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AMICABLE SETS IN ALMOST LATTICES

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ABSTRACT. The concepts of maximal set and amicable set are introduced in an Almost Lattice (AL) and proved certain properties of these concepts.Proved that every maximal set in an AL L is embedded in an amicable set in an AL L. Also, proved that every amicable set in an AL L can be embedded in a maximal set with uni-element.

1. Introduction

After Boole's axiomatisation of the two valued propositional calculus into Boolean algebra, many generalizations of the Boolean algebras have come into being. The class of distributive lattices has occupied in major part of the present lattice theory, since lattices were abstracted from Boolean algebras through the class of distributive lattices and these classes have many interesting properties in which lattices, in general, do not have. For this reason, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy U.M. and Rao G.C. [3], as a common abstraction of existing lattice theoretic and ring theoretic generalizations of Boolean algebra. It was Garett Birkhoff's (1911 - 1996) work in the mid thirties that started the general development of the lattice theory. In a brilliant series of papers, he demonstrated the importance of the lattice theory and showed that it provides a unified frame work for unrelated developments in many mathematical disciplines. V. Glivenko, Karl Menger, John Van Neumann, Oystein Ore, George Gratzer, P. R. Halmos, E. T. Schmidt, G. Szasz, M. H. Stone , R. P. Dilworth and many others have developed enough of this field for making it attractive to the mathematicians and for its further progress. The traditional approach to lattice theory proceeds from partially

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ordered sets to general lattices, semimodular lattices, modular lattices and finally to distributive lattices. The concept of Almost Lattice (AL) was introduced by G. Nanaji Rao and Habtamu Tiruneh Alemu [1] as a common abstraction of almost all lattice theoretic generalizations of Boolean algebra like distributive lattices, almost distributive lattices and lattices.

In this paper, we introduced the concepts of compatible set, maximal set M, M-amicable element and amicable set in an Almost Lattice (AL) and proved that any maximal set in an AL L is a lattice with respect to the induced operations. We proved that if M is a maximal set and $x \in L$ is M-amicable, then there is a smallest element $a \in M$ with the property $a \wedge x = x$ and this element a is denoted by x^M. Also, we proved that for maximal sets M, the set $A_M(L)$, of all M-amicable elements of an AL L is again an AL under the induced operations. Moreover, for any $x, y \in A_M(L)$, we have proved that

$$(x \wedge y)^M = x^M \wedge y^M \text{ and } (x \vee y)^M = x^M \vee y^M.$$

Also, we introduced the concept of a uni-element in a maximal set of an AL L and proved that if M is a maximal set in an AL L with uni-element v, then v is a maximal element of L and $M = \{x \wedge v | \; x \in L\}$. Also, we proved that if an AL L has maximal element, then a maximal set M in an AL L is amicable if and only if M has a uni-element. Finally, we proved that if an AL L has maximal element, then every amicable set can be embedded in a maximal set with uni-element.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. Let $(P, \leqslant)$ be a poset and $a \in P$. Then

(1) a is called the least element of P if $a \leqslant x$ for all $x \in P$.
(2) a is called the greatest element of P if $x \leqslant a$ for all $x \in P$.

It can be easily observed that, if least (greatest) element exists in a poset, then it is unique.

Definition 2.2. Let $(P, \leqslant)$ be a poset and $a \in P$. Then

(1) a is called a minimal element, if $x \in P$ and $x \leqslant a$, then $x = a$.
(2) a is called maximal element, if $x \in P$ and $a \leqslant x$, then $a = x$.

It can be easily verified that least (greatest) element (if exists), then it is minimal (maximal) but, converse need not be true.

Definition 2.3. Let $(P, \leqslant)$ be a poset and S be a non empty subset of P. Then

(1) An element a in P is called a lower bound of S if $a \leqslant x$ for all $x \in S$.
(2) An element a in P is called an upper bound of S if $x \leqslant a$ for all $x \in S$.
(3) An element a in P is called the greatest lower bound (g.l.b or infimum) of S if a is a lower bound of S and $b \in P$ such that b is a lower bound of S, then $b \leqslant a$.

(4) An element a in P ia called the least upper bound (l.u.b or supremum) of S if a is an upper bound of S and $b \in P$ such that b is an upper bound of S, then $a \leqslant b$.

DEFINITION 2.4. (Zorn's Lemma) If every sub chain of a non empty partly ordered set P has an upper bound in P, then P contains a maximal element.

DEFINITION 2.5. Let $(P, \leqslant)$ be a poset. If P has least element 0 and greatest element 1, then P is said to be a bounded poset.

If $(P, \leqslant)$ is a bounded poset with bounds $0, 1$, then for any $x \in P$, we have $0 \leqslant x \leqslant 1$.

DEFINITION 2.6. An algebra $(L, \vee, \wedge)$ of type (2, 2) is called an Almost Lattice(AL) if it satisfies the following axioms. For any $a, b, c \in L$:

A_1. $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
A_2. $(a \vee b) \wedge c = (b \vee a) \wedge c$
A_3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
A_4. $(a \vee b) \vee c = a \vee (b \vee c)$
A_5. $a \wedge (a \vee b) = a$
A_6. $a \vee (a \wedge b) = a$
A_7 $(a \wedge b) \vee b = b$

LEMMA 2.1. *Let L be an AL. Then for any $a, b \in L$ we have the following:*

(1) $a \vee a = a$
(2) $a \wedge a = a$
(3) $a \wedge b = a$ *if and only if* $a \vee b = b$

DEFINITION 2.7. For any $a, b \in L$ in an AL L, we say that a is less than or equal to b and write as $a \leqslant b$ if and only if $a \wedge b = a$ or, equivalently $a \vee b = b$.

THEOREM 2.1. *Let L be an AL such that $a, b, c \in L$. Then we have the following.*

(1) *The relation $\leqslant$ is a partial ordering on L and hence $(L, \leqslant)$ is a poset.*
(2) $a \leqslant b \Longrightarrow a \wedge b = b \wedge a$
(3) $a \leqslant a \vee b$
(4) $a \wedge b \leqslant b$
(5) $(a \vee b) \wedge a = a$
(6) $(a \vee b) \wedge b = b$
(7) $b \vee (a \wedge b) = b$
(8) $a \wedge b = b \Longleftrightarrow a \vee b = a$
(9) $a \leqslant b \Longrightarrow a \vee b = b \vee a$
(10) $a \vee b = b \vee a \Longrightarrow a \wedge b = b \wedge a$
(11) *If $a \leqslant c$ and $b \leqslant c$, then $a \wedge b \leqslant c$ and $a \vee b \leqslant c$*
(12) $(a \vee b) \vee b = a \vee b$
(13) $(a \vee b) \vee a = a \vee b$
(14) $a \vee (a \vee b) = a \vee b$
(15) $a \wedge (a \wedge b) = a \wedge b$

(16) $(a \wedge b) \wedge b = a \wedge b$
(17) $b \wedge (a \wedge b) = a \wedge b$
(18) $a \vee b = a \vee (b \vee a)$.

Definition 2.8. An AL L is said to be directed above if for any $a, b \in L$ there exists $c \in L$ such that $a, b \leqslant c$.

Theorem 2.2. *Let L be an AL. Then the following are equivalent:*

(1) *L is directed above.*
(2) *$\wedge$ is commutative.*
(3) *$\vee$ is commutative.*
(4) *L is a lattice.*

Definition 2.9. Let L be an AL. Then for any $a, b \in L$, we say that a is compatible with b, written as $a \sim b$ if and only if $a \wedge b = b \wedge a$ or, equivalently, $a \vee b = b \vee a$.

Proposition 2.1. *Let L be an AL such that $a, b, c \in L$. Then we have the following.*

(1) $a \sim b \Longleftrightarrow a \wedge b \sim b \wedge a$.
(2) $a \sim b \Longleftrightarrow a \vee b \sim b \vee a$.
(3) $a \sim b$ *and* $a \sim c \Longrightarrow a \sim b \wedge c$.

Definition 2.10. An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an AL with 0 if it satisfying the following axioms. For any $a, b, c \in L$:

(A_1) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
(A_2) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(A_3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
(A_4) $(a \vee b) \vee c = a \vee (b \vee c)$
(A_5) $a \wedge (a \vee b) = a$
(A_6) $a \vee (a \wedge b) = a$
(A_7) $(a \wedge b) \vee b = b$
(0_1) $0 \wedge a = 0$

Lemma 2.2. *Let L be an AL with 0. Then for any $a, b \in L$, we have the following:*

(1) $a \wedge 0 = 0$.
(2) $a \vee 0 = a$.
(3) $0 \vee a = a$
(4) $a \wedge b = 0 \Longleftrightarrow b \wedge a = 0$.
(5) $a \wedge b = b \wedge a$ *whenever* $a \wedge b = 0$

Definition 2.11. Let L be an AL. Then an element $a \in L$ is maximal if for any $x \in L$, $a \leqslant x$ implies $a = x$.

Proposition 2.2. *Let L be an AL. Then for any $m \in L$, the following are equivalent:*

(1) *m is maximal.*

(2) $m \vee x = m, \ \forall x \in L$.
(3) $m \wedge x = x, \ \forall x \in L$.

Corollary 2.1. *L is discrete AL if and only if every element of L is maximal.*

3. Amicable Sets

If $(S, .)$ is a P_1- semigroup, then we have seen that S is an AL and the Birkhoff center $B(S)$ of S has the following property; given $x \in S$ there exists an element $x^0 \in B(S)$ which is the least among a of B(S) such that $a \wedge x = ax = x$. In this section, we introduce the concepts of compatible set, maximal set and M-amicable element and prove certain properties of these concepts. Also, we prove that if M is a maximal set in an AL L and $x \in L$ is an M-amicable element, then there exists a smallest element x^M in M such that $x^M \wedge x = x$. We prove that the set $A_M(L)$ of all M-amicable elements in an AL L is again an AL with respect to induced operations. We introduce the concepts of an amicable set in an AL L and prove that every maximal set in an AL L is embedded in an amicable set.

Recall that for any a, b in an AL L, we say a is compatible with b, write as $a \sim b$ if and only if $a \wedge b = b \wedge a$ or, equivalently, $a \vee b = b \vee a$. First we begin with the following definition:

Definition 3.1. A subset S of an Almost Lattice (AL) L is said to be compatible if $a \sim b$ for all $a, b \in S$.

It can be easily seen that for any a in an AL, $\{a\}$ is a compatible set of L and also seen that the set $\mathcal{F}$ of all compatible sets of L is a poset with respect to the set inclusion. Now, we introduce the following definition.

Definition 3.2. Let L be an AL. Then a maximal set in L is a maximal element in the poset $(\mathcal{F}, \subseteq)$.

In the following we prove certain basic properties of compatible sets.

Lemma 3.1. *Let M be a maximal set in an AL L and $x \in L$ such that $x \sim a$ for all $a \in M$. Then $x \in M$.*

Proof. Let M be a maximal set in an AL L and $x \in L$ such that $x \sim a$ for all $a \in M$. Then clearly, $M \cup \{x\}$ is a compatible and $M \subseteq M \cup \{x\}$. Hence, by maximality of M, we get $M = M \cup \{x\}$. Therefore $x \in M$. □

Theorem 3.1. *Let M is a maximal set in an AL L. Then M is a lattice with respect to induced operations.*

Proof. Let M be a maximal set. It is sufficient to show that M is closed under $\vee$ and $\wedge$. Let $a, b \in M$. Now, for any $x \in M$ consider, $x \wedge (a \wedge b) = (x \wedge a) \wedge b = (a \wedge x) \wedge b = a \wedge (x \wedge b) = a \wedge (b \wedge x) = (a \wedge b) \wedge x$. Hence $a \wedge b \in M$ by lemma 3.1. Similarly we can prove that $a \vee b \in M$. Therefore M is a lattice. □

Now, we have the following corollary whose proof follows by theorem 3.1.

Corollary 3.1. *Let L be an AL. Then the following are equivalent:*

(1) *L is a lattice.*
(2) *L is compatible set.*
(3) *L is maximal set.*

Proposition 3.1. *Let M be a maximal set of an AL L and $a \in M$. Then for any $x \in L$, $x \wedge a \in M$.*

Proof. Suppose M is a maximal set such that $a \in M$ *and* $x \in L$. Then for any $b \in M$ consider, $(x \wedge a) \wedge b = x \wedge (a \wedge b) = x \wedge (b \wedge a) = (x \wedge b) \wedge a = (b \wedge x) \wedge a = b \wedge (x \wedge a)$. Hence by lemma 3.1, we get $x \wedge a \in M$. □

Corollary 3.2. *Let M be a maximal set in an AL L. Then M is an initial segment in the poset $(L, \leqslant)$ (i.e. for any $x \in L$ and $a \in M, x \leqslant a$ implies, $x \in M$).*

Proof. Suppose that $x \in L$ and $a \in M$ such that $x \leqslant a$. Then $x = x \wedge a$. Hence by proposition 3.1, we have $x = x \wedge a \in M$. □

In the following we introduce the concept of an M-amicable element in an AL L.

Definition 3.3. Let M be a maximal set in an AL L. Then an element $x \in L$ is said to be M-amicable if there exists $a \in M$ such that $a \wedge x = x$.

It can be easily observed that every element in discrete AL, is M-amicable. In the following we prove that if M is maximal in an AL L and $x \in L$ is M-amicable, then there exists a smallest element $a \in M$ with the property that $a \wedge x = x$. For this first we need the following lemma.

Lemma 3.2. *Let M be a maximal set in an AL L and $x \in L$ be M- amicable. Then there exists $a \in M$ with the following properties:*

(1) $a \wedge x = x$
(2) *If $b \in L$ such that $b \wedge x = x$, then $b \wedge a = a$.*

Proof. (1):- Suppose M be a maximal set in an AL L and $x \in L$ is M-amicable. Then there exists an element $c \in M$ such that $c \wedge x = x$. Now, put $a = x \wedge c$. Then by proposition 3.1, we get $a \in M$. Now, $a \wedge x = (x \wedge c) \wedge x = (c \wedge x) \wedge x = c \wedge (x \wedge x) = c \wedge x = x$. Therefore $a \wedge x = x$
(2):- Let $b \in L$ such that $b \wedge x = x$. Then $b \wedge a = b \wedge (x \wedge c) = (b \wedge x) \wedge c = x \wedge c = a$. Thus $b \wedge a = a$. □

Note that if $b \in M$ in the above lemma, then $a = b \wedge a = a \wedge b$ so that $a \leqslant b$ and hence we have the following:

Theorem 3.2. *Let M be a maximal set in an AL L and $x \in L$ be M-amicable. Then there is a smallest element $a \in M$ with the property $a \wedge x = x$.*

Proof. Suppose M be a maximal set in an AL L and $x \in L$ be M-amicable. Then there exists $a \in M$ such that $a \wedge x = x$. It is enough to prove a is the smallest element of M with the property $a \wedge x = x$. Suppose $b \in M$ such that $b \wedge x = x$. Then by condition (2) of lemma 3.2, we get $b \wedge a = a$ and hence $a \wedge b = a$. It implies $a \leqslant b$. Therefore a is the smallest element of M with the property $a \wedge x = x$. □

Note that such smallest element a in M is denoted by x^M and observe that x^M depends on M as well as on x.

Corollary 3.3. *Let M be a maximal set in an AL L and* $x \in L$. *Then x is M-amicable and* $x = x^M$ *if and only if* $x \in M$.

Proof. Suppose for any $x \in L, x$ is M-amicable and $x = x^M$. Then $x = x^M \in M$. Hence $x \in M$. Conversely, suppose $x \in M$. Then clearly x is M-amicable. Now, by theorem 3.10, there exists a smallest element $x^M \in M$ such that $x^M \wedge x = x$. Also, we have $x \wedge x = x$. It follows that, $x^M \leqslant x$. Now, $x \wedge x^M = x^M \wedge x = x$, since $x, x^M \in M$ and hence $x \leqslant x^M$. Therefore $x = x^M$. □

Corollary 3.4. *Let M be a maximal set in an AL L and* $x \in L$ *is M-amicable. Let* $a \in L$ *such that* $x \wedge a = a$. *Then a is M-amicable and* $a^M \leqslant x^M$.

Proof. Suppose $x \in L$ is M-amicable and $a \in L$ with the property $x \wedge a = a$. Then by theorem 3.2, there exists a smallest element $x^M \in M$ such that $x^M \wedge x = x$. Now, consider $x^M \wedge a = x^M \wedge (x \wedge a) = (x^M \wedge x) \wedge a = x \wedge a = a$. Hence a is M-amicable. Therefore by theorem 3.2 there exists smallest element a^M of M with the property that $a^M \wedge a = a$. It follows that $a^M \leqslant x^M$. □

Corollary 3.5. *Let M be a maximal set in an AL L and* $x \in M$. *Then* x^M *is the largest element of M with the property* $x \wedge x^M = x^M$.

Proof. Let M be a maximal set in an AL L and $x \in M$. Then by corollary 3.3, we have $x = x^M$ and hence $x \wedge x^M = x^M$. Now, suppose $b \in M$ such that $x \wedge b = b$. Then $b \leqslant x$ and hence $b \leqslant x^M$. Therefore x^M is the largest element of M with the property $x \wedge x^M = x^M$. □

Corollary 3.6. *Let M be a maximal set in an AL L and* $x \in L$ *be M-amicable. Then for any* $a \in L$, $a \wedge x = x$ *and* $x \wedge a = a$ *if and only if a is M-amicable and* $x^M = a^M$.

Proof. Let M be a maximal set in an AL L and $x \in L$ be M-amicable. Suppose for any $a \in L$, $a \wedge x = x$ *and* $x \wedge a = a$. Then there exists $x^M \in M$ such that $x^M \wedge x = x$. Now, $x^M \wedge a = x^M \wedge (x \wedge a) = (x^M \wedge x) \wedge a = x \wedge a = a$. Hence a is M-amicable. Therefore there exists a smallest element $a^M \in M$ with the property that $a^M \wedge a = a$. Hence $a^M \leqslant x^M$. Also, $a^M \wedge x = a^M \wedge (a \wedge x) = (a^M \wedge a) \wedge x = a \wedge x = x$. It follows that, $x^M \leqslant a^M$. Therefore $x^M = a^M$. Conversely, suppose that a is M-amicable and $a^M = x^M$. Then $a \wedge x = a \wedge (x^M \wedge x) = a \wedge (a^M \wedge x) = (a \wedge a^M) \wedge x = a^M \wedge x = x^M \wedge x = x$. Similarly, we can prove that $x \wedge a = a$. □

Corollary 3.7. *Let M be a maximal set in an AL L and* $x \in L$ *be M-amicable. Then* x^M *is the unique element of M such that* $x^M \wedge x = x$ *and* $x \wedge x^M = x^M$.

Proof. Suppose M is a maximal set and $x \in M$ is M-amicable. Since $x^M \in M$, by corollary 3.11, we have x^M is M-amicable and $(x^M)^M = x^M$. Put $a = x^M$. Then $a^M = (x^M)^M = a$. Therefore $a^M = a = x^M$. Then by corollary 3.6, $a \wedge x = x$ *and* $x \wedge a = a$ and hence $x^M \wedge x = x$ *and* $x \wedge x^M = x^M$. Now, we prove x^M is unique element of M satisfying the given condition. Suppose $b \in M$

such that $b \wedge x = x$ *and* $x \wedge b = b$. Then by corollary 3.6 and corollary 3.3, we have $b = b^M = x^M$. Therefore x^M is a unique element of M satisfying the given condition, $x^M \wedge x = x$ *and* $x \wedge x^M = x^M$. □

If M is a maximal set in an AL L, then we denote the set of all M-amicable elements of L by $A_M(L)$. Now, we prove that $A_M(L)$ is an AL with the induced operation on L.

THEOREM 3.3. *Let M be a maximal set in an AL L. Then $A_M(L)$ is an AL with the induced operations on L. More over for any $x, y \in L$ we have $(x \wedge y)^M = x^M \wedge y^M$ and $(x \vee y)^M = x^M \vee y^M$.*

PROOF. Let M be a maximal set in an AL L. Then clearly we have $A_M(L) \subseteq L$. Now, it is suffice to prove that $A_M(L)$ is closed over $\vee$ *and* $\wedge$. Let $x, y \in A_M(L)$. Then we have $x^M, y^M \in M$ such that $x^M \wedge x = x$ *and* $y^M \wedge y = y$. Now, for any $t \in M$ consider, $(x^M \wedge y^M) \wedge t = x^M \wedge (y^M \wedge t) = x^M \wedge (t \wedge y^M) = (x^M \wedge t) \wedge y^M = (t \wedge x^M) \wedge y^M = t \wedge (x^M \wedge y^M)$. Also, $(x^M \vee y^M) \vee t = x^M \vee (y^M \vee t) = x^M \vee (t \vee y^M) = (x^M \vee t) \vee y^M = (t \vee x^M) \vee y^M = t \vee (x^M \vee y^M)$. Therefore $x^M \vee y^M, x^M \wedge y^M \in M$. Now, we have $(x^M \vee y^M) \vee (x \vee y) = ((x^M \vee y^M) \vee (x \vee y)) \wedge ((x^M \vee y^M) \vee (x \vee y)) = ((x^M \vee x) \vee (y^M \vee y)) \wedge ((x^M \vee y^M) \vee (x \vee y)) = (x^M \vee y^M) \wedge ((x^M \vee y^M) \vee (x \vee y)) = x^M \vee y^M$. Hence $(x^M \vee y^M) \wedge (x \vee y) = x \vee y$. Therefore $x \vee y \in A_M(L)$. Also, $(x^M \wedge y^M) \wedge (x \wedge y) = (x^M \wedge x) \wedge (y^M \wedge y) = x \wedge y$. Hence $x \wedge y \in A_M(L)$. Therefore $A_M(L)$ is closed under the operations $\vee$ and $\wedge$ on L and hence $(A_M(L), \vee, \wedge)$ is an AL. It remains to show that $(x \vee y)^M = x^M \vee y^M$ and $(x \wedge y)^M = x^M \wedge y^M$. Now, we have $(x \vee y) \vee (x^M \vee y^M) = ((x \vee y) \vee (x^M \vee y^M)) \wedge ((x \vee y) \vee (x^M \vee y^M)) = (x \vee y) \wedge ((x \vee y) \vee (x^M \vee y^M)) = x \vee y$. Hence $(x \vee y) \wedge (x^M \vee y^M) = x^M \vee y^M$. Similarly, $(x^M \vee y^M) \wedge (x \wedge y) = x \wedge y$. Hence by corollary 3.6 and 3.7, we get $(x \vee y)^M = x^M \vee y^M$. Also, $(x \wedge y) \wedge (x^M \wedge y^M) = x^M \wedge y^M$ *and* $(x^M \wedge y^M) \wedge (x \wedge y) = x \wedge y$. Hence by corollary 3.6 and 3.7, we get $(x \wedge y)^M = x^M \wedge y^M$. □

PROPOSITION 3.2. *Let M be a maximal set in an AL L and $x, y \in L$ be M-amicable such that $x \sim y$. Then $x^M = y^M$ if and only if $x = y$.*

PROOF. Let $x, y \in L$ be M-amicable and $x \sim y$. Then $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$. Now, suppose $x^M = y^M$. Consider, $x = x^M \wedge x = y^M \wedge x = y \wedge y^M \wedge x = y^M \wedge y \wedge x = y \wedge x = x \wedge y$. Hence $x \leqslant y$. Similarly, we can prove that $y \leqslant x$. Therefore $x = y$. Conversely, suppose $x = y$. Since $x, y \in L$ is M-amicable and $x \sim y, x \wedge y = y \wedge y = y$ *and* $y \wedge x = x \wedge x = x$. Then by corollary 3.16, we get $x^M = y^M$. □

It can be easily seen that every element in a maximal set M is M-amicable. Hence we get $M \subseteq A_M(L) \subseteq L$. Now, we prove the following theorem.

THEOREM 3.4. *Let M be a maximal set in an AL L. Then the following are equivalent:*

(1) $M = A_M(L)$
(2) $M = L$
(3) *L is a lattice.*

PROOF. (1) $\Longrightarrow$ (2). Assume (1). Clearly, $M \subseteq L$. Let $x \in L$ and $t \in M = A_M(L)$. Then $x \wedge t \in M$. Now, $(x \wedge t) \wedge (t \wedge x) = (x \wedge t) \wedge x = (t \wedge x) \wedge x = t \wedge x$. Hence $t \wedge x \in A_M(L)$. Then $x \wedge t \sim t \wedge x$. It follows that, $x \sim t$ and hence $x \in M$ (by lemma 3.1). Therefore $L \subseteq M$. Thus $M = L$.

Proof of (2) $\Longrightarrow$ (3) is clear since M is a lattice.

(3) $\Rightarrow$ (1) Suppose L is a lattice. Clearly, $M \subseteq A_M(L)$. Let $x \in A_M(L)$ and $t \in M$. Then we have $t \wedge x = x \wedge t$. Therefore $t \sim x$ and hence $x \in M$. Therefore $M = A_M(L)$. □

Now, we introduce the concept of an amicable set in an AL L and prove that the Birkhoff centre of a P_1- semi group is an amicable set. Also, observe that a maximal set in an AL L need not be amicable by means of example.

DEFINITION 3.4. A maximal set M in an AL is said to be amicable if $A_M(L) = L$ (i.e. if every element of L is M-amicable).

COROLLARY 3.8. *In a discrete AL, every singleton set is amicable.*

PROOF. Suppose L is a discrete AL and $a \in L$. First we shall prove $\{a\}$ is maximal set. Clearly, $\{a\}$ is compatible set. Suppose N is compatible set in L such that $\{a\} \subseteq N$. Let $b \in N$. Then we have $a, b \in N$. It follows $a = b \wedge a = a \wedge b = b$. Thus $\{a\} = N$. Therefore $\{a\}$ is a maximal set. Now, we prove $\{a\}$ is amicable. That is enough to prove that $A_{\{a\}}(L) = L$. Clearly, $A_{\{a\}}(L) \subseteq L$. Let $b \in L$. Then we have $a \wedge b = b$ and hence b is $\{a\}$- amicable. Hence $b \in A_{\{a\}}(L)$. Thus $L \subseteq A_{\{a\}}(L)$. Therefore$A_{\{a\}}(L) = L$. Thus $\{a\}$ is amicable. □

Recall that if $(S, .)$ is a P_1- semi group, then to each $x \in S$, there exists x^0 in the Birkhoff centre $B(S)$ of S which is least among the elements of $B(S)$ with the property $x^0 x = x$. Since $x^0 \in B(S)$, there exists $x^{0'} \in B(S)$ such that the mapping $y \mapsto (x^0 y, x^{0'} y)$ of S onto $x^0 S \times x^{0'} S$ is an isomorphism. Now, if we define for any $x, y \in S$, $x \wedge y = x^0 y$ and $x \vee y$ to be the unique element of S such that $x^0(x \vee y) = x$ and $x^{0'}(x \vee y) = x^{0'} y$. Then it can be easily verified that $(S, \vee, \wedge)$ is an AL.

Now we prove the following theorem.

THEOREM 3.5. *If $(S,\ .)$ is a P_1- semi group, then the Birkhoff centre $B(S)$ of S is an amicable set in S.*

PROOF. Let $(S, .)$ be a P_1- semi group. Let us recall that, for any $x \in S$, there is a smallest element $x^0 \in B(S)$ such that $x^0 x = x$. It is enough if we prove that $B(S)$ is a maximal set and $x^0 \wedge x = x$ and $x \wedge x^0 = x^0$. Let $x \in S$ such that $x \sim a$ for all $a \in B(S)$. In particular $x \sim x^0$ so that $x^0 = x^0.x^0 = x \wedge x^0 = x^0 \wedge x = x^0.x = x$ and hence $x \in B(S)$. Thus $B(S)$ is a maximal set. Now, by the definition of the operation $\wedge$ in S, we have $x^0 \wedge x = x^0 x = x$ and $x \wedge x^0 = x^0 x^0 = x^0$ for all $x \in S$. Hence $B(S)$ is an amicable set where, for any $x \in S$, $x^{B(S)} = x^0$. □

Now, we prove the following theorem which explains the relation between the maximal sets in an AL L and the amicable sets in an AL L. For this first we introduce the concept of AL-homomorphism.

DEFINITION 3.5. Let $(L_1, \vee, \wedge)$ and $(L_2, \vee', \wedge')$ be two ALs. Then a mapping $\psi : L_1 \longrightarrow L_2$ is said to be a *homomorphism* if for any $x, y \in L$, $\psi(x \vee y) = \psi(x) \vee' \psi(y)$ *and* $\psi(x \wedge y) = \psi(x) \wedge' \psi(y)$. A homomorphism ψ is said to be a *monomorphism* (*epimorphism*) if ψ is one-one(onto) and ψ is said to be an *isomorphism* if ψ is a bijection.

THEOREM 3.6. *Let M be a maximal set in an AL L and $M^{'}$ be an amicable set in L. Then the mapping $a \mapsto a^{M^{'}}$ is a mono morphism of the lattice $(M, \vee, \wedge)$ into the lattice $(M^{'}, \vee, \wedge)$. Further if M is also amicable, then the above mapping is an isomorphism.*

PROOF. Define $f : M \longrightarrow M^{'}$ by $f(a) = a^{M^{'}}$ for all $a \in M$. Now, we shall prove that f is a *monomorphism*. Let $a, b \in M$. Then we have $a \sim b$. Also since $a, b \in M \subseteq L$ and $M^{'}$ is amicable, $A_{M'}(L) = L$. Hence a and b are $M^{'}$-amicable. It follows that $a^{M^{'}} = b^{M^{'}}$ if and only if $a = b$. Therefore f is well defined and one-one. Now, let $a, b \in M$. Then $f(a \wedge b) = (a \wedge b)^{M^{'}} = a^{M^{'}} \wedge b^{M^{'}} = f(a) \wedge f(b)$ and $f(a \vee b) = (a \vee b)^{M^{'}} = a^{M^{'}} \vee b^{M^{'}} = f(a) \vee f(b)$. Thus f is a *monomorphism*. Suppose M is an amicable set in L. Let $x \in M^{'}$. Then $x \in A_{M'}(L) = L = A_M(L)$. Hence x is M-amicable. Hence there exists unique element $x^M \in M$ such that $x \wedge x^M = x^M$ and $x^M \wedge x = x$. Now, since $x^M \in M$, x^M is M-amicable. Therefore $x^M \in A_M(L) = L = A_{M'}(L)$. Hence x^M is $M^{'}$ -amicable. It follows that, $(x^M)^{M^{'}} \in M^{'}$ such that $x^M \wedge (x^M)^{M^{'}} = (x^M)^{M^{'}}$ and $(x^M)^{M^{'}} \wedge x^M = x^M$. Then by uniqueness, we get $x = (x^M)^{M^{'}}$. Now, we have$x^M \in M$ and $f(x^M) = (x^M)^{M^{'}} = x$. Therefore f is onto and hence it is an isomorphism from a lattice M on to a lattice $M^{'}$. □

In the following, we introduce the concepts of a uni-element in an AL L and prove certain properties of a uni-element.

DEFINITION 3.6. Let M be a maximal set in an AL L. An element υ of L is said to be a uni-element of M if $a \leqslant \upsilon$ for all $a \in M$.

LEMMA 3.3. *Let L be an AL and M be a maximal set in L. If υ is a uni-element of M, then $\upsilon \in M$.*

PROOF. Suppose M is a maximal set and let $\upsilon \in L$ is a uni-element of M. Then we have $x \leqslant \upsilon$ for all $x \in M$. It follows that, $x \wedge \upsilon = \upsilon \wedge \upsilon$ for all $x \in M$ and hence $x \sim \upsilon$ for all $x \in M$. Therefore $x \in M$. □

LEMMA 3.4. *Let L be an AL and M be a maximal set in L. Then M has at most one uni-element.*

PROOF. Suppose υ_1, $\upsilon_2 \in L$ such that υ_1 *and* υ_2 are uni-elements of M. Then by lemma 3.3, υ_1, $\upsilon_2 \in M$. It follows that, $\upsilon_1 \leqslant \upsilon_2$ and $\upsilon_2 \leqslant \upsilon_1$. Hence $\upsilon_2 = \upsilon_1$. □

In the following we prove some properties of uni-elements and maximal elements. For this, first, we need the following:

LEMMA 3.5. *Let L be an AL and $x, y \in L$. Then $x \vee y$ is maximal if and only if $y \vee x$ is maximal.*

PROOF. Suppose $x, y, t \in L$. Then we have $(x \vee y) \wedge t = (y \vee x) \wedge t$. It follows that, $x \vee y$ is maximal if and only if $y \vee x$ is maximal. □

LEMMA 3.6. *Let L be an AL with maximal element n. Then for each $x \in L$, there exists a maximal element $m \in L$ such that $x \leqslant m$.*

PROOF. Suppose $x \in L$. Now, put $m = x \vee n$. Then clearly m is maximal and $x \wedge m = x \wedge (x \vee n) = x$. Therefore $x \leqslant m$. □

LEMMA 3.7. *Let M be a maximal set in an AL L with uni-element υ. Then υ is a maximal element of L and $M = \{x \wedge \upsilon | x \in L\}$.*

PROOF. Suppose $\upsilon \in L$ is uni element of M. Now, let $x \in L$ and $\upsilon \leqslant x$. Since υ is a uni-element of M, $\upsilon \in M$. Therefore $a \leqslant \upsilon \leqslant x$ for all $a \in M$. This implies $a \leqslant x$ for all $a \in M$. Hence $a \wedge x = x \wedge a$ for all $a \in L$. Therefore $a \sim x$ for all $a \in L$. Thus $x \in M$. It follows that $x \leqslant \upsilon$. Hence $\upsilon = x$. Therefore υ is a maximal element in L. Now, put $M' = \{x \wedge \upsilon | x \in L\}$. We shall prove that $M = M'$. Let $a \in M$. Then we have $a \leqslant \upsilon$. Therefore $a = a \wedge \upsilon$ and hence $a \in M'$. Conversely, suppose $x \wedge \upsilon \in M'$ and $t \in M$. Now, consider $(x \wedge \upsilon) \wedge t = x \wedge (\upsilon \wedge t) = x \wedge (t \wedge \upsilon) = (x \wedge t) \wedge \upsilon = (t \wedge x) \wedge \upsilon = t \wedge (x \wedge \upsilon)$. Hence $x \wedge \upsilon \in M$. Thus $M = M' = \{x \wedge \upsilon | x \in L\}$. □

LEMMA 3.8. *Let m be a maximal element of an AL L. Then the set $M_m = \{x \wedge m | x \in L\}$ is a maximal set in L with m as its uni-element.*

PROOF. Suppose $x \wedge m, y \wedge m \in M_m$. Then $(x \wedge m) \wedge (y \wedge m) = ((x \wedge m) \wedge y) \wedge m = (x \wedge (m \wedge y)) \wedge m = (x \wedge (y \wedge m)) \wedge m = ((x \wedge y) \wedge m) \wedge m = ((y \wedge x) \wedge m) \wedge m = (y \wedge (x \wedge m)) \wedge m = ((x \wedge y) \wedge m) \wedge m = (x \wedge (y \wedge m)) \wedge m = (y \wedge m) \wedge (x \wedge m)$. Therefore $(x \wedge m) \sim (y \wedge m)$ and hence M_m is compatible set in L. Suppose M is a compatible set in L such that $M_m \subseteq M$. Let $x \in M$ and $y \wedge m \in M_m$. Then we have $x,\ y \wedge m \in M$. Therefore $x \wedge (y \wedge m) = (y \wedge m) \wedge x$. Hence $x \sim y \wedge m$ for all $y \wedge m \in M_m$. It follows that, $x \in M_m$. Thus $M \subseteq M_m$. Therefore M_m is a maximal set. Since $x \wedge m \leqslant m$ for all $x \wedge m \in M_m$, m is a uni-element of M_m. □

Using the above two lemmas, we have the following theorem.

THEOREM 3.7. *An element $m \in L$ of an AL L is a maximal element if and only if there exists a maximal set M with m as its uni-element.*

REMARK 3.1. Whether amicable sets in an AL exists or not is not known and it is still under investigation.

In the following example, we describe an *AL* L and exhibit a maximal set M of L for which $A_M(L) \subsetneqq L$. That is, M is a maximal set but not amicable.

EXAMPLE 3.1. Let L be the set of all sequences $\{a_n\}$ of non negative integers whose range is finite. Define two binary operations $\vee$ and $\wedge$ on L as follows: For any $\{a_n\}, \{b_n\} \in L$,

$$\{a_n\} \vee \{b_n\} = \{c_n\} \text{ where } c_n = \begin{cases} a_n & \text{if } a_n \neq 0 \\ b_n & \text{if } a_n = 0 \end{cases}$$

and

$$\{a_n\} \wedge \{b_n\} = \{d_n\} \text{ where } d_n = \begin{cases} b_n & \text{if } a_n \neq 0 \\ 0 & \text{if } a_n = 0 \end{cases}$$

Then it can be verified that $(L, \vee, \wedge)$ is an AL. Also, observe that for any $\{a_n\}$, $\{b_n\} \in L$, $\{a_n\} \sim \{b_n\}$ if and only if $a_n \neq 0 \neq b_n$ implies $a_n = b_n$. Write

$$M = \{\{a_n\} \in L |\ a_n = n \text{ or } a_n = 0 \text{ for all } n\}.$$

Observe that every sequence in M has only a finite number of non zero entries. Clearly, M is a compatible set in L. Now, we prove that M is a maximal set. Let $\{c_n\} \in L$ and let $\{c_n\} \sim \{a_n\}$ for all $\{a_n\} \in M$. Suppose for some m, $c_m \neq 0$.

Now, consider the sequence $\{a_n\}$ where

$$a_n = \begin{cases} m & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Then $\{a_n\} \in M$ so that $\{c_n\} \sim \{a_n\}$. Hence $c_m = a_m = m$ since $c_m \neq 0 \neq a_m$. Thus $\{c_n\} \in M$. Therefore M is a maximal set. Now, consider the constant sequence $\{1\}$. Here $\{1\} \in L$, but $\{1\} \notin A_M(L)$. For, if $\{1\} \in A_M(L)$, then there exists $\{a_n\} \in M$ such that $\{a_n\} \wedge \{1\} = \{1\}$ which means $a_n \neq 0$ for all n which is a contradiction. Hence M is a maximal set in L which is not amicable.

Finally, we give a necessary and sufficient condition for a maximal set to become an amicable set.

THEOREM 3.8. *Let L be an AL with maximal element m. Then a maximal set M of L is amicable if and only if M has a uni-element.*

PROOF. Suppose M is amicable set. Since $m \in L = A_M(L)$, m is M-amicable element. Hence there exists a smallest element $m^M \in M$ such that $m^M \wedge m = m$. Let $a \in M$. Then $a \wedge m^M = m^M \wedge a = m^M \wedge m \wedge a = m \wedge a = a$. Hence $a \leqslant m^M$. Thus m^M is a uni-element of M. Conversely, suppose that a maximal set M has a uni element say, υ. Now, we prove M is amicable. That is enough to prove that $A_M(L) = L$. We have $A_M(L) \subseteq L$. Now, let $x \in L$. Then we have $x \wedge \upsilon \leqslant \upsilon$ *and* $\upsilon \in M$. Therefore $x \wedge \upsilon \in M$. Now, $(x \wedge \upsilon) \wedge x = (\upsilon \wedge x) \wedge x = \upsilon \wedge x = x$. Thus x is M-amicable and hence $x \in A_M(L)$. Therefore $L \subseteq A_M(L)$. Hence we get $A_M(L) = L$. Therefore M is amicable set. □

COROLLARY 3.9. *If an AL L has a maximal element, then every amicable set in L can be embedded in a maximal set with uni-element.*

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