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# ON HOMODERIVATIONS AND COMMUTATIVITY OF RINGS

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Abstract. Let $R$ be a ring admits a homoderivation $h$, $Z(R)$ is the center of $R$, and $I$ be a nonzero left ideal. In this paper, we proved the commutativity of the ring $R$ if $h(xy) - xy \in Z(R)$ and $h(xy) + xy \in Z(R)$ for all $x, y \in I$.

## 1. Introduction

Let $R$ be a ring with a center $Z(R)$ . The ring $R$ is called a prime if $aRb = 0$ either $a = 0$ or $b = 0$ for all $a, b \in R$ and is called semiprime ring if $aRa = 0$ then $a = 0$ for all $a \in R$. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$. An element $a \in R$ is called nilpotent if there exist a positive integer n such that $a^n = 0$. A prime ring is obviously semiprime and the center of a semiprime ring contains no nonzero nilpotent elements. A mapping $f : R \to R$ is said to be centralizing on $R$ if $[f(x), x] \in Z(R)$ for all $x \in R$ and is said to be commuting on $R$ if $[f(x), x] = 0$ for all $x \in R$. A derivation on $R$ is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. El-Sofy [**4**] defined a homderivation on $R$ as an additive map $h$ from $R$ into itself satisfies $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. For a positive integer $n(x) > 1$ in such that $f^{n(x)}(x) = 0$, then a map $f : R \to R$ is called zero-power valued for all $x \in R$ [**4**].

## 2. Centralizing Homoderivations.

Bell and W. S. Martindale [**2**] studied the commutativity of rings admitting centralizing derivation. Our purpose in this section is to prove the commutativity of the rings with centralizing homoderivations.

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Lemma 2.1 ([**6**], Lemma 4). *Let $b$ and $ab$ be in the center of a prime ring $R$. If $b \neq 0$, then $a$ is in $Z(R)$.*

Lemma 2.2. *Let $I$ be a nonzero left ideal of a prime ring $R$ and $h$ be a nonzero homoderivation on $R$, then $h$ is a nonzero on $I$.*

Proof. If $h(x) = 0$ for all $x \in I$ , then $h(rx) = 0$ and $r \in R$, it follows that $h(r)x = 0$ for all $r \in R$ , $x \in I$. Hence $h(R)I = 0$ , since $R$ is a prime ring and $I \neq 0$, then $h(R) = 0$ is a contradiction, so $h$ is a nonzero mapping on $I$. □

El-Sofy [**4**] proved that a prime ring $R$ of $char(R) \neq 2$, $I$ a nonzero right ideal of $R$ and $h$ be a nonzero homoderivation on $R$ such that $[x, d(x)] \in Z(R)$ for all $x \in I$, then $h$ is commuting on $I$. Using other technique of proof, we proved the result for left ideal.

Theorem 2.1. *Let $R$ be a semiprime ring of characteristics not 2, $I$ be a nonzero left ideal of $R$ and $h$ be a nonzero homoderivation on $R$ such that $h$ is centralizing. Then $h$ is commuting on $I$.*

Proof. Let $x \in I$. Then $[x^2, h(x^2)] \in Z(R)$ by hypothesis. Now

$$\begin{aligned}
[x^2, h(x^2)] &= [x^2, h(x)h(x) + xh(x) + h(x)x] \\
&= [x^2, h(x)h(x) + 2xh(x) - [x, h(x)]] \\
&= [x^2, h(x)h(x) + 2xh(x)] + [x^2, -[x, h(x)]] \\
&= [x^2, h(x)h(x) + 2xh(x)] \\
&= [x^2, h(x)h(x)] + [x^2, 2xh(x)] \\
&= x[x, h(x)h(x)] + [x, h(x)h(x)]x + x[x, 2xh(x)] + [x, 2xh(x)]x \\
&= xh(x)[x, h(x)] + x[x, h(x)]h(x) + h(x)[x, h(x)]x + [x, h(x)]h(x)x \\
&\quad + 2x^2[x, h(x)] + 2x[x, h(x)]x \\
&= 2xh(x)[x, h(x)] + 2h(x)x[x, h(x)] + 4x^2[x, h(x)] \\
&= 2x(x + h(x))[x, h(x)] + 2(h(x) + x)x[x, h(x)] \\
&= (2x(x + h(x)) + 2(h(x) + x)x)[x, h(x)] \in Z(R) \\
&= 2(2x^2 + h(x)x + xh(x))[x, h(x)] \in Z(R)
\end{aligned}$$

By lemma 2.1 , if $[x, h(x)] \neq 0$, then

$$2(2x^2 + h(x)x + xh(x)) \in Z(R)$$

$$So \quad 2[2x^2 + h(x)x + xh(x), x] = 0$$

Since $char(R) \neq 2$

$$[2x^2 + h(x)x + xh(x), x] = 0$$

$$[h(x)x, x] + [xh(x), x] = 0$$

$$2x[h(x), x] = 0$$

$$x[h(x), x] = 0$$

$$x[h(x), x]^2 = 0$$

$$[x, h(x)]^3 = 0$$

Since the center of a semiprime ring contains no nonzero nilpotent elements, so

$$[x, h(x)] = 0 \text{ for all } x \in I.$$

So, $h$ is commuting. □

COROLLARY 2.1. *Let $R$ be a prime ring of characteristics not 2, $I$ be a nonzero left ideal of $R$ and $h$ be a nonzero homoderivation on $R$ such that $h$ is centralizing. Then $h$ is commuting on $I$.*

COROLLARY 2.2 ([**4**], Corollary 3.4.8). *Let $R$ be a prime ring of characteristic not 2 and $I$ a two sided ideal of $R$. If $R$ admits a nonzero homoderivation $h$ which is centralizing and zero-power valued on $I$, then $R$ is commutative.*

## 3. Commutative of Prime Ring

Ashraf and Nadeem Ur-Rehman [**1**] proved the commutativity of prime ring $R$ admitting a derivation $d$ that satisfies any one of the properties $d(xy) - xy \in Z(R)$ *and* $d(xy) + xy \in Z(R)$, for all $x, y$ in nonzero ideal $I$. Our purpose in this section is to prove a similar result regarding homoderivations.

LEMMA 3.1. ([**6**], Lemma 3) *If a prime ring $R$ contains a commutative nonzero right ideal,then $R$ is commutative.*

THEOREM 3.1. *Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a homderivation $h$ which is zero-power valued on $I$ such that $h(xy) - xy \in Z(R)$, for all $x, y \in I$, then $R$ is commutative.*

PROOF. We have, $h(xy) - xy \in Z(R)$ for all $x, y \in I$. If $h = 0$, then $xy \in Z(R)$. Then, $[xy, x] = 0$ for all $x, y \in I$, so, $x[y, x] = 0$. Replace $y$ by $yz$ where $z \in R$, we have $xy[z, x] = 0$ for all $x, y \in I$. Then $xRI[z, x] = 0$ for all $x, z \in I$. Since $R$ is a prime ring and $I \neq 0$ then $I[z, x] = 0$ for all $x, z \in I$. Then $[z, x] = 0$ for all $x, z \in I$. By lemma 3.1, $R$ is commutative.

If $h \neq 0$, $h(x)h(y) + xh(y) + h(x)y - xy \in Z(R)$, replacing $y$ by $yz$ , we get

$$h(x)h(yz) + xh(yz) + h(x)yz - xyz \in Z(R)$$

$$h(x)h(y)h(z) + h(x)yh(z) + h(x)h(y)z + xh(y)h(z)$$
$$+xyh(z) + xh(y)z + h(x)yz - xyz \in Z(R)$$

$$[(h(x)h(y) + h(x)y + xh(y) + xy)h(z) + (h(x)h(y) + xh(y) + h(x)y - xy)z, z] = 0$$
$$[(h(xy) + xy)h(z), z] + [(h(xy) - xy)z, z] = 0$$
$$[(h(xy) + xy)h(z), z] = 0$$

Since $h$ is zero-power valued, we get

$$[xyh(z), z] = 0$$

$$xy[h(z), z] + x[y, z]h(z) + [x, z]yh(z) = 0 \text{ for all } x, y, z \in I. \tag{3.1}$$

For any $y_1 \in I$ , replace $x$ by $y_1x$

$$y_1xy[h(z), z] + y_1x[y, z]h(z) + [y_1x, z]yh(z) = 0$$

$$y_1xy[h(z),z] + y_1x[y,z]h(z) + y_1[x,z]yh(z) + [y_1,z]xyh(z) = 0$$
$$y_1(xy[h(z),z] + x[y,z]h(z) + [x,z]yh(z)) + [y_1,z]xyh(z) = 0$$

From (3.1) we get ;

$$[y_1,z]xyh(z) = 0$$
$$[y_1,z]xRIh(z) = 0.$$

Since $R$ is prime ring we get, either $[y_1,z]x = 0$ or $Ih(z) = 0$. The set of $z \in I$ for which these two properties hold are additive subgroups of $I$ whose union is $I$. Therefore either $Ih(z) = (0)$, for all $z \in I$ or $[y_1,z]x = 0$, for all $x, y_1, z \in I$.

If $Ih(z) = 0$, for all $z \in I$, then $IRh(z) = (0)$, for all $z \in I$. Since $I \neq (0)$, and $R$ is prime, then $h(z) = 0$, for all $z \in I$. This implies that $h(zr) = h(z)h(r) + zh(r) + h(z)r = 0$, for all $z \in I$, and $r \in R$. Hence $zh(r) = 0$ that is $IRh(r) = (0)$. Since $I \neq 0$, so, $h = 0$, this is a contradiction. On other hand if $[y_1,z]x = 0$ for all $x,y,z \in I$, $[y_1,z]RI = 0$ for all $x,y,z \in I$. By the primeness of $R$ then $[y_1,z] = 0$ for all $x,y,z \in I$. Hence $R$ is commutative by Lemma 3.1. □

THEOREM 3.2. *Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. If $R$ admits a homderivation $h$ which is zero-power valued on $I$ such that $h(xy) + xy \in Z(R)$, for all $x, y \in I$ , then $R$ is commutative.*

PROOF. If $h$ is a homoderivation satisfying $h(xy) + xy \in Z(R)$ ,for all $x, y \in I$ . Then $(-h)(xy) - xy \in Z(R)$. Then by Theorem 3.1, $R$ is commutative. □

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