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# ABSORBING MAPPINGS AND FIXED POINTS IN $G$-METRIC SPACES

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Abstract. In this paper a general fixed point theorem for two pairs of absorbing mappings satisfying a new type of common limit range property in $G$-metric spaces is proved.

## 1. Introduction

Let $(X, d)$ be a metric space and $S$, $T$ be two self mappings of $X$. In [**12**], Jungck defined $S$ and $T$ to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

This concept has been frequently used to prove existence theorems in fixed point theory.

A point $x \in X$ is a coincidence point of $S$ and $T$ if $Sx = Tx$. The set of all coincidence points of $S$ and $T$ is denoted by $\mathcal{C}(S, T)$.

The study of common fixed points for noncompatible mappings is also interesting. The work in this regard has been initiated by Pant in [**18**], [**19**].

Aamri and El-Moutawakil [**1**] introduced a generalization of noncompatible mappings.

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Definition 1.1 ([**1**]). Let $S$ and $T$ be self mappings of a metric space $(X, d)$. We say that $S$ and $T$ satisfy $(E.A)$-property if there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Remark 1.1. It is clear that two self mappings $S$ and $T$ of a metric space $(X, d)$ will be noncompatible if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or nonexistent. Therefore, two noncompatible self mappings of a metric space $(X, d)$ satisfy $(E.A)$-property.

There exists a vast literature concerning the study of fixed points for pairs of mappings satisfying $(E.A)$-property.

In 2005, Liu et al. [**13**] defined the notion of common $(E.A)$-property.

Definition 1.2 ([**13**]). Two pairs $(A, S)$ and $(B, T)$ of self mappings of a metric space $(X, d)$ satisfy common $(E.A)$-property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$.

In 2011, Sintunavarat and Kumam [**28**] introduced the concept of common limit range property.

Definition 1.3 ([**28**]). A pair $(A, S)$ of mappings of a metric space $(X, d)$ is said to satisfy common limit range property with respect to $S$, denoted $CLR_{(S)}$ - property, if there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in S(X)$.

Thus we can infer that a pair $(A, S)$ satisfying $(E.A)$-property, along with the closedness of the subspace $S(X)$, always have $CLR_{(S)}$ - property.

Recently, Imdad et al. [**8**] extended the notion of common limit range property for two pairs of mappings in metric spaces.

Definition 1.4 ([**8**]). Two pairs $(A, S)$ and $(B, T)$ of self mappings of a metric space $(X, d)$ are said to satisfy common limit range property with respect to $S$ and $T$, denoted $CLR_{(S,T)}$ - property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in S(X) \cap T(X)$.

Some results for pairs of mappings satisfying $CLR_{(S)}$- and $CLR_{(S,T)}$-property are obtained in [**9**]-[**11**] and in other papers.

In [**22**], the first author introduced a new type of common limit range property.

Definition 1.5 ([**22**]). *Let* $A, S$ *and* $T$ *be self mappings of a metric space* $(X, d)$. The pair $(A, S)$ is said to satisfy common limit range property with respect to $T$, denoted $CLR_{(A,S)T}$-property, if there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in S(X) \cap T(X)$.

Example 1.1. Let $X = \mathbb{R}_+$ be the metric space with the usual metric, and $Ax = \frac{x^2+1}{2}$, $Sx = \frac{x+1}{2}$, $Tx = x + \frac{1}{4}$. Then $S(X) = \left[\frac{1}{2}, \infty\right), T(X) = \left[\frac{1}{4}, \infty\right)$, $S(X) \cap T(X) = \left[\frac{1}{2}, \infty\right)$.

Let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} x_n = 0$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} = t,\ t \in S(X) \cap T(X).$$

Remark 1.2 ([**22**]). Let $A, B, S$ and $T$ be self mappings of a metric space $(X, d)$. If $(A, S)$ and $(B, T)$ satisfy $CLR_{(S,T)}$-property, then $A, S$ and $T$ satisfy $CLR_{(A,S)T}$ - property.

The notion of absorbing mappings in metric spaces is introduced in [**5**], [**6**]. Other results are obtained in [**7**], [**14**] and in other papers.

## 2. Preliminaries

In [**3**], [**4**] Dhage introduced a new class of generalized metric spaces named $D$ - metric spaces. Mustafa and Sims [**15**], [**16**] proved that most of claims concerning the fundamental topological structures are incorrect and introduced appropriate notion of generalized metric spaces, named $G$-metric spaces. In fact, Mustafa, Sims and other authors proved many fixed point results for self mappings under certain conditions in [**15**]-[**17**], [**27**] and in other papers.

Definition 2.1 ([**16**]). Let $X$ be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

$(G_3) : G(x, y, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

$(G_4) : G(x, y, z) = G(y, z, x) = ...$ (symmetry in all three variables),

$(G_5) : G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (triangle inequality).

The function $G$ is called a $G$-metric on $X$ and $(X, G)$ is called a $G$-metric space.

Remark 2.1. Let $(X, G)$ be a $G$-metric space. If $y = z$, then $G(x, y, y)$ is a quasi-metric on $X$. Hence, $(X, Q)$, where $Q(x, y) = G(x, y, y)$ is a quasi-metric and since every metric space is a particular case of quasi-metric space it follows that the notion of $G$-metric space is a generalization of a metric space.

Lemma 2.1 ([**16**]). *Let* $(X, G)$ *be a* $G$*-metric space. Then the function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 2.2 ([**16**]). Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ in $X$ is $G$-convergent if for $\varepsilon > 0$, there exists $x \in X$ and $k \in \mathbb{N}$ such that for all $m, n \geqslant k$, $G(x, x_n, x_m) < \varepsilon$.

Lemma 2.2 ([**16**]). *Let $(X, G)$ be a $G$-metric space. Then the following conditions are equivalent:*

1) *$\{x_n\}$ is $G$-convergent to $x$,*
2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$,*
3) *$G(x_n, x, x) \to 0$ as $n \to \infty$,*
4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Quite recently, in [**26**], a general fixed point theorem for two pairs of mappings satisfying $CLR_{(S,T)}$-property is proved.

Let $(A, S)$ and $T$ be self mappings of a $G$-metric space. The notion of common limit range property with respect to $T$ in $G$-metric space is similar to the definition of metric spaces (Definition 1.5).

We introduce the notion of absorbing mapping in $G$-metric spaces.

Definition 2.3. Let $A$ and $S$ be self mappings of a $G$-metric space.

1) $A$ is called $S$ absorbing if there exists $R \geqslant 0$ such that

$$G(Sx, SAx, SAx) \leqslant RG(Sx, Ax, Ax), \ \forall x \in X.$$

Similarly, $S$ is $A$ absorbing.

2) $A$ is called pointwise $S$ absorbing if for given $x \in X$, there exists $R \geqslant 0$ such that for given $x \in X$,

$$G(Sx, SAx, SAx) \leqslant RG(Sx, Ax, Ax).$$

Similarly, $S$ is pointwise $A$ absorbing.

Example 2.1. Let $[0, \infty)$ with

$$G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}.$$

Then $(X, G)$ is a $G$-metric space.

Let $Ax = 0$ and $Sx = \frac{x}{x+1}$. Then

$$G(Sx, SAx, SAx) = \tfrac{x}{x+1} \text{ and } G(Sx, Ax, Ax) = \tfrac{x}{x+1}.$$

Hence,

$$G(Sx, SAx, SAx) \leqslant RG(Sx, Ax, Ax)$$

for $R \geqslant 1$.

Other examples are in Example 4.1.

## 3. Implicit relations in $G$-metric spaces

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [**20**], [**21**] and in other papers. The study of fixed points for mappings satisfying an implicit relation in $G$-metric spaces is initiated in [**23**]. The study of fixed points for a pair of mappings with common limit range property in metric spaces satisfying implicit relations is initiated in [**9**]. The study of fixed points for pairs of mappings with common limit range property in $G$-metric spaces is initiated in [**25**] and [**26**].

In 2008 in [**2**], Ali and Imdad introduced a new class of implicit relation. Let $\mathcal{F}_G$ be the family of lower semi-continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

$(F_1) : F(t, 0, t, 0, 0, t) > 0$, for all $t > 0$.
$(F_2) : F(t, 0, 0, t, t, 0) > 0$, for all $t > 0$,
$(F_3) : F(t, t, 0, 0, t, t) > 0$, for all $t > 0$.

EXAMPLE 3.1. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \geqslant 0$ and $a + b + c + d + e < 1$.

EXAMPLE 3.2. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, where $k \in [0, 1]$.

EXAMPLE 3.3. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, where $k \in [0, 1]$.

EXAMPLE 3.4. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$, where $k \in [0, 1]$.

EXAMPLE 3.5. $F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_5, t_6\}$, where $a, b, c \geqslant 0$ and $a + b + c < 1$.

EXAMPLE 3.6. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $\alpha \in (0, 1)$, $a, b \geqslant 0$ and $a + b < 1$.

EXAMPLE 3.7. $F(t_1, ..., t_6) = t_1 - at_2 - \frac{b(t_5+t_6)}{1+t_3+t_4}$, where $a, b \geqslant 0$ and $a + 2b < 1$.

EXAMPLE 3.8. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $a, b, c \geqslant 0$ and $a + b + c < 1$.

For other examples see [**2**].

The purpose of this paper is to prove a general fixed point theorem for two pairs of absorbing mappings satisfying a new type of common limit range property in $G$ - metric spaces. As applications we obtain some results for a sequence of mappings in $G$-metric spaces and for $\varphi$-contractive mappings.

## 4. Main results

THEOREM 4.1. *Let $A, B, S$ and $T$ be self mappings of a $G$-metric space and such that for all $x, y \in X$*

$$F\left(\begin{array}{c} G(Ax, By, By), G(Sx, Ty, Ty), G(Sx, Sx, Ax), \\ G(Ty, By, By), G(Sx, By, By), G(Ax, Ty, Ty) \end{array}\right) \leqslant 0 \tag{4.1}$$

*for some $G \in \mathcal{F}_G$.*

*If $(A, S)$ and $T$ satisfy $CLR_{(A,S)T}$-property, then:*

1) $\mathcal{C}(A, S) \neq \emptyset$,

2) $\mathcal{C}(B, T) \neq \emptyset$.

*Moreover,*

a) *if $A$ is pointwise $S$ absorbing, then $A$ and $S$ have a common fixed point,*

b) *if $B$ is pointwise $T$ absorbing, then $B$ and $T$ have a common fixed point,*

c) *if the conditions of a) and b) hold, then $A, B, S$ and $T$ have a unique common fixed point.*

Proof. Since $(A, S)$ and $T$ satisfy $CLR_{(A,S)T}$-property, there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \text{ with } z \in S(X) \cap T(X).$$

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$.

By (4.1) for $x = x_n$ and $y = u$ we obtain

$$F\left(\begin{array}{c} G(Ax_n, Bu, Bu), G(Sx_n, Tu, Tu), G(Sx_n, Sx_n, Ax_n), \\ G(Tu, Bu, Bu), G(Sx_n, Bu, Bu), G(Ax_n, Tu, Tu) \end{array}\right) \leqslant 0.$$

Letting $n$ tend to infinity we obtain

$$F\left(G(z, Bu, Bu), 0, 0, G(z, Bu, Bu), G(z, Bu, Bu), 0\right) \leqslant 0,$$

a contradiction of $(F_2)$ if $G(z, Bu, Bu) > 0$. Hence $G(z, Bu, Bu) = 0$, which implies $z = Bu = Tu$. Hence $\mathcal{C}(B, T) \neq \emptyset$.

Since $z \in S(X)$, there exists $v \in X$ such that $z = Sv$. By (4.1) for $x = v$ and $y = u$ we obtain

$$F\left(\begin{array}{c} G(Av, Bu, Bu), G(Sv, Tu, Tu), G(Sv, Sv, Av), \\ G(Tu, Bu, Bu), G(Sv, Bu, Bu), G(Av, Tu, Tu) \end{array}\right) \leqslant 0,$$

$$F\left(G(Av, z, z), 0, G(z, z, Av), 0, 0, G(z, z, Av)\right) \leqslant 0,$$

a contradiction of $(F_1)$ if $G(z, z, Av) > 0$. Hence $G(z, z, Av) = 0$, which implies $z = Av = Sv$. Therefore $\mathcal{C}(A, S) \neq \emptyset$ and $z = Av = Sv = Bu = Tu$.

a) If $A$ is pointwise $S$ absorbing, there exists $R \geqslant 0$ such that

$$G(Sv, SAv, SAv) \leqslant RG(Sv, Av, Av) = 0.$$

Hence $Sv = SAv = z$. Therefore, $z = Sz$ and $z$ is a fixed point of $S$.

By (4.1) for $x = z$ and $y = u$ we obtain

$$F\left(\begin{array}{c} G(Az, Bu, Bu), G(Sz, Tu, Tu), G(Sz, Sz, Az), \\ G(Tu, Bu, Bu), G(Sz, Bu, Bu), G(Az, Tu, Tu) \end{array}\right) \leqslant 0,$$

$$F\left(G(Az, z, z), 0, G(Az, z, z), 0, 0, G(Az, z, z)\right) \leqslant 0,$$

a contradiction of $(F_1)$ if $G(Az, z, z) > 0$. Hence $G(Az, z, z) = 0$, which implies $z = Az = Sz$. Therefore, $z$ is a common fixed point of $A$ and $S$.

b) If $B$ is pointwise $T$ absorbing, there exists $R \geqslant 0$ such that

$$G(Tu, TBu, TBu) \leqslant RG(z, Bu, Bu).$$

Then $z = Tu = TBu = Tz$ and $z$ is a fixed point of $T$.

By (4.1) for $x = v$ and $y = z$ we obtain

$$F\left(\begin{array}{c} G(Av, Bz, Bz), G(Sv, Tz, Tz), G(Sv, Sv, Av), \\ G(Tz, Bz, Bz), G(Sv, Bz, Bz), G(Av, Tz, Tz) \end{array}\right) \leqslant 0,$$

$$F\left(G(z, Bz, Bz), 0, 0, G(z, Bz, Bz), G(z, Bz, Bz), 0\right) \leqslant 0,$$

a contradiction of $(F_2)$ if $G(z, Bz, Bz) > 0$. Hence $G(z, Bz, Bz) = 0$, which implies $z = Bz = Tz$. Hence, $z$ is a common fixed point of $B$ and $T$.

c) If the conditions of a) and b) hold, then $z$ is a common fixed point of $A, B, S$ and $T$.

Suppose that $z_1$ is an other fixed point of $A, B, S$ and $T$. By (4.1) we have

$$F\left(\begin{array}{c} G(Az, Bz_1, Bz_1), G(Sz, Tz_1, Tz_1), G(Sz, Sz, Av), \\ G(Tz_1, Bz_1, Bz_1), G(Sz, Bz_1, Bz_1), G(Az, Tz_1, Tz_1) \end{array}\right) \leqslant 0,$$

$$F\left(G(z, z_1, z_1), G(z, z_1, z_1), 0, 0, G(z, z_1, z_1), G(z, z_1, z_1)\right) \leqslant 0,$$

a contradiction of $(F_3)$ if $G(z, z_1, z_1) > 0$. Hence, $G(z, z_1, z_1) = 0$, which implies $z = z_1$. Therefore, $z$ is a unique common fixed point of $A, B, S$ and $T$. □

EXAMPLE 4.1. Let $X = [0, \infty)$ with $G(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}$. Then $(X, G)$ is a $G$-metric space. Let $Ax = 0$, $Sx = \frac{x}{x+1}$, $Bx = \frac{x}{3}$, $Tx = x$ and $S(X) = [0, \infty)$, $T(X) = [0, \infty)$, $S(X) \cap T(X) = [0, \infty)$. Let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} x_n = 0$. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0 = t \in S(X) \cap T(X)$. $(A, S)$ and $T$ satisfy $CLR_{(A,S)T}$-property. By Example 2.1, $A$ is $S$ pointwise absorbing. Since $G(Tx, TBx, TBx) = |Tx - TBx| = \frac{2x}{3}$ and $G(Tx, Bx, Bx) = \frac{2x}{3}$, it follows that

$$G(Tx, TBx, TBx) \leqslant RG(Tx, Bx, Bx)$$

for $R \geqslant 1$. Therefore, $B$ is $T$ pointwise absorbing.

On the other hand,

$$G(Ax, By, By) = |Ax - By| = By = \tfrac{y}{3},$$
$$G(Ty, By, By) = \left|y - \tfrac{y}{3}\right| = \tfrac{2y}{3}.$$

Hence,

$$G(Ax, By, By) \leqslant kG(Ty, By, By)$$

for $k \in \left[\frac{1}{2}, 1\right)$, which implies

$$G(Ax, By, By) \leqslant k \max\left\{\begin{array}{c} G(Sx, Ty, Ty), G(Sx, Sx, Ax), G(Ty, By, By), \\ G(Sx, By, By), G(Ax, Ty, Ty) \end{array}\right\},$$

for $k \in \left[\frac{1}{2}, 1\right)$. By Example 3.2 and Theorem 4.1, $A, B, S$ is $T$ have a unique common fixed point $z = 0$.

## 5. Applications

### 5.1. Fixed points for a sequence of mappings in $G$-metric spaces.

For a function $f : (X, G) \to (X, G)$ we denote

$$Fix(f) = \{x \in X : x = fx\}.$$

THEOREM 5.1. *Let $A, B, S$ and $T$ be self mappings of a $G$-metric space $(X, G)$. If inequality (4.1) holds for all $x, y \in X$ and $F \in \mathcal{F}_G$, then*

$$[Fix(S) \cap Fix(T)] \cap Fix(A) = [Fix(S) \cap Fix(T)] \cap Fix(B).$$

PROOF. Let $x \in [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A)$. Then by (4.1) we have

$$F\left(\begin{array}{c} G(Ax,Bx,Bx), G(Sx,Tx,Tx), G(Sx,Sx,Ax), \\ G(Tx,Bx,Bx), G(Sx,Bx,Bx), G(Ax,Tx,Tx) \end{array}\right) \leqslant 0$$

$$F\,(G(x,Bx,Bx), 0, 0, G(x,Bx,Bx), G(x,Bx,Bx), 0) \leqslant 0,$$

a contradiction of $(F_2)$ if $G(x,Bx,Bx) > 0$. Hence, $G(x,Bx,Bx) = 0$ which implies $x = Bx$ and $x \in Fix\,(B)$. Therefore

$$[Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A) \subset [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(B)\,.$$

Similarly, by (4.1) and $(F_1)$ we obtain

$$[Fix\,(S) \cap Fix\,(T)] \cap Fix\,(B) \subset [Fix\,(S) \cap Fix\,(T)] \cap Fix\,(A)\,.$$

□

By Theorems 5.1 and 4.1 we obtain

THEOREM 5.2. *Let $S, T$ and $\{A_i\}_{i \in \mathbb{N}^*}$ be self mappings of a $G$-metric space such that for all $x, y \in X$*

$$F\left(\begin{array}{c} G(A_ix,A_{i+1}y,A_{i+1}y), G(Sx,Ty,Ty), G(Sx,Sx,A_ix), \\ G(Ty,A_{i+1}y,A_{i+1}y), G(Sx,A_{i+1}y,A_{i+1}y), G(A_ix,Ty,Ty) \end{array}\right) \leqslant 0,$$

*for some $F \in \mathcal{F}_G$.*

*If $(A_1, S)$ and $T$ satisfy $CLR_{(A_1,S),T}$-property and $A_1$ is pointwise $S$ absorbing and $A_2$ is pointwise $T$ absorbing, then $S, T$ and $\{A_i\}_{i \in \mathbb{N}^*}$ have a unique common fixed point.*

**5.2. Fixed points for mappings satisfying $\varphi$-contractive condition in $G$-metric spaces.**

Let $\Phi$ be the family of continuous nondecreasing functions $\varphi : [0,\infty) \to [0,\infty)$ such that:

1) $\varphi\,(t) < t$ for all $t > 0$,

2) $\varphi\,(0) = 0$.

The following functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfy conditions $(F_1), (F_2), (F_3)$.

EXAMPLE 5.1.

$$F\,(t_1, ..., t_6) = t_1 - \varphi\,(\max\{t_2, t_3, t_4, t_5, t_6\})\,.$$

EXAMPLE 5.2.

$$F\,(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\right\}\right).$$

EXAMPLE 5.3.

$$F\,(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\}\right)$$

EXAMPLE 5.4.

$$F\,(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, \sqrt{t_3t_4}, \sqrt{t_3t_5}, \sqrt{t_4t_6}, \sqrt{t_5t_6}\right\}\right).$$

EXAMPLE 5.5.

$$F\left(t_1,...,t_6\right)=t_1-\varphi\left(at_2+bt_3+ct_4+dt_5+et_6\right),$$

where $a$, $b$, $c$, $d$, $e\geqslant 0$ and $a+b+c+d+e\leqslant 1$.

EXAMPLE 5.6.

$$F\left(t_1,...,t_6\right)=t_1-\varphi\left(at_2+\frac{b\sqrt{t_5t_6}}{1+t_3+t_4}\right),$$

where $a,b\geqslant 0$ and $a+b\leqslant 1$.

EXAMPLE 5.7.

$$F\left(t_1,...,t_6\right)=t_1-\varphi\left(at_2+b\max\{t_3,t_4\}+c\max\left\{\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right),$$

where $a,b,c\geqslant 0$ and $a+b+c\leqslant 1$.

EXAMPLE 5.8.

$$F\left(t_1,...,t_5\right)=t_1-\varphi\left(at_2+b\max\left\{\frac{2t_4+t_5}{3},\frac{2t_4+t_6}{3},\frac{t_3+t_5+t_6}{3}\right\}\right),$$

where $a,b\geqslant 0$ and $a+b\leqslant 1$.

By Theorem 4.1 and Example 3.1, we obtain

THEOREM 5.3. *Let $A,B,S$ and $T$ be self mappings of a $G$-metric space and such that for all $x,y\in X$*

$$G(Ax,By,By)\leqslant\varphi\left(\max\left\{\begin{array}{c}G(Sx,Ty,Ty),G(Sx,Sx,Ax),\\ G(Ty,By,By),G(Sx,By,By),G(Ax,Ty,Ty)\end{array}\right\}\right)$$

*where $\varphi\in\Phi$.*

*If $(A,S)$ and $T$ satisfy $CLR_{(A,S)T}$-property, then:*

1) $\mathcal{C}\left(A,S\right)\neq\emptyset$,

2) $\mathcal{C}\left(B,T\right)\neq\emptyset$.

*Moreover,*

a) *if $A$ is pointwise $S$ absorbing, then $A$ and $S$ have a common fixed point,*

b) *if $B$ is pointwise $T$ absorbing, then $B$ and $T$ have a common fixed point,*

c) *if the conditions of a) and b) hold, then $A,B,S$ and $T$ have a unique common fixed point.*

REMARK 5.1. 1) By Examples 5.2 - 5.8 and Theorem 4.1 we obtain new particular results.

2) By Examples 5.1 - 5.8 and Theorem 4.1 we obtain particular results for sequences of mappings satisfying $\varphi$-contractive conditions.

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