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ON THE STUDY OF FRACTIONAL DIFFERENTIAL EQUATIONS IN A WEIGHTED SOBOLEV SPACE

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Abstract. In this article, we study the existence and uniqueness of solutions for a nonlinear boundary value problem of fractional differential equations with higher order α $(n-1<\alpha\leqslant n)$ involving Riemann-Liouville fractional derivative. The solutions are discussed in a weighted Sobolev space using Banach's fixed point theorem. An illustrative example is also given to embody the main results.

1. Introduction

This work is concerned with the existence and uniqueness of solution to the following initial value problem of the higher-order fractional differential equations (FDEs) with Riemann-Liouville derivative

$$D^{\alpha}u(t) = g\left(t,u(t),D^{\beta}u(t)\right),\ t\in I=[0,T],\ T>0 \tag{1.1}$$

$$D^{\alpha-i}u|_{t=0} = 0,\ i=1,...,n,\ i\neq n-1 \text{ and } u(T)=0, \tag{1.2}$$

where $1\leqslant n-1<\alpha\leqslant n,\ 0<\beta<1,\ g:I\times\mathbb{R}^2\to\mathbb{R}$ is given function, D^{α} denotes the Riemann-Liouville's fractional derivative.

The field of fractional differential equations taken the attention of many researchers, especially in recent decades, because of their importance in exact modeling and the description of several properties of the non stationary or stationary physical phenomena among them: viscoelasticity, electrotechnics, electrochemistry, biophysics, biology, engineering, the theory f the signal, image processing, economy. Therefore, applications of fractional differential equations in modeling of different

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phenomena became broader used than ordinary differential equations. For more clarifications on this theory and its applications, see the monographs of Hilfer [**11**], Kilbas et el. [**14**] and Podlubny [**16**]. Majority of the research focused on the existence and uniqueness of solution for fractional differential equations, where this side of study for nonlinear FDEs have been extensively developed using mostly the fixed point theory and other methods as iterative method, measures of non-compactness technique, Krasnoselskii-Krein and Nagumo uniqueness theorems (see [**4, 19**]). However, the fixed point theorems staying the most used method to study the existence and uniqueness of solutions of nonlinear FDEs and nonlinear fractional differential systems (see [**2, 3, 5, 6, 10, 15, 21**]) and the references therein.

Beside, the mentioned published papers has been devoted to give the existence and uniqueness of solution of various classes of fractional differential and integral equations in the space of continuous functions $C([a,b])$ or $C(\mathbb{R}_+)$. But the discussion on measurable solutions of differential and integral equations remains relatively few compared to continuous solutions, we refer to some papers about this side as [**9, 12, 13**]. Where L^p-solutions of fractional differential equations are discussed in [**9**] by Burton and Zhang to show the belonging of solutions to $L^p(\mathbb{R}_+)$. In [**12**], Schauder's and Darbo's fixed point theorems are employed to study the existence of $L^p(\mathbb{R}_+)-$solutions of nonlinear quadratic integral equations. In [**13**], the authors give different existence results for $L^p[a,b]$ and $C([a,b])-$solutions of some non-linear integral equations of the Hammerstein and Volterra types using some fixed point theorems combined with a general version of Gronwall's inequality. And in [**18**], the authors investigated the existence and uniqueness of weak solutions for a class of initial/boundary-value parabolic problems with nonlinear perturbation term in weighted Sobolev space. By employing the extending Galerkin's method, the authors obtained existence results.

In this paper, motivated by those valuable contributions mentioned above, we mainly discuss the existence and uniqueness of solution for nonlinear FDEs of higher order $\alpha (n-1<\alpha\leqslant n)$ in a measurable weighted fractional Sobolev space using Banach contraction principle. To this end, we first transform the fractional differential equation (1.1) with conditions (1.2) into a equivalent integral equation with Green continuous function by using the technique of Laplace transform of the Riemann-Liouville fractional derivative and some analytical skills, then we present the our study space which is based essentially on the classical concepts of weighted L^p-spaces and Sobolev spaces. Furthermore, we investigate the existence and uniqueness of solution of the system (1.1)-(1.2) by using Banach's fixed point theorem.

The rest of this paper is organized as follows: in section 2 we present some auxiliary definitions and lemmas about fractional calculus theory and measurable functions theory that will be used to prove our main results, also we show the completeness of fractional Sobolev space. Section 3 is devoted to the main result. We present lastly, an illustrate example to show the effectiveness of our main result.

2. Preliminaries

We start by presenting some necessary definitions and lemmas that we will used for investigate our main results. For more details see [**1, 7, 8, 11, 14, 16, 17, 20**].

Definition 2.1 ([**11, 14, 16**]). The Riemann-Liouville fractional integral of the function u of order $\alpha \geqslant 0$ is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-s} s^{\alpha-1} ds$.

Definition 2.2 ([**11, 14, 16**]). The Riemann-Liouville fractional derivative of the function u of order $\alpha \in (n-1, n]$ is defined by

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds.$$

Definition 2.3 ([**17**]). $\Phi : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is called a Carathéodory function if
(i) $t \longmapsto \Phi(t,u)$ is measurable for every $u \in \mathbb{R}$,
(ii) $u \longmapsto \Phi(t,u)$ is continuous almost by all $t \in I$.

Remark 2.1. A first possible definition of solutions of problem (1.1)-(1.2) in the Lebesgue spaces of measurable function $L^p(I)$, is a function $u \in L^p(I)$ which fractional derivative $D^{\beta}u$, $\beta \in (0,1)$ belongs to $L^p(I)$. On the other hand, from definition 1, for some $\beta \in (0,1)$, it is obvious that the Riemann-Liouville fractional derivative of a function u is written in the form: $D^{\beta}u = \left(I^{1-\beta}u\right)'$. That is, if $D^{\beta}u$ exists then the Riemann-Liouville fractional integral $I^{1-\beta}u$ is differentiable almost everywhere. Therefore, we use a more convenient definition of the solutions of (1.1)-(1.2) as the functions $u \in L^p(I)$, $I^{1-\beta}u \in L^p(I)$ and $\left(I^{1-\beta}u\right)' \in L^p(I)$, which takes the structure of a Sobolev space that we denote him by $W_{RL}^{\beta,p}(I)$, defined as follows

$$W_{RL}^{\beta,p}(I) = \left\{u \in L^p(I) \text{ and } I^{1-\beta}u \in W^{1,p}(I)\right\}.$$

Before passing to show the completeness of $W_{RL}^{\beta,p}(I)$, we define the spaces
$\mathfrak{D}'(I)$: space of distributions.
$C_c^1(I)$: space of $C^1(I)$ $-$functions with compact support.

Lemma 2.1. $\left(W_{RL}^{\beta,p}(I), \|u\|_{W_{RL}^{\beta,p}(I)}\right)$ *is a Banach space endowed with the norm*

$$\|u\|_{W_{RL}^{\beta,p}(I)} = \left(\|u\|_p^p + \left\|I^{1-\beta}u\right\|_{W^{1,p}(I)}^p\right)^{\frac{1}{p}}.$$

Proof. It is easy to verify that $\|.\|_{W_{RL}^{\beta,p}(I)}$ defines a norm so we pass to prove the completeness. Let $(u_m) \in W_{RL}^{\beta,p}(I)$ be a Cauchy sequence, this implies that (u_m) and $\left(I^{1-\beta}u_m\right)$ are Cauchy sequences in $L^p(I)$ and $W^{1,p}(I)$ respectively, since $L^p(I)$ and $W^{1,p}(I)$ are completes, there exist functions u and u_{β} such that

$u_m \to u$ in $L^p(I)$ and $I^{1-\beta}u_m \to u_\beta$ in $W^{1,p}(I)$ [*i.e.* $I^{1-\beta}u_m \to u_\beta$ in $L^p(I)$ and $\left(I^{1-\beta}u_m\right)' \to u'_\beta$ in $L^p(I)$].

We have $\left(I^{1-\beta}u_m\right)$ is a Cauchy sequence in $W^{1,p}(I)$, then $\left(I^{1-\beta}u_m\right)$ is a Cauchy sequence in $L^p(I)$, therefore, there exist $v \in L^p(I)$ such that $I^{1-\beta}u_m \to v$ in $L^p(I)$. Beside, we have $u_m \to u$ in $L^p(I)$, then by using the fact $I^{1-\beta} : L^p(I) \to L^p(I)$, $\beta \in (0,1)$, we get $I^{1-\beta}u_m \to I^{1-\beta}u$ in $L^p(I)$, so, $I^{1-\beta}u = v$ and $I^{1-\beta}u = u_\beta$.

It remains to show that $\left(I^{1-\beta}u\right)' = u'_\beta$, where u'_β denotes the first derivatives in distributions sense of u_β. In other term we prove that $\left(I^{1-\beta}u_m\right)' \to \left(I^{1-\beta}u\right)'$ in $L^p(I)$. Clearly, $L^p(I) \subset L^1_{loc}(I)$, then $I^{1-\beta}u_m$ determines a distribution $\check{T}_{I^{1-\beta}u_m} \in \mathfrak{D}'(I)$. For $\Phi \in C^1_c(I)$ and we use Holder inequality we get

$$\begin{aligned} \left|\check{T}_{I^{1-\beta}u_m}(\Phi) - \check{T}_{I^{1-\beta}u}(\Phi)\right| &\leqslant \int_I \left|I^{1-\beta}u_m(t) - I^{1-\beta}u(t)\right| \left|\Phi(t)\right| dt \\ &\leqslant \|\Phi\|_{p'} \left\|I^{1-\beta}u_m - I^{1-\beta}u\right\|_p . \end{aligned}$$

where p' is the exponent conjugate to p. therefore: $\check{T}_{I^{1-\beta}u_m}(\Phi) \to \check{T}_{I^{1-\beta}u}(\Phi)$ as $m \to \infty$.

Also, $\left(I^{1-\beta}u_m\right)'$ determine a distribution $\widehat{T}$, then for $\Phi \in C^1_c(I)$ we have

$$\begin{aligned} \widehat{T}_{(I^{1-\beta}u_m)'}(\Phi) &= \int_I \left(I^{1-\beta}u_m\right)'(t)\,\Phi(t)\,dt \\ &= -\int_I \left(I^{1-\beta}u_m\right)(t)\,\Phi'(t)\,dt = -\widehat{T}_{I^{1-\beta}u_m}(\Phi'), \end{aligned}$$

we pass to the limit when $m \to \infty$, we obtain

$$\widehat{T}_{u'_\beta}(\Phi) = -\widehat{T}_{I^{1-\beta}u}(\Phi') = \widehat{T}_{(I^{1-\beta}u)'}(\Phi),$$

for every $\Phi \in C^1_c(I)$. Thus $u'_\beta = \left(I^{1-\beta}u\right)'$ in the distributional sense on I for $\beta \in (0,1)$.

Consequently, $I^{1-\beta}u \in W^{1,p}(I)$ and $\left(I^{1-\beta}u\right)' = u'_\beta$ in distributional sense. Therefore $I^{1-\beta}u_m \to I^{1-\beta}u$ in $W^{1,p}(I)$. Accordingly, $u_m \to u$ in $W^{\beta,p}_{RL}(I)$, whence $\left(W^{\beta,p}_{RL}(I), \|\cdot\|_{W^{\beta,p}_{RL}(I)}\right)$ is a Banach space. □

Remark 2.2. In [**3**], the authors discussed more broadly about fractional Sobolev space $W^{\beta,p}_{RL}(I)$ in the case where $p = 1$ to make the relation between this spaces and the classical spaces of functions of bounded variation BV. The authors shown also the completeness of the fractional Sobolev spaces $W^{\beta,1}_{RL}(I)$. for more details about Sobolev spaces and their properties see [**11**] and [**5**].

We should note that we can not show the existence of solutions according to Schauder's fixed point theorem in $W^{\beta,p}_{RL}(I)$. To overcome these problem, we can use a more suitable weighted norm.

We define the weighted L^p-space

$$L^{p,\sigma}(I) = \left\{u \in L^p(I),\ \|u\|_{p,\sigma} < +\infty\right\},$$

where, $\|u\|_{p,\sigma}$ is the positive real valued function defined on $L^p(I)$ by

$$\|u\|_{p,\sigma} = \left(\int_I \sigma(t) |u(t)|^p \, dt\right)^{\frac{1}{p}} \text{ for all } u \in L^p(I).$$

Also, we define the weighted fractional Sobolev space with Riemann-Liouville fractional derivative by

$$E_\sigma(I) = \left\{u \in L^{p,\sigma}(I) : I^{1-\beta}u \in W^{1,p,\sigma}(I), \ \beta \in (0,1)\right\},$$

equipped with the norm

$$\|u\|_\sigma = \left(\|u\|_{p,\sigma}^p + \left\|I^{1-\beta}u\right\|_{W^{1,p,\sigma}}^p\right)^{\frac{1}{p}},$$

where

$$W^{1,p,\sigma}(I) = \{v \in L^{p,\sigma}(I) : v' \in L^{p,\sigma}(I)\},$$

σ is a given function defined on I and such that there exists a real number $\sigma_* > 1$ satisfies $1 \leqslant \sigma(t) \leqslant \sigma_*$, for all $t \in I$, and

$$K'(t) \in L^{p,\sigma}(I), \text{ for a.e. } t \in I,$$

where

$$K(t) = \begin{cases} \int_0^t \dfrac{(\sigma(t-s))^{\frac{1}{p}}}{(t-s)^\beta} ds, & t \geqslant s, \\ 0, & t \leqslant s. \end{cases}$$

Clearly

$$\sigma(t-s) \geqslant 1, \text{ for all } t, s \in I \text{ with } t \geqslant s,$$

and $\|.\|_\sigma$ is a norm. Since $1 \leqslant \sigma(t) \leqslant \sigma_*$, then the two norms $\|.\|_{W_{RL}^{\beta,p}(I)}$ and $\|.\|_\sigma$ are equivalent. So, from Lemma 2.1, $(E_\sigma, \|.\|_\sigma)$ is a Banach space.

Definition 2.4. The solutions of the system (1.1)-(1.2) are functions $u \in E_\sigma(I)$ and u satisfies the system (1.1)-(1.2).

Lemma 2.2 ([**17**]). *Let $n-1 \leqslant \alpha < n$ and $q > 0$. The Laplace transform of the Riemann-Liouville fractional derivative $D^\alpha u(t)$ and the power function $t \mapsto t^q$ are given respectively by*

(i) $L\{D^\alpha u(t), z\} = z^\alpha U(z) - \sum_{i=0}^{n-1} z^i \left[D^{\alpha-i-1}u(t)\right]_{t=0},$

(ii) $L\{t^q, z\} = \Gamma(q+1) z^{-(q+1)},$

where $U(z)$ denotes the Laplace transform of $u(t)$.

Lemma 2.3. *Let $n-1 < \alpha \leqslant n$. The unique solution of the linear fractional problem*

$$D^\alpha u(t) = y(t), \ t \in I = [0,T], \ T > 0 \tag{2.1}$$

$$D^{\alpha-i}u|_{t=0} = 0, \ i = 1, ..., n, \ i \neq n-1 \text{ and } u(T) = 0, \tag{2.2}$$

is given by

$$u(t) = \int_0^T G(t,s) y(s) \, ds,$$

where $G(t,s)$ denotes the Green's function defined by
(2.3)
$$G(t,s)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\left[(t-s)^{\alpha-1}-\left(\frac{t}{T}\right)^{\alpha-n+1}(T-s)^{\alpha-1}\right], & 0\leqslant s\leqslant t\leqslant T,\\ \dfrac{1}{\Gamma(\alpha)}\left[-\left(\frac{t}{T}\right)^{\alpha-n+1}(T-s)^{\alpha-1}\right], & 0\leqslant t\leqslant s\leqslant T.\end{cases}$$

PROOF. We take $\left[D^{\alpha-i}u(t)\right]_{t=0}=b_i$. Applying Laplace transform on both side of (2.1) and using Lemma 2.2, we get

$$z^{\alpha}U(z)-\sum_{i=0}^{n-1}z^{i}\left[D^{\alpha-i-1}u(t)\right]_{t=0}=Y(z),$$

where $U(z)$ and $Y(z)$ denote the Laplace transform of $u(t)$ and $y(t)$ respectively. In other words, we can write

$$U(z)=z^{-\alpha}Y(z)+\sum_{i=0}^{n-1}b_{i+1}z^{i-\alpha}.$$

Inverse Laplace transform give us

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+\sum_{i=0}^{n-1}\frac{b_{i+1}}{\Gamma(\alpha-i)}t^{\alpha-i-1}\\&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+\sum_{i=1}^{n}\frac{b_i}{\Gamma(\alpha-i+1)}t^{\alpha-i},\end{aligned}$$

we have $b_i=0$, $i=1,...,n$ for $i\neq n-1$ then

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+\frac{b_{n-1}}{\Gamma(\alpha-n+2)}t^{\alpha-n+1}. \tag{2.4}$$

By condition $u(T)=0$ we obtain

$$\frac{b_{n-1}}{\Gamma(\alpha-n+2)}=\frac{-T^{n-\alpha-1}}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}y(s)\,ds,$$

substituting in (2.4), we get

$$u(t)=\int_0^T G(t,s)\,y(s)\,ds,$$

where $G(.,.)$ the Green's kernel defined by (2.3). The proof is complete. □

Define the integro-differential operator $B:E_\sigma(I)\to E_\sigma(I)$ by

$$(Bu)(t)=\int_0^T G(t,s)\,g\left(s,u(s),D^{\beta}u(s)\right)ds. \tag{2.5}$$

Obviously, all fixed point of B is a solution of system (1.1)-(1.2).

We give in the following, Banach's fixed point theorem which is the main ingredient in the proof of our existence results.

THEOREM 2.1 (Banach contraction principle [**20**]). *Let E be a Banach space. If $B : E \to E$ is a contraction, then B has a unique fixed point in E.*

3. Main results

In this section, we prove the existence and uniqueness of solutions in the Banach space $E_\sigma(I)$.

THEOREM 3.1. *Assume the following hypotheses on g*

(H_1) $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory's condition.

(H_2) There exist a positive real number $p' \geqslant 1$ and a function $\varphi : I \to \mathbb{R}_+$ and such that

(i) $\varphi \in L^{p'}(I)$ a.e. $t \in I$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) For any $s \in I$ and any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$, we have

$$|g(s,u,v) - g(s,\overline{u},\overline{v})| \leqslant \varphi(s)\left[|u-\overline{u}| + |v-\overline{v}|\right].$$

(H_3) The condition

$$G_{**}\left\|\varphi\right\|_{p'}\left[(T\sigma_*)^{\frac{1}{p}} + \frac{T^{1-\beta}(T\sigma_*)^{\frac{1}{p}}}{\Gamma(2-\beta)} + \frac{\|K'\|_{p,\sigma}}{\Gamma(1-\beta)}\right] < 1,$$

holds, where $G_{**} = \max\limits_{(t,s)\in I^2} |G(t,s)|$.

Then the system (1.1)-(1.2) has a unique solution in E_σ.

PROOF. Consider the operator B given by (2.5), we want to show that B is a contraction on $E_\sigma(I)$. To this end, let $u, v \in E_\sigma$ and using (H_1) and (H_2), then for a.e. $t \in I$ we have

$$\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|(Bu)(t) - (Bv)(t)\right| \\
&\leqslant \sigma(t)^{\frac{1}{p}} \int_0^T \frac{|G(t,s)|}{\sigma(s)^{\frac{1}{p}}}\left[\sigma(s)^{\frac{1}{p}}\left|g\left(s,u(s),D^\beta u(s)\right) - g\left(s,v(s),D^\beta v(s)\right)\right|\right] ds \\
&\leqslant G_{**}\sigma(t)^{\frac{1}{p}} \int_0^T \varphi(s)\left[(\sigma(s))^{\frac{1}{p}}\left(|u(s)-v(s)| + \left|D^\beta u(s) - D^\beta v(s)\right|\right)\right] ds \\
&\leqslant G_{**}\sigma_*^{\frac{1}{p}}\left\|\varphi\right\|_{p'}\left\|\sigma^{\frac{1}{p}}\left(|u-v| + \left|D^\beta u - D^\beta v\right|\right)\right\|_p \\
&\leqslant G_{**}\sigma_*^{\frac{1}{p}}\left\|\varphi\right\|_{p'}\left[\|u-v\|_{p,\sigma} + \left\|D^\beta u - D^\beta v\right\|_{p,\sigma}\right] \\
&\leqslant G_{**}\sigma_*^{\frac{1}{p}}\left\|\varphi\right\|_{p'}\left[\|u-v\|_{p,\sigma} + \left\|\left(I^{1-\beta}u\right)' - \left(I^{1-\beta}v\right)'\right\|_{p,\sigma}\right] \\
&\leqslant G_{**}\sigma_*^{\frac{1}{p}}\left\|\varphi\right\|_{p'}\|u-v\|_\sigma,
\end{aligned}$$

applying L^p−norm, we get

$$\|Bu - Bv\|_{p,\sigma} \leqslant G_{**}(T\sigma_*)^{\frac{1}{p}}\left\|\varphi\right\|_{p'}\|u-v\|_\sigma. \tag{3.1}$$

Also

$$
\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|I^{1-\beta}(Bu)(t)-I^{1-\beta}(Bv)(t)\right| \\
&\leqslant \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\int_0^T \frac{|G(s,\theta)|}{\sigma(\theta)^{\frac{1}{p}}}\left[\sigma(\theta)^{\frac{1}{p}}\left|g\left(\theta,u(\theta),D^{\beta}u(\theta)\right)\right.\right. \\
&\left.\left.-g\left(\theta,v(\theta),D^{\beta}v(\theta)\right)\right|\right] d\theta ds \\
&\leqslant \frac{G_{**}\sigma(t)^{\frac{1}{p}}}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\int_0^T \varphi(\theta)\left[\sigma(\theta)^{\frac{1}{p}}\left(|u(\theta)-v(\theta)|\right.\right. \\
&\left.\left.+\left|D^{\beta}u(\theta)-D^{\beta}v(\theta)\right|\right)\right] d\theta ds \\
&\leqslant \frac{G_{**}\sigma_*^{\frac{1}{p}}}{\Gamma(1-\beta)}\left(\int_0^t (t-s)^{-\beta}ds\right)\|\varphi\|_{p'}\left\|\sigma^{\frac{1}{p}}\left(|u-v|+\left|D^{\beta}u-D^{\beta}v\right|\right)\right\|_p \\
&\leqslant \frac{G_{**}\sigma_*^{\frac{1}{p}}}{\Gamma(2-\beta)}T^{1-\beta}\|\varphi\|_{p'}\left[\|u-v\|_{p,\sigma}+\left\|\left(I^{1-\beta}u\right)'-\left(I^{1-\beta}v\right)'\right\|_{p,\sigma}\right],
\end{aligned}
$$

then

$$
\left\|I^{1-\beta}Bu-I^{1-\beta}Bv\right\|_{p,\sigma} \leqslant \frac{T^{1-\beta}G_{**}(T\sigma_*)^{\frac{1}{p}}}{\Gamma(2-\beta)}\|\varphi\|_{p'}\|u-v\|_{\sigma}. \tag{3.2}
$$

Moreover

$$
\begin{aligned}
&\sigma(t)^{\frac{1}{p}}\left|\left(I^{1-\beta}Bu\right)'(t)-\left(I^{1-\beta}Bv\right)'(t)\right| \\
&\leqslant \frac{\sigma(t)^{\frac{1}{p}}}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-s)^{-\beta}\int_0^T \frac{|G(s,\theta)|}{\sigma(\theta)^{\frac{1}{p}}}\left[\sigma(\theta)^{\frac{1}{p}}\left|g\left(\theta,u(\theta),D^{\beta}u(\theta)\right)\right.\right. \\
&\left.\left.+g\left(\theta,v(\theta),D^{\beta}v(\theta)\right)\right|\right] d\theta ds \\
&\leqslant \frac{G_{**}\sigma(t)^{\frac{1}{p}}}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-s)^{-\beta}\int_0^T \varphi(\theta)\left[(\sigma(\theta))^{\frac{1}{p}}\left(|u(\theta)-v(\theta)|\right.\right. \\
&\left.\left.+\left|D^{\beta}u(\theta)-D^{\beta}v(\theta)\right|\right)\right] d\theta ds \\
&\leqslant \frac{G_{**}}{\Gamma(1-\beta)}\left[\sigma(t)^{\frac{1}{p}}\frac{d}{dt}\int_0^t \frac{(\sigma(t-s))^{\frac{1}{p}}}{(t-s)^{\beta}}ds\right]\|\varphi\|_{p'} \\
&\times\left\|\sigma^{\frac{1}{p}}\left(|u-v|+\left|D^{\beta}u-D^{\beta}v\right|\right)\right\|_p,
\end{aligned}
$$

using some precedent method and applying L^p-norm on both sides of previous inequality, we get

$$
\left\|\left(I^{1-\beta}Bu\right)'-\left(I^{1-\beta}Bv\right)'\right\|_{p,\sigma} \leqslant \frac{G_{**}}{\Gamma(1-\beta)}\|K'\|_{p,\sigma}\|\varphi\|_{p'}\|u-v\|_{\sigma}. \tag{3.3}
$$

Combining inequalities (3.1)-(3.3) then we obtain

$$
\|Bu-Bv\|_{\sigma} \leqslant G_{**}\|\varphi\|_{p'}\left[(T\sigma_*)^{\frac{1}{p}}+\frac{T^{1-\beta}(T\sigma_*)^{\frac{1}{p}}}{\Gamma(2-\beta)}+\frac{\|K\|_{p,\sigma}}{\Gamma(1-\beta)}\right]\|u-v\|_{\sigma}, \tag{3.4}
$$

this means that the operator is a contraction from condition (H_3). Hence, by using Banach contraction principle and according to the theorem 3.1, we conclude that B has a unique fixed point in E_σ. This fixed point is a solution of system (1.1)-(1.2). □

4. Example

Consider the following boundary value problem of fractional differential equations with $p = 4$

$$\left\{ \begin{array}{ll} D^{\alpha}u(t) = \overline{\varphi}(t)\left[e^{-1}\sin(tu) + t^2 h\left(D^{\beta}u\right)\right], & t \in I = [0,1], \\ D^{(\alpha-i)}u|_{t=0} = 0, \ i = 1,2,3,5, & u(1) = 0, \end{array} \right. \tag{4.1}$$

where $\alpha = \frac{9}{2}$, $\beta = \frac{1}{6}$, $g(t,x,y) = \overline{\varphi}(t)\left[e^{-1}\sin(tu) + t^2 h\left(D^{\beta}u\right)\right]$ with $h(x) = e^{-e^{-x}}$, $\overline{\varphi}(t) = \dfrac{1}{(9t)^2(1+t)}$. By the finite increments theorem we get

$$|h(x) - h(y)| \leqslant e^{-1}|x-y|,$$

for $x, y \in \mathbb{R}$ (since $z + e^{-z} \geqslant 1$ for all real z), also

$$|\sin(tx) - \sin(ty)| \leqslant t^2|x-y|,$$

then

$$\begin{array}{rcl} |g(t,x,y) - g(t,\overline{x},\overline{y})| & \leqslant & \overline{\varphi}(t)\left[e^{-1}|\sin(tx) - \sin(t\overline{x})| + t^2|h(y) - h(\overline{y})|\right] \\ & \leqslant & \varphi(t)\left[|x-\overline{x}| + |y-\overline{y}|\right], \end{array}$$

so, condition (H_3) holds with $\varphi(t) = \frac{e^{-1}}{9^2(1+t)}$, obviously $\varphi \in L^{\frac{3}{4}}([0,1])$ and

$$\|\varphi\|_{3/4} = 0.0363,$$

$\sigma(t) = (1+t)^4$, it is clear that $\sigma(t) \geqslant 1$ for $t \in [0,1]$, and the Banach space is

$$E_{\sigma}^{*}(I) = \left\{u \in L^{4,\sigma}(I) : I^{\frac{5}{6}}u \in W^{1,4,\sigma}(I)\right\},$$

also

$$K(t) = \int_0^t \frac{(\sigma(t-s))^{\frac{1}{p}}}{(t-s)^{\beta}}ds = \int_0^t \frac{1+z}{z^{\frac{1}{6}}}dz = \frac{6t^{\frac{5}{6}}(5t+11)}{55},$$

and

$$K'(t) = (t^{\frac{1}{6}} + t^{-\frac{1}{6}}),$$

then, some computations give us

$$\|K'\|_{4,\sigma} \simeq 3.187991075720807,$$

and

$$G_{**}\|\varphi\|_{3/4}\left[(T\sigma_*)^{\frac{1}{p}} + \frac{T^{1-\beta}(T\sigma_*)^{\frac{1}{p}}}{\Gamma(2-\beta)} + \frac{\|K'\|_{p,\sigma}}{\Gamma(1-\beta)}\right] \simeq 0.5046 < 1,$$

this means that condition (H_3) is also holds. So, using Theorem 3.1, we deduce that the nonlinear functional boundary value problem 4.1 has a unique solution in $E_{\sigma}^{*}(I) \subset L^{4,\sigma}(I)$.

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