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THE EXACT AND THE SHARP UPPER BOUND FOR MULTIPLICATIVE ZAGREB INDICES OF GRAPH PRODUCT

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Abstract. In this paper, we determine the exact formula for the multiplicative Zagreb indices of tensor product. Also we find the sharp upper bound for the multiplicative first Zagreb index of strong product of two connected graphs and using this result we compute the exact formula for the multiplicative first Zagreb index of strong product of two complete graphs.

1. Introduction

In this paper, all graphs considered are simple and connected graphs. We denote the vertex and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex v in G. The number of elements in the vertex set of a graph G is called the order of G and is denoted by $v(G)$. The number of elements in the edge set of a graph G is called the size of G and is denoted by $e(G)$. A graph with order n and size m is called a (n, m)-graph. For any $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. A graph G is complete, if every pair of its vertices are adjacent. A complete graph on n vertices is denoted by K_n.

A topological index of a graph is a parameter related to the graph, it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacological, toxicological, biological and other properties of chemical compounds [**5**]. Several types of such indices exist, especially those based on vertex and edge distance. One of the oldest intensively studied

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topological indices is the Wiener index. In 1947, Wiener [**9**] introduced the first distance-based topological index which is named as Wiener index and it is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v).$$

Its chemical applications and Mathematical properties are well studied in [**3**].

There are some topological indices based on degrees known as the first and second Zagreb indices of molecular graphs. The first and second kinds of Zagreb indices are introduced by Gutman et al. in [**4**]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a graph G are defined as

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)] = \sum_{v\in V(G)} d_G^2(v).$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$$

In 2010, Todeschini et al. [**7, 8**] have proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows:

$$\prod_1 = \prod_1(G) = \prod_{v\in V(G)} d_G^2(v), \quad \prod_2 = \prod_2(G) = \prod_{uv\in E(G)} d_G(u)d_G(v).$$

Mathematical properties and applications of multiplicative Zagreb indices are reported in [**2**].

The strong product [1] of graphs G_1 and G_2 is denoted by $G_1 \boxtimes G_2$, and it is the graph with vertex set $V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) are adjacent if $(i)u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$, or $(iii)u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. The tensor product of the graphs G_1 and G_2, denoted by $G_1\times G_2$, has the vertex set $V(G_1\times G_2)$ and $E(G_1\times G_2) = \big\{(u_1,v_1)(u_2,v_2)|u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\big\}$.

LEMMA 1.1 ([**2**]). *Let $x_1, x_2, ..., x_n$ be non-negative numbers. Then*

$$\frac{x_1 + x_2 + ... + x_n}{n} \geqslant \sqrt[n]{x_1x_2...x_n}.$$

LEMMA 1.2 ([**6**]). (a) *The degree of a vertex (u_i, v_j) of $G_1\times G_2$ is given by*

$$d_{G_1\times G_2}(u_i,v_j) = d_{G_1}(u_i)d_{G_2}(v_j).$$

(b) *Let x_{ij} denote the vertex (u_i,u_j) of $G\boxtimes K_r$. Now $d_{G\boxtimes K_r}(x_{ij}) = rd_G(u_i)+(r-1)$ and*

$$d_{G\boxtimes K_r}(x_{ij},x_{kp}) = \begin{cases} 1, & i=k, j\neq p \\ d_G(u_i,u_k), & i\neq k, j = p \\ d_G(u_i,u_k), & i\neq k, j\neq p. \end{cases}$$

The degree of the vertex (u_i,v_j) of $V(G_1\boxtimes G_2)$ is

$$d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j),$$

that is

$$d_{G_1 \boxtimes G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) d_{G_2}(v_j).$$

LEMMA 1.3. *Let G be a graph. Then*

$$\sum_{xy \in G} 1 = 2e(G)$$

PROOF.

$$\sum_{xy \in G} 1 = 2 \sum_{xy \in E(G)} 1 = 2e(G)$$

□

2. The Multiplicative Zagreb indices of $G_1 \times G_2$

In this section, we compute the multiplicative Zagreb indices of the tensor product of graphs.

THEOREM 2.1. *Let G_1 be a graph with n- vertices and G_2 be a graph with r-vertices. Then*

$$\prod_1 (G_1 \times G_2) = \Big[\prod_1 (G_1)\Big]^r \Big[\prod_1 (G_2)\Big]^n.$$

PROOF. Let $V(G_1) = \{u_0, u_1, u_2, ..., u_{n-1}\}$, $V(G_2) = \{v_0, v_1, v_2, ..., v_{r-1}\}$ and $w_{ij} = (u_i, v_j)$.

$$\begin{aligned}
\prod_1 (G_1 \times G_2) &= \prod_{w_{ij} \in V(G_1 \times G_2)} d^2_{G_1 \times G_2}(w_{ij}) \\
&= \prod_{i=0}^{n-1} \prod_{j=0}^{r-1} \Big[d_{G_1 \times G_2}(u_i, v_j)\Big]^2 \\
&= \prod_{i=0}^{n-1} \prod_{j=0}^{r-1} \Big[d_{G_1}(u_i) d_{G_2}(v_j)\Big]^2 \\
&= \prod_{i=0}^{n-1} \prod_{j=0}^{r-1} \Big[d^2_{G_1}(u_i) d^2_{G_2}(v_j)\Big] \\
&= \Big[\prod_{i=0}^{n-1} d^2_{G_1}(u_i)\Big]^r \Big[\prod_{j=0}^{r-1} d^2_{G_2}(v_j)\Big]^n \\
&= \Big[\prod_1 (G_1)\Big]^r \Big[\prod_1 (G_2)\Big]^n
\end{aligned}$$

□

THEOREM 2.2. *Let G_1 be a graph with n- vertices and G_2 be a graph with r-vertices. Then*

$$\prod_2 (G_1 \times G_2) = \Big[\prod_2 (G_1)\Big]^{2e(G_2)} \Big[\prod_2 (G_2)\Big]^{2e(G_1)}$$

PROOF. Let $V(G_1) = \{u_0, u_1, u_2, ..., u_{n-1}\}$, $V(G_2) = \{v_0, v_1, v_2, ..., v_{r-1}\}$ and $w_{ij} = (u_i, v_j)$. Then

$\prod_2(G_1 \times G_2) = \prod_{(w_{ij},w_{pq})\in E(G_1\times G_2)} d_{G_1\times G_2}(w_{ij})d_{G_1\times G_2}(w_{pq})$

$= \prod_{(u_i,u_p)\in E(G_1)} \quad \prod_{(v_j,v_q)\in E(G_2)} d_{G_1\times G_2}(w_{ij})d_{G_1\times G_2}(w_{pq})d_{G_1\times G_2}(w_{pj})d_{G_1\times G_2}(w_{iq})$

$= \prod_{(u_i,u_p)\in E(G_1)} \quad \prod_{(v_j,v_q)\in E(G_2)} d^2_{G_1}(u_i)d^2_{G_1}(u_p)d^2_{G_2}(v_j)d^2_{G_2}(v_q)$

$= \Big\{ \prod_{(u_i,u_p)\in E(G_1)} d^2_{G_1}(u_i)d^2_{G_1}(u_p)\Big\}^{e(G_2)} \quad \Big\{\prod_{(v_j,v_q)\in E(G_2)} d^2_{G_2}(v_j)d^2_{G_2}(v_q)\Big\}^{e(G_1)}$

$= \Big\{\Big[\prod_{(u_i,u_p)\in E(G_1)}[d_{G_1}(u_i)d_{G_1}(u_p)\Big]^2\Big\}^{e(G_2)} \quad \Big\{\Big[\prod_{(v_j,v_q)\in E(G_2)} d_{G_2}(v_j)d_{G_2}(v_q)\Big]^2\Big\}^{e(G_1)}$

$= \Big[\prod_2(G_1)\Big]^{2e(G_2)} \quad \Big[\prod_2(G_2)\Big]^{2e(G_1)}$. □

3. The Multiplicative Zagreb indices of $G_1 \boxtimes G_2$

In this section, we compute the multiplicative first Zagreb index of $G_1 \boxtimes G_2$.

THEOREM 3.1. *Let G_1 be a graph with n- vertices and G_2 be a graph with r-vertices. Then*

$$\prod_1(G_1 \boxtimes G_2) \leqslant \Big\{ \frac{1}{nr}\Big[rM_1(G_1) + nM_1(G_2) + M_1(G_1)M_1(G_2) + 8e(G_1)e(G_2) + 4M_1(G_1)e(G_2) + 4M_1(G_2)e(G_1)\Big]\Big\}^{nr}$$

PROOF. Let $V(G_1) = \{u_0, u_1, u_2, ..., u_{n-1}\}$, $V(G_2) = \{v_0, v_1, v_2, ..., v_{r-1}\}$ and $w_{ij} = (u_i, v_j)$. $\prod_1(G_1 \boxtimes G_2) = \prod_{w_{ij}\in V(G_1\boxtimes G_2)} d^2_{G_1\boxtimes G_2}(w_{ij})$

$= \prod_{i=0}^{n-1} \quad \prod_{j=0}^{r-1} d_{G_1\boxtimes G_2}((u_i, v_j))^2$

$= \prod_{i=0}^{n-1} \quad \prod_{j=0}^{r-1} \Big[d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i)d_{G_2}(v_j)\Big]^2$

$= \prod_{i=0}^{n-1} \quad \prod_{j=0}^{r-1} \Big[d^2_{G_1}(u_i) + d^2_{G_2}(v_j) + d^2_{G_1}(u_i)d^2_{G_2}(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j)$

$+2d^2_{G_1}(u_i)d_{G_2}(v_j) + 2d_{G_1}(u_i)d^2_{G_2}(v_j)\Big]$

$\leqslant Big[\frac{1}{nr}\Big\{ \sum_{i=0}^{n-1} \quad \sum_{j=0}^{r-1} \Big(d^2_{G_1}(u_i) + d^2_{G_2}(v_j) + d^2_{G_1}(u_i)d^2_{G_2}(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j)$

$+2d^2_{G_1}(u_i)d_{G_2}(v_j) + 2d_{G_1}(u_i)d^2_{G_2}(v_j)\Big)\Big\}\Big]^{nr}$

$= \Big[\frac{1}{nr}\Big\{ \sum_{i=0}^{n-1} d^2_{G_1}(u_i) \quad \sum_{j=0}^{r-1} 1 + \sum_{i=0}^{n-1} 1 \quad \sum_{j=0}^{r-1} d^2_{G_2}(v_j)$

$+ \sum_{i=0}^{n-1} d^2_{G_1}(u_i) \quad \sum_{j=0}^{r-1} d^2_{G_2}(v_j)$

$+2\sum_{i=0}^{n-1} d_{G_1}(u_i) \quad \sum_{j=0}^{r-1} d_{G_2}(v_j) + 2\sum_{i=0}^{n-1} d^2_{G_1}(u_i) \quad \sum_{j=0}^{r-1} d_{G_2}(v_j)$

$+2\sum_{i=0}^{n-1} d_{G_1}(u_i) \quad \sum_{j=0}^{r-1} d^2_{G_2}(v_j)\Big\}\Big]^{nr}$

Thus

[3pt] $\prod_1(G_1 \boxtimes G_2) \leqslant \Big\{ \frac{1}{nr}\Big[rM_1(G_1) + nM_1(G_2) + M_1(G_1)M_1(G_2)$

$$+8e(G_1)e(G_2)+4M_1(G_1)e(G_2)+4M_1(G_2)e(G_1)\Big]\Big\}^{nr}. \qquad \square$$

THEOREM 3.2.

$$\prod_1(K_n \boxtimes K_r) = (nr-1)^{2nr}.$$

PROOF. The degree of every vertex in $K_n \boxtimes K_r$ is

$$r(n-1)+(r-1)=(rn-1).$$

Therefore $K_n \boxtimes K_r$ is a complete graph. Hence

$$\prod_1(K_n \boxtimes K_r) = (nr-1)^{2nr}. \tag{3.1}$$

$\square$

REMARK 3.1. Using Theorem 3.2, we show that the upper bound in Theorem 3.1 is sharp. Clearly, $M_1(K_n)=n(n-1)^2$, $e(K_n)=\frac{n(n-1)}{2}$, when $G_1=K_n$ and $G_2=K_r$, the upper bound in Theorem 3.1 becomes

$$\prod_1(K_n \boxtimes K_r) \leqslant \Big\{ \frac{1}{nr}\Big[rM_1(K_n)+nM_1(K_r)+M_1(K_n)M_1(K_r)$$

$$+8e(K_n)e(K_r)+4M_1(K_n)e(K_r)+4M_1(K_r)e(K_n)\Big]\Big\}^{nr}$$

$$=\Big\{ \frac{1}{nr}\Big[rn(n-1)^2+nr(r-1)^2+nr(n-1)^2(r-1)^2$$

$$+8\ \frac{n(n-1)}{2}\ \frac{r(r-1)}{2}+4n(n-1)^2\ \frac{r(r-1)}{2}+4\ \frac{n(n-1)}{2}r(r-1)^2\Big]\Big\}^{nr}$$

So,

$$\prod_1(K_n \boxtimes K_r) \quad \leqslant \quad (nr-1)^{2nr} \tag{3.2}$$

From (3.1) and (3.2), we conclude that the upper bound is sharp.

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