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THE EFFECTIVE MODIFICATION OF SOME ANALYTICAL TECHNIQUES FOR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper mainly focuses on the recent advances in the homotopy approximated methods for solving Fredholm integro-differential equations of the second kind. This study shows the Homotopy Perturbation Method (HPM) and Direct Homotopy Analysis Method (DHAM), the reliability of the methods and reduction in the size of the computational work give this methods wider applicability. Convergence analysis of the exact solution of the proposed methods will be established. Moreover, we proved the existence and uniqueness of the solution. To illustrate the methods, some examples are presented.

1. Introduction

In this paper, we consider Fredholm integro-differential equation of the form:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \lambda \int_a^b K(x,t)G(u(t))dt \tag{1.1}$$

with the initial conditions

$$u^{(r)}(a) = b_r, \quad r = 0, 1, 2, \cdots, (k-1), \tag{1.2}$$

where $u^{(j)}(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ and $p_j(x)$ are an analytic function, G is nonlinear function of u and a, b, λ, and b_r are real finite constants.

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The Fredholm integro-differential equations arise in many scientific applications. It was also shown that these equations can be derived from boundary value problems. Erik Ivar Fredholm (1866–1927) is best remembered for his work on integral equations and spectral theory.

The application of homotopy techniques in linear and non-linear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem which is easy to solve into the under study problem which is difficult to solve. This method was proposed first by He in 1997 and systematical description in 2000 which is, in fact, a coupling of the traditional perturbation method and homotopy in topology [**19**]. This method was further developed and improved by He and applied to non-linear oscillators with discontinuities [**20**]. After that many researchers applied the method to various linear and non-linear problems. For example, it was applied to the quadratic Ricatti differential equation by Abbasbandy [**1**], to the axisymmetric flow over a stretching sheet by Ariel et al. [**3**], to the Helmholtz equation and fifth-order KdV equation by Rafei and Ganji [**24**], for the thin film flow of a fourth grade fluid down a vertical cylinder by Siddiqui et al. [**25**], to the non-linear Volterra-Fredholm integral equations by Hamoud and Ghadle [**6, 7**], to integro-differential equation [**4, 5, 8, 16**], to system of Fredholm integral equations [**22**], Alao et al. [**2**] studied the Adomian decomposition method and the variational iteration method on various types of integro-differential equation. Moreover, many methods for solving integro-differential equations have been studied by several authors [**9, 10, 11, 12, 13, 14, 15, 17, 18**].

The main objective of the present chapter is to study the behavior of the solution that can be formally determined by semi-analytical approximated methods as the homotopy perturbation method and direct homotopy analysis method. Moreover, we proved the existence and uniqueness results of the Fredholm integro-differential equations.

2. Homotopy Perturbation Method (HPM)

The homotopy perturbation method first proposed by He [**19, 20**]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

under the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (2.1) therefore can be rewritten as follows [**21**]:

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

By the homotopy technique (Liao 1992, 1997) [**23**]. We construct a homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1]. \tag{2.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0)] + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq.(2.1) which satisfies the boundary conditions. From Eqs.(2.4), (2.5) we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{2.6}$$
$$H(v,1) = A(v) - f(r) = 0. \tag{2.7}$$

The changing in the process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$, and $A(v) - f(r)$ are called homotopic. Now, assume that the solution of Eqs. (2.4), (2.5) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{2.8}$$

The approximate solution of Eq.(2.1) can be obtained by setting $p = 1$.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{2.9}$$

Then equating the terms with identical power of P, we obtain the following series of linear equations:

$$\begin{aligned}
P^0 : u_0(x) &= \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r, \\
P^1 : u_1(x) &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u_0(t))(t)dt\right) \\
&- \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_0^{(j)}(x)\right), \\
P^2 : u_2(x) &= \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u_1(t))(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_1^{(j)}(x)\right), \\
P^3 : u_3(x) &= \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u_2(t))dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_2^{(j)}(x)\right), \\
&\vdots
\end{aligned}$$

3. Direct Homotopy Analysis Method (DHAM)

Consider Fredholm integro-differential equation (1.1) and substitute the kernel $K(x,t) = g(x)h(t)$ we obtain

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \lambda g(x)\int_a^b h(t)G(u(t))dt.$$

To obtain the approximate solution, we integrating (k)-times in the interval $[a,x]$ with respect to x we obtain,

$$\begin{aligned} u(x) &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r + \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}\int_a^b h(t)G(u(t))dt\right) \\ &\quad - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}u_n^{(j)}(x)\right), \end{aligned} \tag{3.1}$$

Setting

$$\begin{aligned} Q &= \int_a^b h(t)G(u(t))dt \\ F &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}u_n^{(j)}(x)\right), \end{aligned} \tag{3.2}$$

therefore, Eq. (3.1) can be written as

$$u(x) = F(x) + \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}Q\right),$$

we define the nonlinear homotopy operator as:

$$N[u(x)] = u(x) - F(x) - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}Q\right),$$

The corresponding mth-order deformation equation is as follows

$$L[u_m(x) - \chi_m u_{m-1}(x)] = BH(x)R_m(\overrightarrow{u_{m-1}(x)})$$

where

$$R_m(\overrightarrow{u_{m-1}(x)}) = u_{m-1}(x) - F(x)(1-\chi_m) - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}Q\right), \tag{3.3}$$

and

$$\chi_m = \begin{cases} 1, & m > 1. \\ 0, & m \leqslant 1. \end{cases}$$

choosing the auxiliary linear operator $L[u] = u$, we obtain

$$\begin{aligned}
u_0(x) & \quad \text{Choosing initial guess} \\
u_1(x) &= BH(x)\Big[u_0(x) - L^{-1}\left(\frac{f(x)}{p_k(x)}\right) - \sum_{r=0}^{k-1}\frac{1}{r!}(x-a)^r b_r \\
&\quad -\lambda L^{-1}\left(\frac{g(x)}{p_k(x)}\int_a^b h(t)G(u_0(t))dt\right) + \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}u_0^{(j)}(x)\right)\Big], \\
u_m(x) &= \chi_m u_{m-1}(x) + BH(x)\Big[u_{m-1}(x) \\
&\quad -\lambda L^{-1}\left(\frac{g(x)}{p_k(x)}\int_a^b h(t)G(u_{m-1}(t))dt\right) \\
&\quad +\sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)}u_{m-1}^{(j)}(x)\right)\Big],\ m > 1.
\end{aligned}$$

with auxiliary function $H(x)$ and auxiliary parameter B. Then, $u(x) = \sum_{i=0}^{m} u_i(x)$ as the approximate solution.

4. Existence and Uniqueness Results

In this section, we shall give an existence and uniqueness results of Eq. (1.1), with the initial condition (1.2) and prove it.

We can be written equation (1.1) in the form of:

$$\begin{aligned}
u(x) &= L^{-1}\Big[\frac{f(x)}{p_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r + \lambda_1 L^{-1}\Big[\int_a^b \frac{1}{p_k(x)}K(x,t)G(u_n(t))dt\Big] \\
&\quad -L^{-1}\Big[\sum_{j=0}^{k-1}\frac{p_j(x)}{p_k(x)}u^{(j)}(x)\Big].
\end{aligned}$$

Such that,

$$\begin{aligned}
&L^{-1}\Big[\int_a^b \frac{1}{p_k(x)}K(x,t)G(u_n(t))dt\Big] = \int_a^b \frac{(x-t)^k}{k!p_k(x)}K(x,t)G(u_n(t))dt \\
&\sum_{j=0}^{k-1} L^{-1}\Big[\frac{p_j(x)}{p_k(x)}\Big]u^{(j)}(x) = \sum_{j=0}^{k-1}\int_a^b \frac{(x-t)^{k-1}p_j(t)}{k-1!p_k(t)}u^{(j)}(t)dt.
\end{aligned}$$

We set,

$$\Psi(x) = L^{-1}\Big[\frac{f(x)}{p_k(x)}\Big] + \sum_{r=0}^{k-1}\frac{(x-a)^r}{r!}b_r.$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1): There exist two constants α and $\gamma_j > 0,\ j = 0, 1, \cdots, k$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|G(u_1)) - G(u_2))| \leqslant \alpha\,|u_1 - u_2|$$

and

$$\left|D^j(u_1) - D^j(u_2)\right| \leqslant \gamma_j\,|u_1 - u_2|\,,$$

we suppose that the nonlinear terms $G(u(x))$ and $D^j(u) = (\frac{d^j}{dx^j})u(x) = \sum_{i=0}^{\infty} \gamma_{i_j}$, ($D^j$ is a derivative operator), $j = 0, 1, \cdots, k$, are Lipschitz continuous.

(H2): we suppose that for all $a \leqslant t \leqslant x \leqslant b$, and $j = 0, 1, \cdots, k$:

$$\left|\frac{\lambda(x-t)^k K(x,t)}{k! p_k(x)}\right| \leqslant\ \theta_1, \qquad \left|\frac{\lambda(x-t)^k K(x,t)}{k!}\right| \leqslant \theta_2,$$

$$\left|\frac{(x-t)^{k-1} p_j(t)}{(k-1)! p_k(t)}\right| \leqslant \theta_3, \qquad \left|\frac{(x-t)^{k-1} p_j(t)}{(k-1)!}\right| \leqslant \theta_4,$$

(H3): There exist three functions $\theta_3^*, \theta_4^*,$ and $\gamma^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \leqslant t \leqslant x \leqslant 1\}$ such that:

$$\theta_3^* = \max|\theta_3|\,,\ \theta_4^* = \max|\theta_4|\,,\ \text{and}\ \gamma^* = \max|\gamma_j|\,.$$

(H4): $\Psi(x)$ is bounded function for all x in $J = [a, b]$.

THEOREM 4.1. *Assume that (H1)–(H4) hold. If*

$$0 < \psi = (\alpha\theta_1 + k\gamma^*\theta_3^*)(b-a) < 1, \tag{4.1}$$

then there exists a unique solution $u(x) \in C(J)$ to IVB $(1.1) - (1.2)$.

PROOF. Let u_1 and u_2 be two different solutions of IVB $(1.1) - (1.2)$. Then

$$\begin{aligned}
\Big|u_1 - u_2\Big| \ &= \ \Big|\int_a^b \frac{\lambda(x-t)^k K(x,t)}{p_k(x) k!}[G(u_1) - G(u_2))]dt \\
&\quad - \sum_{j=0}^{k-1}\int_a^b \frac{(x-t)^{k-1} p_j(t)}{p_k(t)(k-1)!}[D^j(u_1) - D^j(u_2))]dt\Big| \\
&\leqslant \ \int_a^b \Big|\frac{\lambda(x-t)^k K(x,t)}{p_k(x) k!}\Big|\Big|G(u_1) - G(u_2))\Big|dt \\
&\quad - \sum_{j=0}^{k-1}\int_a^b \Big|\frac{(x-t)^{k-1} p_j(t)}{p_k(t)(k-1)!}\Big|\Big|D^j(u_1) - D^j(u_2))\Big|dt \\
&\leqslant \ (\alpha\theta_1 + k\gamma^*\theta_3^*)(b-a)|u_1 - u_2|,
\end{aligned}$$

we get $(1 - \psi)|u_1 - u_2| \leqslant 0$. Since $0 < \psi < 1$, so $|u_1 - u_2| = 0$. therefore, $u_1 = u_2$ and the proof is completed. $\square$

5. Illustrative Example

In this section, we present the semi-analytical techniques based on HPM and DHAM to solve Fredholm integro-differential equations. To show the efficiency of the present methods for our problem in comparison with the exact solution we report absolute error.

EXAMPLE 5.1. Consider the following Fredholm integro-differential equation.

$$u'(x) = e^x(1+x) - x + \int_0^1 xu(t)dt, \tag{5.1}$$

with the initial condition

$$u(0) = 0, \tag{5.2}$$

and the the exact solution is $u(x) = xe^x$.

TABLE 1. Numerical Results of the Example 1.

x	Exact	HPM	DHAM
0.1	0.1105170	0.1103782	0.1105170
0.2	0.2442805	0.2437249	0.2442805
0.3	0.4049576	0.4037076	0.4049576
0.4	0.5967298	0.5945076	0.5967298
0.5	0.8243606	0.8208884	0.8233606
0.6	1.0932712	1.0882712	1.0932712
0.7	1.4096268	1.4028213	1.4096268
0.8	1.7804327	1.7715438	1.7804327
0.9	2.2136428	2.2023928	2.2136428

6. Discussion and Conclusion

We discussed the HPM and DHAM for solving Fredholm integro-differential equations of the second kind. To assess the accuracy of each method, the test example with known exact solution are used. In this work, the above methods have been successfully employed to obtain the approximate solution of a Fredholm integro-differential equation. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of these methods are similar approximately. Table 1 shows that the numerical results obtained with DHAM coincide with the exact solutions.

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