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DOMINATION IN CARTESIAN PRODUCT OF THREE GRAPHS

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Abstract. Consider a simple connected graph $G(V, E)$, a set S is a dominating set if for every vertex $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. i.e. for every vertex $u \in V - S$, $d(u, S) = 1$. A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The minimum cardinality among all the minimal dominating sets is called domination number of the graph G denoted by $\gamma(G)$. Let $G_1\left(V_{G_1}, E_{G_1}\right)$, $G_2\left(V_{G_2}, E_{G_2}\right)$, $G_3\left(V_{G_3}, E_{G_3}\right)$ be three simple non trivial connected graph. The Cartesian product of three graphs G_1, G_2, G_3 is denoted by $G_1 \times G_2 \times G_3$ is a graph with vertex set $V_{G_1 \times G_2 \times G_3} = V_{G_1} \times V_{G_2} \times V_{G_3}$. Such that two vertices $a = (u_1, v_1, w_1)$ and $b = (u_2, v_2, w_2)$ are said to be adjacent if $u_1 = u_2$ where $v_1, v_2 \in E_{G_2}$ and $w_1 w_2 \in E_{G_3}$ or $v_1 = v_2$ where $u_1, u_2 \in E_{G_1}$ and $w_1 w_2 \in E_{G_3}$ or $w_1 = w_2$ where $u_1, u_2 \in E_{G_1}$ and $v_1, v_2 \in E_{G_2}$. In this paper we explore the various possibilities of connecting the vertices in three graphs and also obtaining dominating set in Cartesian product of three graphs

1. Introduction

The theory of domination was introduced by Claude Berge in 1958. The inspiration for this concept was drawn from the classical problem of covering chessboard with minimum number of chess pieces. A set S is a dominating set if for every vertex $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v.

A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The minimum cardinality among all the minimal dominating sets is called domination number of the graph G denoted by $\gamma(G)$.

Recalling the definition of cartesian product of two graphs, Let $G_1\,(V_{G_1}, E_{G_1})$, $G_2\,(V_{G_2}, E_{G_2})$ be two simple connected graph. The cartesian product of G_1 and

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G_2 denoted by $G_1 \times G_2$ is a graph with vertex set $V_{G_1} \times V_{G_2}$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 = u_2$ and $v_1, v_2 \in E_{G_2}$ or $v_1 = v_2$ and $u_1, u_2 \in E_{G_1}$. This definition was further extended as 2nd and 3rd dimensional product of vertex measurable graphs in [**7, 1**], i.e., let $G_1\,(V_{G_1}, E_{G_1})$, $G_2\,(V_{G_2}, E_{G_2})$ and $G_3\,(V_{G_3}, E_{G_3})$ be three simple connected graph. The 3rd dimensional product of vertex measurable graphs of G_1, G_2 and G_3 denoted by $G_1 \times G_2 \times G_3$ is a graph with vertex set $V_{G_1} \times V_{G_2} \times V_{G_3}$, where two vertices (u_1, v_1, w_1) and (u_2, v_2, w_2) are adjacent if $u_1 = u_2$ and $v_1, v_2 \in E_{G_2}$ and $w_1 w_2 \in E_{G_3}$ or $v_1 = v_2$ and $u_1, u_2 \in E_{G_1}$ and $w_1 w_2 \in E_{G_3}$ or $w_1 = w_2$ and $u_1, u_2 \in E_{G_1}$ and $v_1, v_2 \in E_{G_2}$. Also if $u_1 \neq u_2$ and $v_1, v_2 \in E_{G_2}$ and $w_1 w_2 \in E_{G_3}$ or $v_1 \neq v_2$ and $u_1, u_2 \in E_{G_1}$ and $w_1 w_2 \in E_{G_3}$ or $w_1 \neq w_2$ and $u_1, u_2 \in E_{G_1}$ and $v_1, v_2 \in E_{G_2}$. The above definitions do not cover all the possibilities of connecting the vertices. Hence in this paper we cover all the possibilities of connecting the vertices using the concept of Cartesian product of three graphs. Dominating set has been widely studied from different perspectives in [**2, 6, 1, 4**]. The applications of Cartesian product can be found in coding theory. Throughout this paper, by a graph G we mean a finite, undirected graph without multiple edges or loops. Also K_n, P_n, C_n represents complete graph, paths and cycles with n vertices respectively.

2. Definition of Cartesian product of three graphs

The Cartesian product of three graphs $G_1\,(U, E_1)$, $G_2\,(V, E_2)$ and $G_3\,(W, E_3)$ is denoted by $G_1 \times G_2 \times G_3$ is a graph with vertex set $V_{G_1} \times V_{G_2} \times V_{G_3}$, where two vertices (u_1, v_1, w_1) and (u_2, v_2, w_2) are adjacent as follows:

Case 1. if $u_1 = u_2$ and $v_1, v_2 \in E_{G_2}$ and $w_1 w_2 \in E_{G_3}$ or $v_1 = v_2$ and $u_1, u_2 \in E_{G_1}$ and $w_1 w_2 \in E_{G_3}$ or $w_1 = w_2$ and $u_1, u_2 \in E_{G_1}$ and $v_1, v_2 \in E_{G_2}$.

Case 2. if $u_1 = u_2$ and $v_1 = v_2$ and $w_1 w_2 \in E_{G_3}$ or $v_1 = v_2$ and $w_1 = w_2$ and $u_1, u_2 \in E_{G_1}$ or $w_1 = w_2$ and $u_1 = u_2$ and $v_1, v_2 \in E_{G_2}$.

Case 3. if $u_1 \neq u_2$ and $v_1, v_2 \in E_{G_2}$ and $w_1 w_2 \in E_{G_3}$ or $v_1 \neq v_2$ and $u_1, u_2 \in E_{G_1}$ and $w_1 w_2 \in E_{G_3}$ or $w_1 \neq w_2$ and $u_1, u_2 \in E_{G_1}$ and $v_1, v_2 \in E_{G_2}$.

Case 4. if $u_1 \neq u_2$ and $v_1 \neq v_2$ and $w_1 w_2 \in E_{G_3}$ or $v_1 \neq v_2$ and $w_1 \neq w_2$ and $u_1, u_2 \in E_{G_1}$ or $w_1 \neq w_2$ and $u_1 \neq u_2$ and $v_1 v_2 \in E_{G_2}$.

Case 5. Union of case 1 and case 2.

Case 6. Union of case 1 and case 3.

Case 7. Union of case 2 and case 3.

Case 8. Union of case 3 and case 4.

Case 9. Union of case1, case 2 and case 3.

It was also found that in the above definitions the conjunction can be replaced by disjunction with respect to the adjacency of vertices. The graphs so obtained as isomorphic to any one of the above cases. Also the compliment of graphs in each of the following case is also isomorphic to any one of the above cases.

Case 10. if $u_1 = u_2$ and $v_1 = v_2$ or $w_1w_2 \in E_{G_3}$ or $v_1 = v_2$ and $w_1 = w_2$ or $u_1, u_2 \in E_{G_1}$ or $w_1 = w_2$ and $u_1 = u_2$ or $v_1, v_2 \in E_{G_2}$ is isomorphic to case 5.

Case 20. if $u_1 \neq u_2$ and $v_1, v_2 \in E_{G_2}$ or $w_1w_2 \in E_{G_3}$ or $v_1 \neq v_2$ and $u_1, u_2 \in E_{G_1}$ or $w_1w_2 \in E_{G_3}$ or $w_1 \neq w_2$ and $u_1, u_2 \in E_{G_1}$ or $v_1, v_2 \in E_{G_2}$ is isomorphic to case 6.

Case 30. if $u_1 = u_2$ or $v_1 = v_2$ and $w_1w_2 \in E_{G_3}$ or $v_1 = v_2$ or $u_1 = u_2$ and $w_1w_2 \in E_{G_3}$ or $w_1 = w_2$ or $u_1 = u_2$ and $v_1, v_2 \in E_{G_2}$ is isomorphic to case 5.

Case 40. if $u_1 \neq u_2$ or $v_1 \neq v_2$ or $w_1w_2 \in E_{G_3}$ or $v_1 \neq v_2$ or $u_1 \neq u_2$ or $w_1w_2 \in E_{G_3}$ or $w_1 \neq w_2$ or $u_1 \neq u_2$ or $v_1, v_2 \in E_{G_2}$ is isomorphic to case 6.

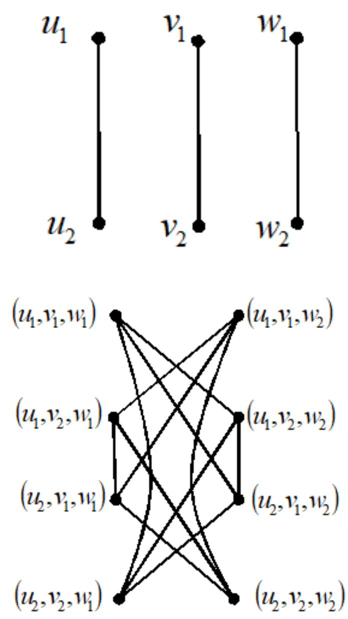
Case 50. if $u_1 = u_2$ and $v_1 \neq v_2$ and $w_1w_2 \in E_{G_3}$ or $v_1 = v_2$ and $w_1 \neq w_2$ and $u_1, u_2 \in E_{G_1}$ or $w_1 = w_2$ and $u_1 \neq u_2$ and $v_1, v_2 \in E_{G_2}$ is isomorphic to case 1.

Compliment of case 1 is isomorphic to case 7, Compliment of case 2 is isomorphic to case 6, Compliment of case 3 is isomorphic to case 5, Compliment of case 5 is isomorphic to case 3, Compliment of case 6 is isomorphic to case 2 and Compliment of case 7 is isomorphic to case 1.

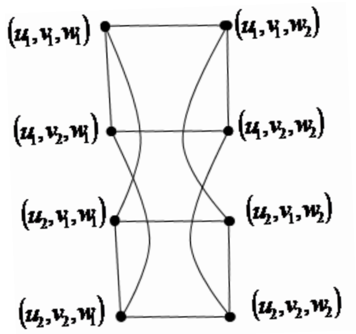
3. Examples

Consider the graphs $G_1 = G_2 = G_3 = K_2$. Then the Cartesian product of three graphs G_1, G_2 and G_3 is given by the following cases:

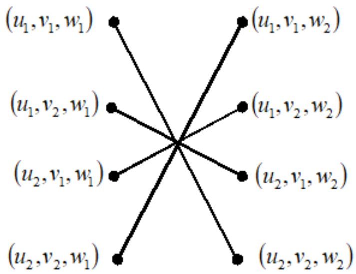
Case 1:



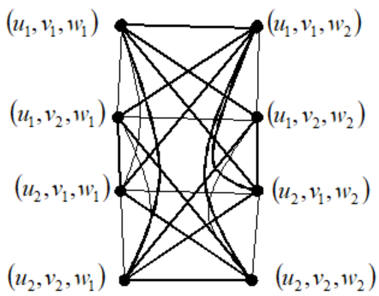
Case 2:



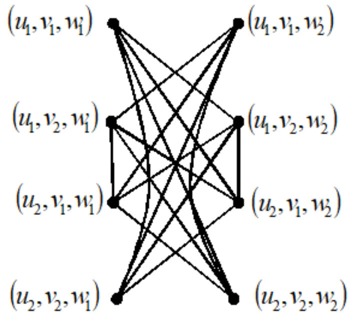
Case 3 and Case 4:



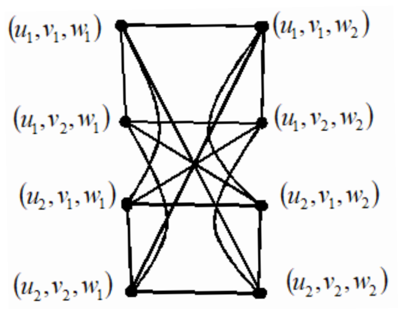
Case 5: The graph obtained is not a complete graph.



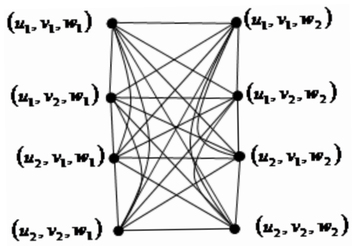
Case 6:



Case 7:



Case 9: The graph obtained is a complete graph.



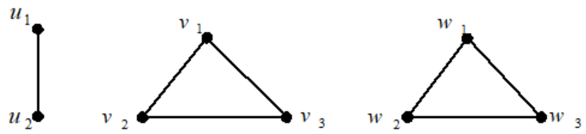
The graphs obtained in the above cases are non-isomorphic in nature. This shows that each case is a unique way of connecting the vertices given graphs G_1, G_2, G_3.

4. Main Results

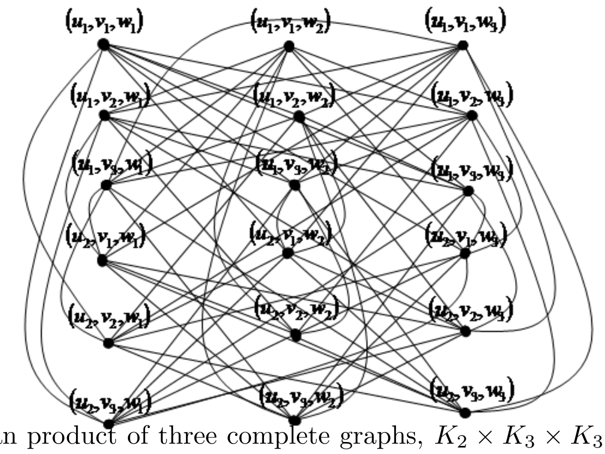
It would be enough to find the domination number for the graph obtained in case 1 and case 2. As case 3 and case 4 are not considered due to disconnectedness in graph. In remaining cases the graph contains more number of edges hence the domination number obtained will be less than or equal to the domination number obtained in case 1 or case 2.

NOTATION 4.1. The following terminologies will be used through the rest of this paper. The domination number obtained for $G_1 \times G_2 \times G_3$ in case 1 is denoted by $\gamma_1(G)$. The domination number obtained for $G_1 \times G_2 \times G_3$ in case 2 is denoted by $\gamma_2(G)$.

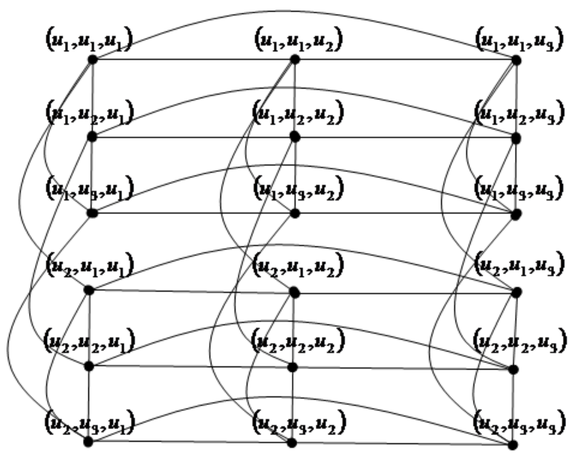
EXAMPLE 4.1. $G_1 = K_2$, $G_2 = K_3$, $G_3 = K_3$.



Cartesian product of three complete graphs, $K_2 \times K_3 \times K_3$ in case 1 is given by



Cartesian product of three complete graphs, $K_2 \times K_3 \times K_3$ in case 2 is given by



THEOREM 4.1. *If $G_1 = K_a$, $G_2 = K_b$, and $G_3 = K_c$ for all $a, b, c \in \mathbb{N}$, then*

$$\gamma_1 (K_a \times K_b \times K_c) = \min\{a, b, c\}.$$

PROOF. In Cartesian product of three graphs each vertex will be of the form (u_l, v_m, w_n), where $1 \leqslant l \leqslant a$, $1 \leqslant m \leqslant b$, $1 \leqslant n \leqslant c$. The dominating set is given by (u_l, v_m, w_n), where $1 \leqslant l \leqslant a$, $m = 1$, $n = 1$ assuming a to be minimum. The vertex (u_1, v_1, w_1) in the dominating set covers all vertices (u_l, v_m, w_n) with anyone of l, m, n is equal to 1 and other two variables not equal to 1. The vertex (u_2, v_1, w_1) covers all vertices (u_l, v_m, w_n), $l = 2$, $1 < m \leqslant b$, $1 < n \leqslant c$. The vertex (u_3, v_1, w_1) covers all vertices (u_l, v_m, w_n), $l = 3$, $1 < m \leqslant b$, $1 < n \leqslant c$ Continuing the process the dominating set is given by (u_l, v_m, w_n), where $1 \leqslant l \leqslant a$, $m = 1$, $n = 1$. i.e., the dominating set is given by $\{(1,1,1), (2,1,1), (3,1,1), \ldots, (a,1,1)\}$. Similarly the dominating set can be considered for other cases of m, n considering b and c as minimum. Hence the proof. □

COROLLARY 4.1. *If $G_1 = G_2 = G_3 = K_n$, for all $n \in \mathbb{N}$, then*

$$\gamma_1 (K_n \times K_n \times K_n) = n$$

.

THEOREM 4.2. *If $G_1 = K_a$, $G_2 = K_b$ and $G_3 = K_c$, for all $a, b, c \in \mathbb{N}$, then*

$$\gamma_2 (K_a \times K_b \times K_c) \geqslant \left\lceil \frac{a \times b \times c}{a + b + c - 2} \right\rceil.$$

PROOF. From [**5**] it is known that $\left\lceil \frac{n}{\Delta+1} \right\rceil \leqslant \gamma(G) \leqslant n - \Delta$. Using this inequality we find the lower bound for the case 2 Cartesian product of three graphs. The number of vertices for Cartesian product of three graphs given $G_1 = K_a$, $G_2 = K_b$, $G_3 = K_c$ is equal to $a \times b \times c$. The construction results in a regular graph. Using the concept of permutation it can be found that the degree of each vertex in this regular graph of case 2 Cartesian product of three graphs is $(a-1) + (b-1) + (c-1) = (a + b + c - 3)$. Therefore $\Delta = a + b + c - 3$.
Hence the result. □

COROLLARY 4.2. *If $G_1 = G_2 = G_3 = K_n$, for all $n \in \mathbb{N}$, then*

$$\gamma_2 (K_n \times K_n \times K_n) \geqslant \left\lceil \frac{n^3}{3n - 2} \right\rceil.$$

THEOREM 4.3. *If $G_1 = P_a$, $G_2 = P_b$, and $G_3 = P_c$, ll $a, b, c \in \mathbb{N}$, $a \leqslant b \leqslant c$, then*

$$\gamma_1 (P_a \times P_b \times p_c) \leqslant \left\lceil \frac{a}{3} \right\rceil \times \min\left(\left\lceil \frac{b}{3} \right\rceil \times c \left\lceil \frac{c}{3} \right\rceil \times b \right).$$

PROOF. Without loss of generality the Cartesian product of three path graphs P_a, P_b, and P_c will have $a \times b$ rows and c columns. All the vertices will be of the form (u_i, v_j, w_k) where $1 \leqslant i \leqslant a$, $1 \leqslant j \leqslant b$, $1 \leqslant k \leqslant c$. In $a \times b$ with c columns, $\left\lceil \frac{b}{3} \right\rceil$ rows or $\left\lceil \frac{c}{3} \right\rceil$ columns are dominant. If $\left\lceil \frac{b}{3} \right\rceil$ rows are dominant then out of $a \times b$ rows $\left\lceil \frac{a}{3} \right\rceil \times \left\lceil \frac{b}{3} \right\rceil \times c$ vertices will be the dominating set. If $\left\lceil \frac{c}{3} \right\rceil$ columns are dominant then out of $a \times b$ rows with c columns $\left\lceil \frac{a}{3} \right\rceil \times \left\lceil \frac{c}{3} \right\rceil \times b$ vertices will be the dominating set. Hence by definition the minimum cardinality among all the dominating get is the domination number hence domination number is given $\gamma_1 \leqslant \left\lceil \frac{a}{3} \right\rceil \times \min\left(\left\lceil \frac{b}{3} \right\rceil \times c \left\lceil \frac{c}{3} \right\rceil \times b\right)$. □

COROLLARY 4.3. *If* $G_1 = G_2 = G_3 = P_n$*, ll* $n \in \mathbb{N}$*, then*

$$\gamma_1 \left(P_n \times P_n \times P_n\right) \leqslant n \left(\left\lceil \frac{n}{3} \right\rceil\right)^2.$$

THEOREM 4.4. *If* $G_1 = P_a$*,* $G_2 = P_b$ *and* $G_3 = P_c$*, ll* $a, b, c \in \mathbb{N}$*, then*

$$\gamma_2 \left(P_a \times P_b \times p_c\right) \geqslant \left\lceil \frac{a \times b \times c}{7} \right\rceil.$$

PROOF. From [**5**] it is known that $\left\lceil \frac{n}{\Delta+1} \right\rceil \leqslant \gamma(G) \leqslant n - \Delta$. Using this inequality we find the lower bound for the case 2 Cartesian product of three graphs. The number of vertices for Cartesian product of three graphs given $G_1 = P_a$, $G_2 = P_b$, $G_3 = P_c$ is equal to $a \times b \times c$. In construction of these graphs, we can use the concept of permutation and found that the degree of each vertex in the graph of case 2 has maximum degree 6. Therefore $\Delta = 6$. Hence the result. □

COROLLARY 4.4. *If* $G_1 = G_2 = G_3 = P_n$*, ll* $n \in \mathbb{N}$*, then*

$$\gamma_2 \left(P_n \times P_n \times P_n\right) \geqslant \left\lceil \frac{n^3}{7} \right\rceil.$$

Construction of Cartesian product of three graphs in case 1 and case 2 leads to complex graphs, investigating the degree of these graphs it can be easily found that degree of case1 graph is greater than or equal to degree of graphs in case2. Hence $\gamma_1(G) \leqslant \gamma_2(G)$.

5. Vizing conjecture

In general, we can show the vizing conjecture

$$\gamma\left(P_a \times P_b \times P_c\right) \geqslant \gamma\left(P_a\right) \gamma\left(P_b\right) \gamma\left(P_c\right)$$

satisfies for both Case 1 and Case 2 graphs.

We have $\gamma_1 \leqslant \left\lceil \frac{a}{3} \right\rceil \times \min\left(\left\lceil \frac{b}{3} \right\rceil \times c \left\lceil \frac{c}{3} \right\rceil \times b\right)$, $\gamma\left(P_a\right) \gamma\left(P_b\right) \gamma\left(P_c\right) = \left\lceil \frac{a}{3} \right\rceil \left\lceil \frac{b}{3} \right\rceil \left\lceil \frac{c}{3} \right\rceil$. Hence $\left\lceil \frac{a}{3} \right\rceil \times \min\left(\left\lceil \frac{b}{3} \right\rceil \times c \left\lceil \frac{c}{3} \right\rceil \times b\right) \geqslant \left\lceil \frac{a}{3} \right\rceil \left\lceil \frac{b}{3} \right\rceil \left\lceil \frac{c}{3} \right\rceil$. Therefore vizing conjecture satisfies for the classes of paths in Case 1.

Considering for Case 2 graphs we have

$\gamma_2\left(P_a \times P_b \times P_c\right) \geqslant \left\lceil \frac{abc}{\Delta(P_a \times P_b \times P_c)+1} \right\rceil = \left\lceil \frac{abc}{7} \right\rceil$, $\gamma\left(P_a\right) \gamma\left(P_b\right) \gamma\left(P_c\right) = \left\lceil \frac{a}{3} \right\rceil \left\lceil \frac{b}{3} \right\rceil \left\lceil \frac{c}{3} \right\rceil$. Hence $\left\lceil \frac{abc}{7} \right\rceil \geqslant \left\lceil \frac{a}{3} \right\rceil \left\lceil \frac{b}{3} \right\rceil \left\lceil \frac{c}{3} \right\rceil$. Therefore vizing conjecture satisfies for the classes of paths in Case 2. Therefore $\gamma\left(P_a \times P_b \times P_c\right) \geqslant \gamma\left(P_a\right) \gamma\left(P_b\right) \gamma\left(P_c\right)$ for 3rd dimensional product of vertex measurable graphs given the graphs are paths.

Similarly we can prove the results and vizing conjecture for the classes of cycles in case 1 and case 2.

THEOREM 5.1. *If $G_1 = C_a$, $G_2 = C_b$, and $G_3 = C_c$, ll $a, b, c \in \mathbb{N}$, $a \leqslant b \leqslant c$, then*

$$\gamma_1 (C_a \times C_b \times C_c) \leqslant \left\lceil \frac{a}{3} \right\rceil \times \min \left(\left\lceil \frac{b}{3} \right\rceil \times c \left\lceil \frac{c}{3} \right\rceil \times b \right).$$

COROLLARY 5.1. *If $G_1 = G_2 = G_3 = C_n$, ll $n \in \mathbb{N}$, then*

$$\gamma_1 (C_n \times C_n \times C_n) \leqslant n \left(\left\lceil \frac{n}{3} \right\rceil \right)^2.$$

THEOREM 5.2. *If $G_1 = C_a$, $G_2 = C_b$ and $G_3 = C_c$, ll $a, b, c \in \mathbb{N}$, then*

$$\gamma_2 (C_a \times C_b \times C_c) \geqslant \left\lceil \frac{a \times b \times c}{7} \right\rceil.$$

COROLLARY 5.2. *If $G_1 = G_2 = G_3 = C_n$, for all $n \in \mathbb{N}$, then*

$$\gamma_2 (C_n \times C_n \times C_n) \geqslant \left\lceil \frac{n^3}{7} \right\rceil.$$

Given $\gamma (C_a \times C_b \times C_c) \geqslant \left\lceil \frac{abc}{7} \right\rceil$, $\gamma (C_a) \gamma (C_b) \gamma (C_c) = \left\lceil \frac{a}{3} \right\rceil \left\lceil \frac{b}{3} \right\rceil \left\lceil \frac{c}{3} \right\rceil$ and $\left\lceil \frac{abc}{7} \right\rceil \geqslant \left\lceil \frac{a}{3} \right\rceil \left\lceil \frac{b}{3} \right\rceil \left\lceil \frac{c}{3} \right\rceil$. Therefore $\gamma (C_a \times C_b \times C_c) \geqslant \gamma (C_a) \gamma (C_b) \gamma (C_c)$.

6. Open Problem

(1) Find the dominating number for Cartesian triple product of various classes of graphs including arbitrary graph with n vertices.
(2) Analyze the concept of domatically critical with respect to Cartesian product of three graphs.
(3) Various graph parameters can be studied on Cartesian product of three graphs.

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