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HOMODERIVATION OF PRIME RINGS WITH INVOLUTION

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Abstract. Let R be a ring with involution $*$. An additive mapping h from R into itself is called homoderivation if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. In this paper we investigate the commutativity of a ring R with involution $*$ which admits a homoderivation satisfying certain algebraic identities.

1. Introduction

Throughout this paper, R will represent a ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. A ring R is a 2-torsion free if whenever $2x = 0$, $x \in R$, implies $x = 0$. A ring R is called prime if $aRb = 0$, where $a, b \in R$, implies $a = 0$ or $b = 0$, and is called a semiprime ring in case $aRa = 0$ implies $a = 0$. An additive mapping $* : R \to R$ is called an involution if $*$ is an antihomomorphism of order 2, that is, $(a^*)^* = a$, $(a + b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ for all $a, b \in R$. An element x in a ring R with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denote by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$; otherwise it is said to be of the second kind. In the later case, $S(R) \cap Z(R) \neq (0)$. If char$(R)\neq 2$, then $R = S(R) + H(R)$ and $S(R) \cap H(R) = (0)$. Note that in this case x is normal, i.e. $xx^* = x^*x$, if and only if S and h commute. If all elements in R are normal, then R is called a normal ring. A mapping $f : R \to R$ is said to be $*$-centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in S$ and $f : R \to R$ is said to be $*$-commuting on S if $[f(x), x^*] = 0$ for all $x \in S$. A derivation on R is an additive mapping $d : R \to R$ such that

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$d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. El-Sofy [**3**] defined a homoderivation on R as an additive map h on R such that $h(xy) = h(x)h(y) + h(x)y + xh(y)$, for all $x, y \in R$. For a positive integer $n(x) > 1$ such that $f^{n(x)}(x) = 0$ for all $x \in R$, the mapping $f : R \to R$ is called a zero-power valued on R [**3**]. Over the last few decades, several authors have describe the structure of additive mappings that are $*$-commuting on a prime or semiprime ring with involution and study the commutativity of rings with involution satisfying some algebraic conditions(see, [**2**], [**12**]). In this paper, we study the commutativity of rings with involution admitting a homoderivation satisfying some algebraic identities In [**11**], the authors proved the commutativity of $*$-prime rings admitting homoderivations that commute with $*$ and satisfy some conditions on $*$-ideals.

2. preliminaries

In [**8**], for any $x, y, z \in R$, the following identities of anticommutators are obvious

- $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$.
- $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

An important results will be listed in this section.

Lemma 2.1 ([**10**], Lemma 4). *Let b and ab be elements in the center of a prime ring R. If b is not zero, then a is in $Z(R)$.*

Lemma 2.2 ([**9**], Lemma 2). *Suppose $2R = 0$ and U is a commutative Lie ideal of R. Then $u^2 \in Z(R)$ for all $u \in U$.*

Lemma 2.3 ([**1**], Lemma 2.1). *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.*

Lemma 2.4 ([**12**], Lemma 2.1). *Let R be a prime ring with involution of the second kind. Then $*$ is centralizing if and only if R is commutative.*

Lemma 2.5 ([**12**], Lemma 2.2). *Let R be a prime ring with involution of the second kind. Then $x \circ x^* \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Lemma 2.6 ([**11**], Lemma 2.3.1). *Let R be a ring and let h be a zero power valued homoderivation on R. Then h preserves $Z(R)$.*

Lemma 2.7 ([**11**], Lemma 2.3.2). *Let R be a prime ring, and $h \neq 0$ a homoderivation of R such that $[h(x), h(y)] = 0$ for all $x, y \in R$. If char(R)≠2. R is commutative .*

Lemma 2.8 ([**3**], Theorem 3.3.1). *Let R be a prime ring with $char(R) \neq 2$ and $h \neq 0$ be a homoderivation of R. An element $a \in R$ is such that $ah(x) = h(x)a$ for all $x \in R$. Then a must be in $Z(R)$.*

Lemma 2.9 ([**3**], Theorem 3.4.7). *let R be a prime ring and $I \neq 0$ a two sided ideal of R. If R admits a non-zero homoderivation h which is commuting and zero-power valued on I. Then R is a commutative.*

LEMMA 2.10 ([**3**], Corollary 3.4.8). *let R be a prime ring and $I \neq 0$ a two sided ideal of R. If R admits a non-zero homoderivation h which is centralizing and zero-power valued on I. Then R is a commutative.*

LEMMA 2.11 ([**4**], Lemma 1). *Let R be any ring with involution $*$ such that $R = S + K$. Then K^2, the addition subgroup generated by all products k_1k_2 for $k_1, k_2 \in K$, is a Lie ideal of R.*

3. Main Result

LEMMA 3.1. *Let $(R, *)$ be a 2-torsion free prime ring with involution provided with a homoderivation h. If $h(t) = 0$ for all $t \in H(R) \cap Z(R)$, then $h(z) = 0$ for all $z \in Z(R)$.*

PROOF. If $t = 0$, then $h(t) = 0$. Assume that $t \neq 0$, and

$$h(t) = 0 \text{ for all } t \in H(R) \cap Z(R). \tag{3.1}$$

Then replacing t by $tk^2 \in H(R) \cap Z(R)$ where $k \in S(R) \cap Z(R)$ and applying (3.1) we get

$$0 = h(tk^2) = h(t)h(k^2) + th(k^2) + h(t)k^2.$$

$$0 = th(k^2).$$

Now replace k by $s + r$ such that $0 \neq s, r \in S(R) \cap Z(R)$

$$th((s + r)^2) = th(s^2 + 2sr + r^2) = th(s^2) + 2th(sr) + th(r^2) = 2th(sr) = 0.$$

$$2th(sr) = 0.$$

Since R is 2-torsion free, so

$$th(sr) = 0.$$

$$th(s)h(r) + tsh(r) + th(s)r = 0.$$

$$tsh(r) = 0$$

Since the center of a prime ring is free zero divisors this assures that $h(r) = 0$ for all $r \in S(R) \cap Z(R)$. Since each element $z \in Z(R)$ can be uniquely represented in the form $2z = g + k$ where $g \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$then,

$$2h(z) = h(2z) = h(g + k) = h(g) + h(k) = 0.$$

Since char$(R) \neq 2$, so $h(z) = 0$ for all $z \in Z(R)$. □

THEOREM 3.1. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. Let h be a homoderivation of R such that $[h(x), h(x^*)] = 0$ for all $x \in R$, then R is commutative.*

PROOF. By the assumption, we have

$$[h(x), h(x^*)] = 0 \text{ for all } x \in R. \tag{3.2}$$

By lineralization (3.2) yields that

$$[h(x), h(y^*)] + [h(y), h(x^*)] = 0 \text{ for all } x, y \in R. \tag{3.3}$$

Replacing y by xx^* in (3.3),

$$\begin{aligned} 0 &= [h(x), h((xx^*)^*)] + [h(xx^*), h(x^*)]. \\ &= [h(x), h(xx^*)] + [h(xx^*), h(x^*)]. \\ &= [h(x), h(x)h(x^*)] + [h(x), xh(x^*)] + [h(x), h(x)x^*] + [h(x)h(x^*), h(x^*)] \\ &\quad + [xh(x^*), h(x^*)] + [h(x)x^*, h(x^*)]. \\ &= h(x)[h(x), h(x^*)] + [h(x), h(x)]h(x^*) + x[h(x), h(x^*)] + [h(x), x]h(x^*) \\ &\quad + h(x)[h(x), x^*] + [h(x), h(x)]x^* + h(x)[h(x^*), h(x^*)] + [h(x), h(x^*)]h(x^*) \\ &\quad + x[h(x^*), h(x^*)] + [x, h(x^*)]h(x^*) + h(x)[x^*, h(x^*)] + [h(x), h(x^*)]x^*. \end{aligned}$$

$$0 = [h(x), x]h(x^*) + h(x)[h(x), x^*] + [x, h(x^*)]h(x^*) + h(x)[x^*, h(x^*)]. \tag{3.4}$$

Replacing x by $x + t$, where $t \in H(R) \cap Z(R)$, we obtain

$$0 = [h(x+t), x+t]h(x^*+t) + h(x+t)[h(x+t), x^*+t] + [x+t, h(x^*+t)]h(x^*+t) \\ +h(x+t)[x^*+t, h(x^*+t)].$$

$$\begin{gathered} 0 = [h(x), x]h(x^*) + [h(x), x]h(t) + [h(x), t]h(x^*) + [h(x), t]h(t) + [h(t), t]h(x^*) \\ +[h(t), t]h(t) + [h(t), x]h(x^*) + [h(t), x]h(t) + h(x)[h(x), x^*] + h(x)[h(t), t] \\ +h(x)[h(x), t] + h(x)[h(t), x^*] + h(t)[h(x), x^*] + h(t)[h(t), t] + h(t)[h(x), t] \\ +h(t)[h(t), x^*] + h(x)[x^*, h(x^*)] + h(x)[t, h(t)] + h(x)[x^*, h(t)] + h(x)[t, h(x^*)] \\ +h(t)[x^*, h(x^*)] + h(t)[t, h(t)] + h(t)[x^*, h(t)] + h(t)[t, h(x^*)] \\ +[x, h(x^*)]h(x^*) + [t, h(t)]h(x^*) + [x, h(t)]h(x^*) + [t, h(x^*)]h(x^*) \\ +[x, h(x^*)]h(t) + [t, h(t)]h(t) + [x, h(t)]h(t) + [t, h(x^*)]h(t). \end{gathered}$$

By using (3.4) we get

$$0 = h(t)([h(x), x] + [h(x), x^*] + [x^*, h(x^*)] + [x, h(x^*)]).$$

for all $t \in H(R) \cap Z(R)$ and $x \in R$. Since the center of a prime ring is free from zero divisors we get either $h(t) = 0$ for all $t \in H(R) \cap Z(R)$ or $[h(x), x] + [h(x), x^*] + [x^*, h(x^*)] + [x, h(x^*)] = 0$ for all $x \in R$. Suppose

$$h(t) = 0 \text{ for all } \ t \in H(R) \cap Z(R). \tag{3.5}$$

By lemma 3.1 we get

$$h(x) = 0 \text{ for all } x \in Z(R). \tag{3.6}$$

Replacing y by ky in (3.3), where $k \in S(R) \cap Z(R)$ and using (3.6), we get

$$\begin{gathered} [h(x), h((ky)^*)] + [h(ky), h(x^*)] = 0 \text{ for all } x, y \in R. \\ [h(x), h(y^*k^*)] + [h(ky), h(x^*)] = 0 \text{ for all } x, y \in R. \\ [h(x), h(y^*(-k))] + [h(ky), h(x^*)] = 0 \text{ for all } x, y \in R. \\ -[h(x), h((y)^*k)] + [h(ky), h(x^*)] = 0 \text{ for all } x, y \in R. \\ -[h(x), h(y^*)h(k)] - [h(x), y^*h(k)] - [h(x), h(y^*)k] + [h(k)h(y), h(x^*)] + [kh(y), h(x^*)] \\ +[h(k)y, h(x^*)] = 0 \text{ for all } x, y \in R. \\ -h(y^*)[h(x), h(k)] - [h(x), h(y^*)]h(k) - y^*[h(x), h(k)] - [h(x), y^*]h(k) - h(y^*)[h(x), k] \\ -[h(x), h(y^*)]k + h(k)[h(y), h(x^*)] + [h(k), h(x^*)]h(y) + k[h(y), h(x^*)] \end{gathered}$$

$$+[k,h(x^*)]h(y)+h(k)[y,h(x^*)]$$
$$+[h(k),h(x^*)]y=0 \text{ for all } x,y\in R.$$
$$k(-[h(x),h(y^*)]+[h(y),h(x^*)])=0$$

for all $k\in S(R)\cap Z(R)$ and $x,y\in R$. Using the primeness of R and the fact that $S(R)\cap Z(R)\neq(0)$, we get

$$-[h(x),h(y^*)]+[h(y),h(x^*)]=0 \text{ for all } x,y\in R. \tag{3.7}$$

for all $x,y\in R$. On comparing (3.3) and (3.7), we obtain $2[h(x),h(y^*)]=0$ Replacing y by y^* and using the fact that $char(R)\neq 2$, we conclude that $[h(x),h(y)]=0$ for all $x,y\in R$. Therefore, by Lemma 2.7, we get that R is commutative.
Now we consider the case

$$[h(x),x]+[h(x),x^*]+[x^*,h(x^*)]+[x,h(x^*)]=0 \text{ for all } x\in R. \tag{3.8}$$

Replacing x by $t+k$, where $t\in H(R)$ and $k\in S(R)$,

$$[h(t+k),t+k]+[h(t+k),(t+k)^*]+[(t+k)^*,h((t+k)^*))]+[t+k,h((t+k)^*))]=0.$$

$$[h(t),t]+[h(t),k]+[h(k),t]+[h(k),k]+[h(t),t^*]+[h(t),k^*]+[h(k),t^*]+[h(k),k^*]$$
$$+[t^*,h(t^*)]+[t^*,h(k^*)]+[k^*,h(t^*)]+[k^*,h(k^*)]+[t,h(t^*)]+[t,h(k^*)]+[k,h(t^*)]$$
$$+[k,h(k^*)]=0.$$

$$[h(t),t]+[h(t),t^*]+[t^*,h(t^*)]+[t,h(t^*)]+[h(k),k]+[h(k),k^*]+[k^*,h(k^*)]+[k,h(k^*)]$$
$$+[h(t),k]+[h(k),t]+[h(t),k^*]+[h(k),t^*]+[t^*,h(k^*)]+[k^*,h(t^*)]+[t,h(k^*)]$$
$$+[k,h(t^*)]=0.$$

$$+[h(t),k]+2[h(k),t]-[h(t),k]+[h(k),t]-[k,h(t)]+[h(k),t]+[k,h(t^*)]=0$$

for all $x\in R$. By (3.8), we get $4[h(k),t]=0$. Since $char(R)\neq 2$, we obtain

$$[h(k),t]=0 \text{ for all } t\in H(R) \textit{ and } k\in S(R). \tag{3.9}$$

Replacing t by $k_0k^{'}$, where $k_0\in S(R)$ and $k^{'}\in S(R)\cap Z(R)$, we arrive at $([h(k),k_0])k=0$. Using the primeness of R and since $S(R)\cap Z(R)\neq(0)$, we get

$$[h(k),k_0]=0 \text{ for all } k,k_0\in S(R). \tag{3.10}$$

Since $char(R)\neq 2$, every $x\in R$ can be represented as $2x=t+k$, where $t\in H(R), k\in S(R)$, so in equations (3.9) and (3.10),

$$[h(k),2x]=[h(k),t+k]=[h(k),t]+[h(k),k]=0 \text{ for all } k\in S(R)\ x\in R.$$
$$2[h(k),x]=0 \text{ for all } k\in S(R)\ x\in R.$$

Since $char(R)\neq 2$ we conclude that

$$[h(k),x]=0 \text{ for all } k\in S(R)\ x\in R. \tag{3.11}$$

That is $h(k)\in Z(R)$ for all $k\in S(R)$. Assume that $h(S(R))=(0)$, so $(h(x-x^*))=0$ for all $x\in R$. That is $h(x)=h(x^*)$ for all $x\in R$. Now for $k\in S(R)$ and $x\in R$, we have $0=h(kx+x^*k)=h(k)h(x)+kh(x)+h(k)x+h(x^*)h(k)+x^*h(k)+h(x^*)k=kh(x)+h(x^*)k=kh(x)+h(x)k$ for all $x\in R$. This further implies that $k^2h(x)=h(x)k^2$ for all $x\in R$. Thus, by the Lemma 2.8, we conclude that $k^2\in Z(R)$ for all $k\in Z(R)$. Since $S(R)\cap Z(R)\neq(0)$, let $0\neq k_0\in S(R)\cap Z(R)$

and k be an arbitrary element of $S(R)$. Then $(k+k_0)^2 = k^2 + k_0^2 + 2kk_0 \in Z(R)$ hence $2kk_0 \in Z(R)$. Since $char(R) \neq 2$, we get $kk_0 \in Z(R)$ for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$ implies that $k \in Z(R)$ for all $k \in S(R)$ R is normal. Thus, R is commutative by Lemma 2.3.

Now suppose $h(S(R)) \neq (0)$. For $k_0 \in S(R)$ with $h(k_0) \neq 0$ and $k \in [S(R), S(R)]$, we have

$$h(kk_0k) \in Z(R)$$

$$h(k)h(k_0k) + h(k)k_0k + kh(k_0k) \in Z(R)$$

$$h(k)h(k_0)h(k) + h(k)k_0h(k) + h(k)h(k_0)k + h(k)k_0k + kh(k_0)h(k) + kk_0h(k)$$
$$+ kh(k_0)k \in Z(R)$$

Since $h([S(R), S(R)]) = 0$

$$k^2h(k_0) \in Z(R)$$

Thus, by the Lemma 2.8, we conclude that $k^2 \in Z(R)$ for all $k \in Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, let $0 \neq k_0 \in S(R) \cap Z(R)$ and let k be an arbitrary element of $S(R)$. Then $(k+k_0)^2 = k^2 + k_0^2 + 2kk_0 \in Z(R)$ and hence $2kk_0 \in Z(R)$. Since $char(R) \neq 2$, we get $kk_0 \in Z(R)$ for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. This further implies that $k \in Z(R)$ for all $k \in S(R)$. That is, $[S(R), S(R)] \subseteq Z(R)$.

Suppose $[S(R), S(R)] \neq (0)$ and let $k, k_0 \in S(R)$ such that$[k, k_0] \neq 0$. Since $kk_0k \in S(R)$, we have

$$[k, kk_0k] = [k, k]k_0k + k[k, k_0]k + kk_0[k, k] = k^2[k, k_0] \in Z(R).$$

This implies that $k \in Z(R)$ for all $k \in S(R)$. Therefore, R is commutative by Lemma 2.3.

Now suppose $[S(R), S(R)] = (0)$. Since by lemma 2.11 $\overline{S(R)^2}$ is a Lie ideal and a commutative subring of R, by lemma 2.2, $k^2 \in Z(R)$ for all $k \in S(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$. Thus, R is normal . Hence R is commutative by Lemma 2.3. □

THEOREM 3.2. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. Let h be a homoderivation which is zero-power valued on R, then the following are equivalent:*

(i) $h(x) \circ h(x^*) - x \circ x^* \in Z(R)$ *for all* $x \in R$.
(ii) $h(x) \circ h(x^*) + x \circ x^* \in Z(R)$ *for all* $x \in R$.
(iii) R is commutative.

Moreover, if $h \neq 0$ and $h(x) \circ h(x^*) \in Z(R)$ for all $x \in R$, implies that R is commutative.

PROOF. It is clear that (iii) implies both of (i) and (ii). So, we need to prove that (i) ⇒ (iii) and (ii) ⇒(iii).

If $h = 0$ we get $x \circ x^* \in Z(R)$ for all $x \in R$. Using Lemma 2.5 we conclude that R is commutative. Assume that $h \neq 0$.

We have:

$$h(x) \circ h(x^*) - x \circ x^* \in Z(R) \text{ for all } x \in R. \tag{3.12}$$

Replacing x by $x+y$ and applying (3.12), we get

(3.13) $\quad h(x)\circ h(y^*)+h(y)\circ h(x^*)-x\circ y^*-y\circ x^*\in Z(R)$ for all $x,y\in R$.

Replacing y by yd where $d\in Z(R)\cap H(R)$, and using (3.13) yields

$$h(x)\circ h((yd)^*)+h(yd)\circ h(x^*)-x\circ((yd)^*)-(yd)\circ x^*\in Z(R).$$

$$h(x)\circ h(dy^*)+h(yd)\circ h(x^*)-x\circ(dy^*)-(yd)\circ x^*\in Z(R).$$

$$h(x)\circ(h(d)h(y^*))+h(x)\circ(dh(y^*)+h(x)\circ(h(d)y^*)+(h(y)h(d))\circ h(x^*)+(yh(d))\circ h(x^*)$$
$$+(h(y)d)\circ h(x^*)-d(x\circ y^*)-[x,d]y^*-(y\circ x^*)d-y[d,x^*]\in Z(R).$$

$$h(d)(h(x)\circ h(y^*))+[h(x),h(d)]h(y^*)+d(h(x)\circ h(y^*)+[h(x),d]h(y^*)$$
$$+h(d)(h(x)\circ y^*)+[h(x),h(d)]y^*+(h(y)\circ h(x^*))h(d)+h(y)[h(d),h(x^*)]$$
$$+(y\circ h(x^*)h(d)+y[h(d),h(x^*)]+(h(y)\circ h(x^*))d+h(y)[d,h(x^*)]$$
$$-d(x\circ y^*)-[x,d]y^*-(y\circ x^*)d-y[d,x^*]\in Z(R).$$

$$[h(x)\circ h(y^*)+h(y)\circ h(x^*)-x\circ y^*-y\circ x^*,r]d+[h(x)\circ(y^*+h(y^*))$$
$$+(y+h(y))\circ h(x^*),r]h(d)=0$$

for all $x,y\in R$.

(3.14) $\quad [h(x)\circ(y^*+h(y^*))+(y+h(y))\circ h(x^*),r]h(d)=0$ for all $x,y\in R$.

Since h is zero-power valued on R, we get

$$[h(x)\circ y^*+y\circ h(x^*),r]h(d)=0 \text{ for all } x,y,r\in R.$$

thus,

(3.15) $\quad [h(x)\circ y^*+y\circ h(x^*),r]Rh(d)=0$ for all $x,y,r\in R$.

Since R is prime, so, either $h(d)=0$ or $[h(x)\circ y^*+y\circ h(x^*),r]=0$.
If $h(d)=0$ for all $d\in Z(R)\cap H(R)$, by Lemma 3.1, we have that

(3.16) $\quad h(z)=0$ for all $z\in Z(R)$.

Replacing y by yz in (3.13) where $z\in Z(R)$

$$h(x)\circ h(z^*y^*)+h(yz)\circ h(x^*)+(yz)\circ x^*+x\circ(z^*y^*)\in Z(R).$$

$$h(x)\circ(h(z^*)h(y^*))+h(x)\circ(z^*h(y^*))+h(x)\circ(h(z^*)y^*)+(h(y)h(z))\circ h(x^*)$$
$$+(yh(z))\circ h(x^*)+(h(y)z)\circ h(x^*)+(y\circ x^*)z+y[x^*,z]+$$
$$z^*(x\circ y^*)+[x,z^*]y^*\in Z(R).$$

$$h(z^*)(h(x)\circ h(y^*))+[h(x),h(z^*)]h(y^*)+z^*(h(x)\circ h(y^*))+[h(x),z^*]h(y^*)$$
$$+h(z^*)(h(x)\circ y^*)+[h(x),h(z^*)]y^*+(h(y)\circ h(x^*))h(z)+h(y)[h(z),h(x^*)]+$$
$$(y\circ h(x^*))h(z)+y[h(z),h(x^*)]+(h(y)\circ h(x^*))z+h(y)[z,h(x^*)]$$
$$(y\circ x^*)z+y[x^*,z]+z^*(x\circ y^*)+[x,z^*]y^*\in Z(R).$$

$$[h(x)\circ h(y)-x\circ y,r](z^*-z)=0.$$

So that

(3.17) $\quad [h(x)\circ h(y)-x\circ y,r]R(z^*-z)=0$ for all $r,x,y\in R$.

By the primeness of R, either $[h(x)\circ h(y)-x\circ y,r]=0$ or $z^*-z=0$. Since the involution is of the second kind so $z^*-z\neq 0$. Thus, $[h(x)\circ h(y)-x\circ y,r]=0$ for all $r,x,y\in R$, that is,

$$h(x)\circ h(y)-x\circ y\in Z(R) \text{ for all } x,y\in R. \tag{3.18}$$

Taking $y\in Z(R)\backslash\{0\}$ and using (3.16), we have $xy\in Z(R)$ for all $x\in R$, $y\in Z(R)$. By Lemma 2.1, we have $x\in Z(R)$ for all $x\in R$. Hence, R is commutative.

Now suppose that

$$[h(x)\circ y^*+y\circ h(x^*),r]=0 \text{ for all } r,x,y\in R. \tag{3.19}$$

Replacing y by yz where $z\in Z(R)$ in (3.19) , we get

$$[h(x)\circ(yz)^*+(yz)\circ h(x^*),r]=0.$$

$$[h(x)\circ(z^*y^*)+(yz)\circ h(x^*),r]=0.$$

$$[z^*(h(x)\circ y^*),r]+[[h(x),z^*]y^*,r]+[(y\circ h(x^*))z,r]+[y[z,h(x^*],r]=0.$$

$$[z^*(h(x)\circ y^*),r]+[(y\circ h(x^*))z,r]=0.$$

$$-[z^*(y^*\circ h(x)),r]+[(y\circ h(x^*))z,r]=0.$$

$$-[z^*(y^*h(x)+h(x)y^*),r]+[(yh(x^*)+h(x^*)y)z,r]=0.$$

Using (3.19), we get

$$[h(x)y+yh(x),r]R(z-z^*)=0 \text{ for all } r,x,y\in R \textit{ and } z\in Z(R). \tag{3.20}$$

Since R is prime and the involution is of the second kind, so, (3.20) implies

$$[h(x)y,r]+[yh(x),r]=0 \text{ for all } r,x,y\in R. \tag{3.21}$$

Substituting yr for y and using (3.21), we find that

$$[h(x)yr,r]+[yrh(x),r]=0.$$

$$[h(x)y,r]r+yr[h(x),r]+[y,r]rh(x)=0.$$

$$-[yh(x),r]r+yr[h(x),r]+[y,r]rh(x)=0.$$

$$-y[h(x),r]r-[y,r]h(x)r+yr[h(x),r]+[y,r]rh(x)=0.$$

$$[y,r](rh(x)-h(x)r)+y(r[h(x),r]-[h(x),r]r)=0.$$

$$y[[h(x),r],r]-[y,r][h(x),r]=0 \text{ for all } r,x,y\in R$$

$$[y[h(x),r],r]=0 \text{ for all } r,x,y\in R. \tag{3.22}$$

Replacing y by ty where $t\in R$, yields

$$[ty[h(x),r],r]=0.$$

$$ty[[h(x),r],r]+t[y,r][h(x),r]+[t,r]y[h(x),r]=0.$$

$$t(y[h(x),r],r]+[y,r][h(x),r])+[t,r]y[h(x),r]=0.$$

$$t[y[h(x),r],r]+[t,r]y[h(x),r]=0.$$

$$[t,r]y[h(x),r]=0 \text{ for all } r,x,y\in R.$$

Since R is prime, either $[t, r] = 0$ or $[h(x), x] = 0$ for all $x \in R$. By Lemma 2.9 R is commutative.
(ii)⇒(iii) Suppose that,

$$h(x) \circ h(x^*) + x \circ x^* \in Z(R) \text{ for all } x \in R. \tag{3.23}$$

Replacing x by $x + y$ and using (3.23), we find that

$$h(x) \circ h(y^*) + h(y) \circ h(x^*) + y \circ x^* + x \circ y^* \in Z(R) \text{ for all } x, y \in R. \tag{3.24}$$

Replacing y by yd where $d \in Z(R) \cap H(R)$ and using (3.24), we obtain

$$h(x) \circ h(d^*y^*) + h(yd) \circ h(x^*) + (yd) \circ x^* + x \circ (d^*y^*) \in Z(R).$$

Since $d \in Z(R) \cap H(R)$

$$h(x) \circ h(dy^*) + h(yd) \circ h(x^*) + (yd) \circ x^* + x \circ (dy^*) \in Z(R).$$

$$h(x) \circ (h(d)h(y^*)) + h(x) \circ (dh(y^*)) + h(x) \circ (h(d)y^*) + (h(y)h(d)) \circ h(x^*)$$
$$+(yh(d)) \circ h(x^*) + (h(y)d) \circ h(x^*) + (yd) \circ x^* + x \circ (dy^*) \in Z(R).$$

$$h(d)(h(x) \circ h(y^*)) + [h(x), h(d)]h(y^*) + d(h(x) \circ h(y^*)) + [h(x), d]h(y^*)$$
$$+h(d)(h(x) \circ y^*) + [h(x), h(d)]y^* + (h(y) \circ h(x^*))h(d) + h(y)[h(d), h(x^*)]$$
$$+(y \circ h(x^*))h(d) + y[h(d), h(x^*)] + (h(y) \circ h(x^*))d + h(y)[d, h(x^*)] + (y \circ x^*)d + y[d, x^*]$$
$$+d(x \circ y^*) + [x, d]y^* \in Z(R).$$

$$[h(x) \circ h(y^*) + h(x) \circ y^*, r]h(d) + [h(y) \circ h(x^*) + y \circ h(x^*), r]h(d)$$
$$+[h(x) \circ h(y^*) + h(y) \circ h(x^*) + y \circ x^* + x \circ y^*, r]d = 0.$$

$$[h(x) \circ (h(y^*) + y^*) + (y + h(y)) \circ h(x^*), r]h(d) = 0 \text{ for all } x, y, r \in R. \tag{3.25}$$

Since h is zero-power valued on R, we have

$$[h(x) \circ y^* + y \circ h(x^*), r]Rh(d) = 0 \text{ for all } x, y, r \in R. \tag{3.26}$$

Since R is prime, either $h(d) = 0$ or $[h(x) \circ y^* + y \circ h(x^*), r] = 0$.
If $h(d) = 0$ for all $d \in Z(R) \cap H(R)$ by Lemma 3.1, we have that

$$h(z) = 0 \text{ for all } z \in Z(R). \tag{3.27}$$

Replacing y by yz in (3.24) where $z \in Z(R)$

$$h(x) \circ h(z^*y^*) + h(yz) \circ h(x^*) + (yz) \circ x^* + x \circ (z^*y^*) \in Z(R).$$

$$h(x) \circ (h(z^*)h(y^*)) + h(x) \circ (z^*h(y^*)) + h(x) \circ (h(z^*)y^*) + (h(y)h(z)) \circ h(x^*)$$
$$+(yh(z)) \circ h(x^*) + (h(y)z) \circ h(x^*) + (y \circ x^*)z + y[x^*, z]+$$
$$z^*(x \circ y^*) + [x, z^*]y^* \in Z(R).$$

$$h(z^*)(h(x) \circ h(y^*)) + [h(x), h(z^*)]h(y^*) + z^*(h(x) \circ h(y^*)) + [h(x), z^*]h(y^*)$$
$$+h(z^*)(h(x) \circ y^*) + [h(x), h(z^*)]y^* + (h(y) \circ h(x^*))h(z) + h(y)[h(z), h(x^*)]+$$
$$(y \circ h(x^*))h(z) + y[h(z), h(x^*)] + (h(y) \circ h(x^*))z + h(y)[z, h(x^*)]$$
$$(y \circ x^*)z + y[x^*, z] + z^*(x \circ y^*) + [x, z^*]y^* \in Z(R).$$

$$[h(x) \circ h(y) - x \circ y, r](z^* - z) = 0.$$

So that

$$[h(x) \circ h(y) - x \circ y, r]R(z^* - z) = 0 \text{ for all } r, x, y \in R. \tag{3.28}$$

By the primeness of R, either $[h(x)\circ h(y)-x\circ y,r]=0$ or $z^*-z=0$. Since the involution is of the second kind ,so $z^*-z\neq 0$. Then $[h(x)\circ h(y)-x\circ y,r]=0$ for all $r\in R$, that is

$$h(x)\circ h(y)-x\circ y\in Z(R) \text{ for all } r,x,y\in R. \tag{3.29}$$

Taking $y\in Z(R)\backslash\{0\}$ and using (3.27), we have $xy\in Z(R)$ for all $x\in R$, $y\in Z(R)$. By Lemma 2.1, we have $x\in Z(R)$ for all $x\in R$. Hence, R is commutative.

Now suppose that

$$[h(x)\circ y^*+y\circ h(x^*),r]=0 \text{ for all } r,x,y\in R. \tag{3.30}$$

Replacing y by yz where $z\in Z(R)$ in (3.30) , we get

$$[h(x)\circ(yz)^*+(yz)\circ h(x^*),r]=0.$$
$$[h(x)\circ(z^*y^*)+(yz)\circ h(x^*),r]=0.$$
$$[z^*(h(x)\circ y^*),r]+[[h(x),z^*]y^*,r]+[(y\circ h(x^*))z,r]+[y[z,h(x^*],r]=0.$$
$$[z^*(h(x)\circ y^*),r]+[(y\circ h(x^*))z,r]=0.$$
$$-[z^*(y^*\circ h(x)),r]+[(y\circ h(x^*))z,r]=0.$$
$$-[z^*(y^*h(x)+h(x)y^*),r]+[(yh(x^*)+h(x^*)y)z,r]=0.$$

Replace x^* *by* x *and* y^* *by* y we get

$$[h(x)y+yh(x),r]R(z-z^*)=0 \text{ for all } r,x,y\in R \textit{ and } z\in Z(R). \tag{3.31}$$

Since R is prime and the involution is of the second kind, so (3.31) implies

$$[h(x)y,r]+[yh(x),r]=0 \text{ for all } r,x,y\in R. \tag{3.32}$$

Substituting yr for y and using (3.32), we find that

$$[h(x)yr,r]+[yrh(x),r]=0.$$
$$[h(x)y,r]r+yr[h(x),r]+[y,r]rh(x)=0.$$
$$-[yh(x),r]r+yr[h(x),r]+[y,r]rh(x)=0.$$
$$-y[h(x),r]r-[y,r]h(x)r+yr[h(x),r]+[y,r]rh(x)=0.$$
$$[y,r](rh(x)-h(x)r)+y(r[h(x),r]-[h(x),r]r)=0.$$
$$y[[h(x),r],r]-[y,r][h(x),r]=0 \text{ for all } r,x,y\in R$$

$$[y[h(x),r],r]=0 \text{ for all } r,x,y\in R. \tag{3.33}$$

Replacing y by ty where $t\in R$, yields

$$[ty[h(x),r],r]=0.$$
$$ty[[h(x),r],r]+t[y,r][h(x),r]+[t,r]y[h(x),r]=0.$$
$$t(y[h(x),r],r]+[y,r][h(x),r])+[t,r]y[h(x),r]=0.$$
$$t[y[h(x),r],r]+[t,r]y[h(x),r]=0.$$
$$[t,r]y[h(x),r]=0 \text{ for all } r,x,y\in R.$$

Since R is prime, either $[t,r]=0$ or $[h(x),x]=0$ for all $x\in R$. By lemma 2.9 R is commutative.

□

COROLLARY 3.1. *Let R be a 2-torsion free prime ring with involution $*$ of the second kind. Let h be a homoderivation which is zero-power valued on R, then the following are equivalent:*

(i) $h(x) \circ h(y) - x \circ y \in Z(R)$ *for all* $x, y \in R$.
(ii) $h(x) \circ h(y) + x \circ y \in Z(R)$ *for all* $x, y \in R$
(iii) *R is commutative.*

Moreover, if $h \neq 0$ and $h(x) \circ h(y) \in Z(R)$ for all $x, y \in R$, implies that R is commutative.

In [**12**], Theorem 3.7, the authors proved that if R is a 2-torsion free prime ring with involution of the second kind, and d be a non-zero derivation on R. Then R is commutative if and only if $h(x) \circ x^* \in Z(R)$ for all $x \in R$ which is also equivalent to h is $*$-centralizing on R. Applying theses conditions on homoderivation, we get the following theorem.

THEOREM 3.3. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind, and h be a non-zero homoderivation which is zero-power valued on R. Then the following are equivalent:*

(i) *h is $*$-centralizing on R.*
(ii) $h(x) \circ x^* \in Z(R)$ *for all* $x \in R$..
(iii) *R is commutative.*

PROOF. It is obvious that (iii) implies both of (i) and (ii). Now, to prove that $(i) \Rightarrow (iii)$ suppose that

$$[h(x), x^*] \in Z(R) \text{ for all } x \in R. \tag{3.34}$$

Replacing x by $x + y$ and using (3.34), we find that

$$[h(x), y^*] + [h(y), x^*] \in Z(R) \text{ for all } x, y \in R. \tag{3.35}$$

Replacing y by yd, where $d \in Z(R) \cap H(R)$, yields

$$[h(x), d^*y^*] + [h(yd), x^*] \in Z(R) \text{ for all } x, y \in R.$$
$$[h(x), dy^*] + [h(yd), x^*] \in Z(R) \text{ for all } x, y \in R.$$
$$d[h(x), y^*] + [h(x), d]y^* + [h(y)h(d), x^*] + [yh(d), x^*] + [h(y)d, x^*] \in Z(R)$$
for all $x, y \in R$.
$$d[h(x), y^*] + [h(x), d]y^* + h(y)[h(d), x^*] + [h(y), x^*]h(d) + y[h(d), x^*] + [y, x^*]h(d)$$
$$+h(y)[d, x^*] + [h(y), x^*]d \in Z(R) \text{ for all } x, y \in R.$$

(3.36)
$$[h(x), y^*]d + [y, x^*]h(d) + [h(y), x^*]d + [h(y), x^*]h(d) \in Z(R) \text{ for all } x, y \in R.$$

The relation (3.35), (3.36) reduces to

$$[y, x^*]h(d) + [h(y), x^*]h(d) \in Z(R) \text{ for all } x, y \in R.$$
$$[y + h(y), x^*]h(d) \in Z(R) \text{ for all } x, y \in R.$$

Since h is zero-power valued on R, we have

$$[y, x^*]h(d) \in Z(R) \text{ for all } x, y \in R. \tag{3.37}$$

Hence $[[y,x]h(d),r]=0$, for all $r\in R$, so

$$[y,x^*][h(d),r]+[[y,x^*],r]h(d)=0$$

Since $h(Z(R))\subseteq Z(R)$, so $h(d)\in Z(R)$ and

$$[[y,x^*],r]h(d)=0.$$

Replace r by rt for all $r,t\in R$, we have

$$[[y,x^*],r]th(d)=0 \text{ for all } r,t,x,y\in R.$$

thus,

$$[[y,x^*],r]Rh(d)=0 \text{ for all } r,x,y\in R. \tag{3.38}$$

By the primeness of R, we get $h(d)=0$ or $[[y,x^*],r]=0$. If $h(d)=0$, for all $d\in Z(R)\cap H(R)$, by lemma 3.1, we conclude that

$$h(z)=0 \text{ for all } z\in Z(R). \tag{3.39}$$

Substituting yz for y where $z\in Z(R)$ in (3.35), we get

$$[h(x),(yz)^*]+[h(yz),x^*]\in Z(R) \text{ for all } x,y\in R.$$

$$[h(x),z^*y^*]+[h(y)h(z),x^*]+[yh(z),x^*]+[h(y)z,x^*]\in Z(R) \text{ for all } x,y\in R.$$

$$z^*[h(x),y^*]+[h(x),z^*]y^*+h(y)[h(z),x^*]+[h(y),x^*]h(z)+y[h(z),x^*]+[y,x^*]h(z)$$
$$+h(y)[z,x^*]+[h(y),x^*]z\in Z(R) \text{ for all } x,y\in R.$$

$$z^*[h(x),y^*]+[h(y),x^*]h(z)+[y,x^*]h(z)+[h(y),x^*]z\in Z(R) \text{ for all } x,y\in R.$$

$$[h(x),y^*]z^*+[h(y),x^*]z\in Z(R) \text{ for all } x,y\in R. \tag{3.40}$$

From (3.35) , we have

$$[[h(x),y^*],r]+[[h(y),x^*],r]=0 \text{ for all } r,x,y\in R.$$

$$inoh_1 0[[h(y),x^*],r]=-[[h(x),y^*],r] \text{ for all } r,x,y\in R. \tag{3.41}$$

Using (3.35), (3.40) yields

$$[[h(x),y^*]z^*,r]+[[h(y),x^*]z,r]=0 \text{ for all } r,x,y\in R.$$

$$[[h(x),y^*],r]z^*+[[h(y),x^*],r]z=0 \text{ for all } r,x,y\in R.$$

$$[[h(x),y^*],r]z^*-[[h(x),y^*],r]z=0 \text{ for all } r,x,y\in R.$$

$$[[h(x),y^*],r](z^*-z)=0 \text{ for all } r,x,y\in R.$$

Replacing y^* by y , so

$$[[h(x),y],r](z^*-z)=0 \text{ for all } r,x,y\in R. \tag{3.42}$$

Since R is prime ring, either $[[h(x),y],r]=0$ or $z^*-z=0$ Since the involution is of the second kind we have $z^*-z\neq 0$, then

$$[[h(x),y],r]=0 \text{ for all } r,x,y\in R. \tag{3.43}$$

That is, $[h(x),x]\in Z(R)$ for all $x\in R$, thus, h is centralizing. By lemma 2.10 R is commutative.

If $[[y,x],r]=0$, then $[x,x^*]\in Z(R)$ for all $x\in R$. By Lemma 2.4 R is commutative. To prove that (ii)$\Rightarrow$ (iii). By hypothesis, we have

$$h(x)\circ x^*\in Z(R) \text{ for all } x\in R. \tag{3.44}$$

Replacing x by $x+y$ and using (3.44), we obtain

$$h(x)\circ y^*+h(y)\circ x^*\in Z(R) \text{ for all } x,y\in R. \tag{3.45}$$

Accordingly, we get

$$[h(x)\circ y^*,r]+[h(y)\circ x^*,r]=0 \text{ for all } r,x,y\in R. \tag{3.46}$$

Replacing y by yd, where $d\in Z(R)\cap H(R)$, and using (3.46), we obtain

$$[h(x)\circ(d^*y^*),r]+[h(yd)\circ x^*,r]=0.$$

$$[h(x)\circ(dy^*),r]+[(h(y)h(d)\circ x^*,r]+[(yh(d)\circ x^*,r]+[(h(y)d)\circ x^*,r]=0.$$

$$[d(h(x)\circ y^*),r]+[[h(x),d]y^*),r]+[(h(y)\circ x^*)h(d),r]+[h(y)[x^*,h(d)],r]$$

$$+[(y\circ x^*)h(d),r]+[y[h(d),x^*],r]+[(h(y)\circ x^*)d,r]+[h(y)[d,x^*],r]=0.$$

$$d[h(x)\circ y^*,r]+[h(y)\circ x^*,r]h(d)+[y\circ x^*,r]h(d)+[h(y)\circ x^*,r]d=0.$$

Using (3.46) we get

$$[(h(y)+y)\circ x^*,r]h(d)=0 \text{ for all } r,x,y\in R. \tag{3.47}$$

Since h is zero-power valued on R, we get

$$[y\circ x^*,r]h(d)=0 \text{ for all } r,x,y\in R. \tag{3.48}$$

And thus

$$[y\circ x,r]Rh(d)=0 \text{ for all } r,x,y\in R. \tag{3.49}$$

Since R is a prime, so either $[y\circ x,r]=0$ or $h(d)=0$ Assume $h(d)=0$, for all $d\in Z(R)\cap H(R)$. Using Lemma 3.1, we conclude that

$$h(z)=0 \text{ for all } z\in Z(R). \tag{3.50}$$

Replacing y by z in (3.46), we obtain

$$[h(x)\circ z^*,r]+[h(z)\circ x^*,r]=0.$$

$$[h(x)z^*,r]+[z^*h(x),r]=0.$$

$$2[h(x)z^*,r]=0.$$

$$[h(x)z^*,r]=0.$$

$$[h(x),r]z^*=0 \text{ for all } r,x\in R \textit{ and } z\in Z(R).$$

$$So;\ \ [h(x),r]z=0 \text{ for all } r,x\in R \textit{ and } z\in Z(R). \tag{3.51}$$

Taking $r=x$ and using the primeness of R, (3.51) yields

$$[h(x),x]=0 \text{ for all } x\in R. \tag{3.52}$$

By Lemma 2.9, we conclude that R is commutative.
If $[y\circ x,r]=0$ for all $r,x,y\in R$, then replacing y by z where $z\in Z(R)\backslash\{0\}$,

$$[z\circ x,r]=0.$$

$$[zx+xz,r]=0.$$

$$[zx, r] + [xz, r] = 0.$$
$$z[x, r] + [x, r]z = 0.$$
$$2[x, r]z = 0 \text{ for all } r, x \in R, z \in Z(R).$$

Since R is 2-torsion free, we get $[x, r]z = 0$ for all $r, x \in R$ and $z \in Z(R)\backslash\{0\}$. Using the primeness of R, we get $[x, r] = 0$ for all $r, x \in R$ that gives the commutativity of R. □

Example 3.1. Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$. The set R with matrix addition and multiplication is a prime ring . Let $h : R \to R$ be a zero homoderivation on R and $* : R \to R$ is a mapping defined as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $*$ is an involution of the first kind since $x^* = x$ for all $x \in Z(R)$ and $Z(R) \subseteq H(R)$. Now, $(h(x) \circ h(x^*)) \pm (x \circ x^*) \in Z(R)$ for all $x \in R$ Hence, the zero homoderivation satisfies the conditions of Theorem 3.2 but R is a not commutative. Hence the hypothesis of the second kind of involution is crucial in Theorem 3.2

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