graph with $4n$ vertices and $6n$ edges, where $V = \{b_i : 0 \leq i \leq n - 1\} \cup \{c_i : 0 \leq i \leq 2n - 1\}$ and $E = \{b_ib_{i+1}(\text{mod } n) : 0 \leq i \leq n - 1\} \cup \{c_ic_{i+1}(\text{mod } 2n) : 0 \leq i \leq 2n - 1\} \cup \{a_ib_i, a_ic_i, a_i(2n-i) : 0 \leq i \leq n - 1\}$. For $n = 3$ or even $n \geq 4$, $F_n$ is called the related graph of a flower snark. Figure 1 shows the general form of the flower snark $F_5$.

Cranston [9] proved that for $k > 2$, every $k$-regular bipartite graph is antimagic. For non-bipartite regular graphs, Liang and Zhu [15] proved that every cubic graph is antimagic. That result was generalized by Cranston, Liang and Zhu [10], who proved that odd degree regular graphs are antimagic. Hartsfield and Ringel [13] proved that every 2-regular graph is anti-magic. Chang, Liang, Pan, and Zhu [8] proved that every even degree regular graph is antimagic. The graphs considered in this work are antimagic because they are 3-regular (or cubic) graphs.

Since flower snark is a 3-regular graph, according to the definition of super vertex-magic total labeling, flower snark $F_n$ is super vertex-magic only if the magic constant $C$ is $23n + 2$.

In this paper we will investigate the existence of $(a, d)$-antimagic and super $(a, d)$-VAT labeling for a union of $m$ copies of Flower snark and related graphs.

![Figure 1. Flower snark $F_5$](image)

2. Necessary Conditions for Antimagic Labeling

Assume that $F_n$ is $(a, d)$-antimagic on $|V(F_n)| = 4n$ vertices and $|E(F_n)| = 6n$ edges. Let $f : E(F_n) \rightarrow \{1, 2, 3, \ldots, 6n\}$ be an edge labeling and $W = \{w(v)|v \in V(F_n)\} = \{a, a+d, a+2d, \ldots, a+(4n-1)d\}$ be the set of weights. Thus

$$\sum_{e \in E(F_n)} f(e) = \frac{6n(6n + 1)}{2}$$
\[
\sum_{v \in V(F_n)} w(v) = 4na + 2nd(4n - 1).
\]

Clearly, the following equations (2.1), (2.2) hold

(2.1) \[ 2 \sum_{e \in E(F_n)} f(e) = \sum_{v \in V(F_n)} w(v) \]

(2.2) \[ 6n(6n + 1) = 4na + 2nd(4n - 1). \]

From the linear Diophantine equation (2.2) we have

(2.3) \[ d = \frac{3(6n + 1) - 2a}{4n - 1}. \]

The minimal value of a weight assigned to a vertex of degree three is \( a = 6 \). Thus we get the upper bound on the value \( d \), as \( 0 < d < \frac{9}{2} \). This implies that,

1. If \( d \) is odd, we get exactly two different solutions \((a, d) = (7n + 2, 1)\) or \((a, d) = (3n + 3, 3)\) respectively.

2. If \( d \) is even, the associated value of \( a \) is not an integer. So we will not get the desired labeling when \( d = 2 \) or \( 4 \).

3. If \( d \geq 6 \) we will get the desired labeling of the graph, since the upper bound of \( d \) is \( \frac{9}{2} \).

2.1. Basic Properties of Super \((a, d)\)-VAT labeling. Suppose that graph \( G \) has a super \((a, d)\)-VAT labeling. If \( \delta \) is the smallest degree in \( G \) then the minimal possible vertex-weight is

(2.4) \[ 1 + (p + 1) + (p + 2) + \cdots + (p + \delta) \geq 1 + p\delta + \frac{\delta(\delta + 1)}{2} \]

Let \( \Delta \) be the largest degree of \( G \) then

(2.5) \[ d \leq 1 + \frac{\Delta(2p + 2q - \Delta + 1) - \delta(2p + \delta + 1)}{2(p - 1)} \]

The minimal value of \( a \) when we apply \( \delta = \Delta = 3 \) is 10 and the upper bound on the value \( d \), as \( d < \frac{13}{2} \). This implies that,

1. If \( d \) is even, we get exactly three different solutions \((a, d) = (23n + 2, 0)\) or \((a, d) = (19n + 3, 2)\) or \((a, d) = (15n + 4, 4)\) respectively.

2. If \( d \) is odd, we will get the value of \( a \) in decimal. So we will not get the desired labeling when \( d = 1, 3 \) or \( 5 \).

2.2. Basic Counting for Super Vertex Magic Labeling. Set \( M = p + q \) and let \( S_p \) be the sum of the vertex labels and \( S_q \) the sum of the edge labels. Since the labels are the numbers \( 1, 2, \ldots, M \) we have as the sum of all labels

\[
S_p + S_q = \sum_{i=1}^{M} i = \binom{M+1}{2}
\]

At each vertex \( v_i \) we have

(2.6) \[ f(v_i) + \sum f(v_iu) = k. \]
We sum this over all $p$ vertices $v_i$. This adds each vertex label once and each edge label twice, so that
\[ S_p + 2S_q = pk \] (2.7)
\[ S_q + \binom{M+1}{2} = pk \] (2.8)
The edge labels are all distinct (as are all the vertex labels). The edges could conceivably receive the $q$ smallest labels or, at the other extreme, the $q$ largest labels, or anything between. Consequently we have
\[ \sum_{1}^{e} i \leq S_q \leq \sum_{p+1}^{M} i \] (2.9)
A similar result holds for $S_p$. Combining (2.8) and (2.9), we get
\[ \binom{M+1}{2} + \binom{q+1}{2} \leq pk \leq 2\binom{M+1}{2} - \binom{p+1}{2} \] (2.10)
which will give the range of feasible values for $k$.

It is clear from (2.6) that when $k$ is given and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. Surprisingly, however, the vertex labels do not completely determine the labeling. Having assigned the vertex labels to a graph, it may be possible to assign the edge labels to the graph in several different ways.

In following theorems, we prove that $mF_n$ and related graphs are $(7nm + 2, 1)$-antimagic, super $(23nm + 2, 0)$-VAT, and super $(19nm + 3, 2)$-VAT graphs.

3. Antimagic Labeling of Flower Snark and Related Graphs

**Theorem 3.1.** If $n \geq 3$, then $mF_n$ and related graphs are $(7nm + 2, 1)$-antimagic.

**Proof. Case 1:** When $n$ is odd.
The labeling $f^j$ of the $j$th copy of $F_n$, is given by
\[ f^j(b_ib_{i+1}(\mod n)) = 4nm - mi - j + 1, \ 0 \leq i \leq n - 1. \]
\[ f^j(c_ic_{i+1}(\mod 2n)) = \begin{cases} 
2nm + mi + j; & 0 \leq i \leq n - 1, \ for \ all \ i \ even \\
2nm - mi - m + j; & 1 \leq i \leq n - 2, \ for \ all \ i \ odd \\
mi + j; & n \leq i \leq 2n - 1, \ for \ all \ i \ odd \\
4nm - mi - m + j; & n + 1 \leq i \leq 2n - 2, \ for \ all \ i \ even 
\end{cases} \]
\[ f^j(a_ib_i) = mi + j, \ 0 \leq i \leq n - 1 \]
\[ f^j(a_ic_i) = 6nm - mi - j + 1, \ 0 \leq i \leq n - 1 \]
\[ f^j(a_ic_{n+i}) = 7nm - mi - j + 1, \ 0 \leq i \leq n - 1 \]
Hence it is clear that all edge labels $\{1, 2, \ldots, 6nm\}$ are distinct.
Then we can easily see that the set of weights are,

\[ W^j(h_i) = \begin{cases} 
7nm + m + 2 - j; & i = 0 \\
8nm - mi - j + m + 2; & 1 \leq i \leq n - 1
\end{cases} \]

\[ W^j(a_i) = 11nm - mi - j + 2, \quad 0 \leq i \leq n - 1 \]

\[ W^j(c_i) = \begin{cases} 
10nm - m + j + 1; & i = 0 \\
10nm - mi + j + 1 - 2m; & 1 \leq i \leq n - 2, \text{ for all } i \text{ odd} \\
10nm - mi + j + 1; & n + 2 \leq i \leq 2n - 1, \text{ for all } i \text{ even} \\
10nm - mi + j + 1 - 2m; & n + 1 \leq i \leq 2n - 2, \text{ for all } i \text{ even} \\
9nm - m + j + 1; & i = n
\end{cases} \]

Hence it is clear that vertex weights of \( mF_n \) and related graphs constitute the set of consecutive integers \( \{7nm + 2, 7nm + 3, \ldots, (11nm + 1)\} \) when \( n \) is odd.

**Case 2:** When \( n \) is even.

The labeling \( f^j \) of the \( j \)th copy of \( F_n \), is given by

\[ f^j(b_i b_{i+1}^{\text{mod } n}) = 4nm - mi + 1 - j, \quad 0 \leq i \leq n - 1. \]

\[ f^j(c_i c_{i+1}^{\text{mod } 2n}) = \begin{cases} 
2nm - mi + j - m; & 0 \leq i \leq n - 2, \text{ for all } i \text{ even} \\
2nm + mi + j - m; & 1 \leq i \leq n - 1, \text{ for all } i \text{ odd} \\
nm + mi + m + j; & n \leq i \leq 2n - 2, \text{ for all } i \text{ even} \\
3nm - mi - 3m + j; & n + 1 \leq i \leq 2n - 3, \text{ for all } i \text{ odd} \\
2nm + j - 2m; & i = 2n - 1
\end{cases} \]

\[ f^j(a_i b_i) = mi + j, \quad 0 \leq i \leq n - 1 \\
\]

\[ f^j(a_i c_i) = 6nm - mi - j + 1, \quad 0 \leq i \leq n - 1 \\
\]

\[ f^j(a_i c_{i+1}) = 5nm - mi - j + 1, \quad 0 \leq i \leq n - 1 \]

Hence it is clear that all edge labels \( \{1, 2, \ldots, 6nm\} \) are distinct.

Then we can easily see that the set of weights of vertices are,

\[ W^j(h_i) = \begin{cases} 
7nm + m + 2 - j; & i = 0 \\
8nm - mi - j + m + 2; & 1 \leq i \leq n - 1
\end{cases} \]

\[ W^j(a_i) = 11nm - mi - j + 2, \quad 0 \leq i \leq n - 1 \]

\[ W^j(c_i) = \begin{cases} 
10nm - mi + j - 3m + 1; & 0 \leq i \leq n - 2, \text{ for all } i \text{ even} \\
10nm - mi + j - m + 1; & n + 2 \leq i \leq 2n - 2, \text{ for all } i \text{ even} \\
10nm - mi + j - m + 1; & 1 \leq i \leq n - 1, \text{ for all } i \text{ odd} \\
10nm - mi + j - 3m + 1; & n + 1 \leq i \leq 2n - 3, \text{ for all } i \text{ odd} \\
9nm + j + 1 - 2m; & i = 2n - 1 \\
10nm + j - m + 1; & i = n
\end{cases} \]
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Hence it is clear that vertex weights of \( mF_n \) and related graphs constitute the set of consecutive integers \( \{7nm + 2, 7nm + 3, \ldots, (11nm + 1)\} \) when \( n \) is even. Hence \( mF_n \) and related graphs are \((7nm + 2, 1)\)- antimagic graphs.

4. Super Vertex Magic Total Labeling of Flower Snark and Related Graphs

**Theorem 4.1.** If \( n \geq 3 \), then \( mF_n \) and related graphs are super \((23nm + 2, 0)\)-vertex antimagic total graphs. Otherwise, it can be stated, if \( n \geq 3 \), then \( mF_n \) and related graphs are super vertex magic total graphs with magic constant \( 23nm + 2 \).

**Proof.** **Case 1:** When \( n \) is odd.

The labeling \( f^j \) of the \( j \)th copy of \( F_n \), is given by

\[
f^j(b_{i(b(i+1) \mod n)} = 8nm - mi - j + 1, \quad 0 \leq i \leq n - 1.
\]

\[
f^j(c_{i(c(i+1) \mod 2n)} = \begin{cases} 
6nm + mi + j; & 0 \leq i \leq n - 1, \text{ for all } i \text{ even} \\
6nm - mi + j - m; & 1 \leq i \leq n - 2, \text{ for all } i \text{ odd} \\
4nm + mi + j; & n \leq i \leq 2n - 1, \text{ for all } i \text{ odd} \\
8nm - mi + j - m; & n + 1 \leq i \leq 2n - 2, \text{ for all } i \text{ even}
\end{cases}
\]

Hence it is clear that all edge labels \( \{4nm + 1, 4nm + 2, \ldots, 10mn\} \) are distinct. Recalling equation (2.10) we get the magic sum as,

\[
\left(\frac{M + 1}{2}\right) + \left(\frac{q + 1}{2}\right) \leq pk \leq 2\left(\frac{M + 1}{2}\right) - \left(\frac{p + 1}{2}\right)
\]

\[
\left(\frac{10nm + 1}{2}\right) + \left(\frac{6nm + 1}{2}\right) \leq pk \leq 2\left(\frac{10nm + 1}{2}\right) - \left(\frac{4nm + 1}{2}\right)
\]

\[
(10nm + 1)(5nm) + (6nm + 1)(3nm) \leq 4nmk \leq (10nm + 1)(10nm) - (4nm + 1)(2nm)
\]

\[
\frac{(50n^2m^2 + 5nm)}{4nm} \leq k \leq \frac{(100n^2m^2 + 10nm)}{4nm} = \frac{8n^2m^2 + 2nm}{4nm}
\]

\[
\frac{(68n^2m^2 + 8nm)}{4nm} \leq k \leq \frac{(92n^2m^2 + 8nm)}{4nm}
\]

\[
(17nm + 2) \leq k \leq (23nm + 2)
\]

Here, we get the magic constant as \( k = 23nm + 2 \). Then labelling of vertices are given by,

\[
g^j_f(v) = k - \Sigma f^j(uv)
\]

\[W^j = \{g^j_f(v) | v \in V(G)\}\]
Hence we get distinct labels for the vertices starting from \( \{1, 2, \ldots, 8nm\} \). According to the definition of super vertex-magic labeling, we thus conclude that \( mF_n \) and related graphs are super vertex-magic total graphs with magic sum \( 23nm + 2 \) when \( n \) is odd.

**Case 2:** When \( n \) is even.

The labeling \( f^j \) of the \( j \)th copy of \( F_n \), is given by

\[
\begin{align*}
\text{Case 1: When } n \text{ is odd.} \\
\text{The labeling } f^j \text{ of the } j \text{th copy of } F_n, \text{ is given by} \\
\end{align*}
\]

\[
f^j(b_ib_{i+1 \mod n}) = 8nm - mi - j + 1, \quad 0 \leq i \leq n - 1.
\]

\[
f^j(c_ic_{i+1 \mod 2n}) =
\begin{cases}
6nm - mi + j; & 0 \leq i \leq n - 2, \text{ for all } i \text{ even} \\
6nm + mi + j + m; & 1 \leq i \leq n - 1, \text{ for all } i \text{ odd} \\
5nm + mi + j + m; & n \leq i \leq 2n - 2, \text{ for all } i \text{ even} \\
7nm - mi + j - 3m; & n + 1 \leq i \leq 2n - 3, \text{ for all } i \text{ odd} \\
6nm + j - 2m; & i = 2n - 1
\end{cases}
\]

\[
f^j(a_ib_i) = 4nm + mi + j, \quad 0 \leq i \leq n - 1
\]

\[
f^j(a_ic_i) = 10nm - mi - j + 1, \quad 0 \leq i \leq n - 1
\]

\[
f^j(a_ic_{n+i}) = 9nm - mi - j + 1, \quad 0 \leq i \leq n - 1
\]

Hence it is clear that all edge labels \( \{4nm + 1, 4nm + 2, \ldots, 10mn\} \) are distinct. Then labeling of vertices are given by,

\[
g^j_f(v) = k - \sum f^j(uv)
\]

\[
W^j = \{g^j_f(v) \mid v \in V(G)\}
\]

Hence we get distinct labels for the vertices starting from \( \{1, 2, \ldots, 8nm\} \). According to the definition of super vertex-magic labeling, we thus conclude that \( mF_n \) and related graphs are super vertex-magic total graphs with magic sum \( 23nm + 2 \) when \( n \) is even.

5. **Super Vertex Antimagic Total Labeling of Flower Snark and Related Graphs**

**Theorem 5.1.** If \( n \geq 3 \), then \( mF_n \) and related graphs are super \((19nm + 3, 2)\)-vertex antimagic total graphs.

**Proof. Case 1:** When \( n \) is odd. The labeling \( f^j \) of the \( j \)th copy of \( F_n \), is given by

\[
\begin{align*}
\text{Case 2: When } n \text{ is even.} \\
\text{The labeling } f^j \text{ of the } j \text{th copy of } F_n, \text{ is given by} \\
\end{align*}
\]

\[
f^j(b_ib_{i+1 \mod n}) = 8nm - mi - j + 1, \quad 0 \leq i \leq n - 1.
\]

\[
f^j(c_ic_{i+1 \mod 2n}) =
\begin{cases}
6nm + mi + j; & 0 \leq i \leq n - 1, \text{ for all } i \text{ even} \\
6nm - mi + j - m; & 1 \leq i \leq n - 2, \text{ for all } i \text{ odd} \\
4nm + mi + j; & n \leq i \leq 2n - 2, \text{ for all } i \text{ odd} \\
8nm - mi + j - m; & n + 1 \leq i \leq 2n - 2, \text{ for all } i \text{ even}
\end{cases}
\]

\[
f^j(a_ib_i) = 4nm + mi + j, \quad 0 \leq i \leq n - 1
\]

\[
f^j(a_ic_i) = 10nm - mi - j + 1, \quad 0 \leq i \leq n - 1
\]

\[
f^j(a_ic_{n+i}) = 9nm - mi - j + 1, \quad 0 \leq i \leq n - 1
\]
The labeling \( f^j(a_i) = 4nm + mi + j, \ 0 \leq i \leq n - 1 \)

\( f^j(a_i, c_i) = 10nm - mi - j + 1, \ 0 \leq i \leq n - 1 \)

\( f^j(a_i, c_{n+i}) = 9nm - mi - j + 1, \ 0 \leq i \leq n - 1 \)

Hence it is clear that all edge labels \( \{4nm + 1, 4nm + 2, \ldots, 10mn\} \) are distinct. Then labelling of vertices are given by,

\[
f^j(b_i) = \begin{cases} 
m + 1 - j; & i = 0 \\
3nm + j - m; & i \text{ is even} \\
3nm - mi + j - 2m; & 1 \leq i \leq n - 2, \text{ for all } i \text{ odd} \\
3nm - mi + j; & n + 2 \leq i \leq 2n - 1, \text{ for all } i \text{ odd} \\
2nm + j - m; & i = n \\
\end{cases}
\]

\[
f^j(c_i) = \begin{cases} 
m - mi - j + 1; & 0 \leq i \leq n - 1 \end{cases}
\]

Thus the weights of vertices of \( mF_n \) constitute the set of integers \( \{19nm + 3, 19nm + 5, \ldots, (27nm + 1)\} \) when \( n \) is odd.

**Case 2:** When \( n \) is even.

The labeling \( f^j \) of the \( j \)th copy of \( F_n \), is given by

\[
f^j(b_i b_{(i+1) \mod n}) = 8nm - mi - j + 1, \ 0 \leq i \leq n - 1.
\]

\[
f^j(c_i c_{(i+1) \mod 2n}) = \begin{cases} 
6nm - mi + j - m; & 0 \leq i \leq n - 2, \text{ for all } i \text{ even} \\
6nm + mi + j - m; & 1 \leq i \leq n - 1, \text{ for all } i \text{ odd} \\
5nm + mi + j + m; & n \leq i \leq 2n - 2, \text{ for all } i \text{ even} \\
7nm - mi + j - 3m; & n + 1 \leq i \leq 2n - 3, \text{ for all } i \text{ odd} \\
6nm + j - 2m; & i = 2n - 1
\end{cases}
\]
\[ f^j(a,b_i) = 4nm + mi + j, \quad 0 \leq i \leq n - 1 \]
\[ f^j(a,c_i) = 10nm - mi - j + 1, \quad 0 \leq i \leq n - 1 \]
\[ f^j(a,c_{n+i}) = 9nm - mi - j + 1, \quad 0 \leq i \leq n - 1 \]

Hence it is clear that all edge labels \{4nm + 1, 4nm + 2, \ldots, 10mn\} are distinct. Then labeling of vertices are given by,

\[
\begin{align*}
f^j(b_i) &= \begin{cases} 
m - j + 1; & i = 0 \
2nm - mi - j + m + 1; & 1 \leq i \leq n - 1 \end{cases} \\
f^j(a_i) &= 4nm - mi - j + 1, \quad 0 \leq i \leq n - 1 \\
f^j(c_i) &= \begin{cases} 
3nm - mi + j - 3m; & 0 \leq i \leq n - 2, \text{ for all } i \text{ even} \\
3nm - mi + j - m; & n + 2 \leq i \leq 2n - 2, \text{ for all } i \text{ even} \\
3nm - mi + j - m; & 1 \leq i \leq n - 1, \text{ for all } i \text{ odd} \\
3nm - mi + j - 3m; & n + 1 \leq i \leq 2n - 3, \text{ for all } i \text{ odd} \\
2nm + j - 2m; & i = 2n - 1 \\
3nm + j - m; & i = n 
\end{cases}
\end{align*}
\]

Then we can easily see that the set of weights of vertices are,

\[
\begin{align*}
W^j(b_i) &= \begin{cases} 
19nm - 2j + 2m + 3; & i = 0 \\
21nm - 2mi - 2j + 2m + 3; & 1 \leq i \leq n - 1 
\end{cases} \\
W^j(a_i) &= 27nm - 2mi - 2j + 3; \quad 0 \leq i \leq n - 1 \\
W^j(c_i) &= \begin{cases} 
25nm - 2mi + 2j - 6m + 1; & 0 \leq i \leq n - 2, \text{ for all } i \text{ even} \\
25nm - 2mi + 2j - 2m + 1; & n + 2 \leq i \leq 2n - 2, \text{ for all } i \text{ even} \\
25nm - 2mi + 2j - 2m + 1; & 1 \leq i \leq n - 1, \text{ for all } i \text{ odd} \\
25nm - 2mi + 2j - 6m + 1; & n + 1 \leq i \leq 2n - 3, \text{ for all } i \text{ odd} \\
23nm + 2j - 4m + 1; & i = 2n - 1 \\
25nm + 2j - 2m + 1; & i = n 
\end{cases}
\end{align*}
\]

Thus the weights of vertices of \(mF_n\) and related graphs constitute the set of integers \{19nm+3, 19nm+5, \ldots, 27nm+1\} when \(n\) is even. Hence \(mF_n\) and related graphs are \((19nm+3, 2)\)-super vertex antimagic total graphs.

5.1. Open Problem. In this paper we have shown the existence of a \((7nM + 2, 1)\)-antimagic, \((23nM + 2, 0)\)-VAT and \((19nM + 3, 2)\)-VAT labeling of flower snark and related graphs. Now we put forward the following two conjectures.

**Conjecture 1.** For \(n \geq 3\), then \(MF_n\) and related graphs are \((3n + 3, 3)\)-antimagic.

**Conjecture 2.** For \(n \geq 3\), then \(MF_n\) and related graphs are super \((15n+4, 4)\)-VAT.
Illustrative example of the labeling given in the proof of Theorem 3.1 case:1 is displayed in the figure

Figure 2. (100, 1)-Antimagic labeling of $F_7$

References


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