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CERTAIN PROPERTIES OF THE NIELSEN's β-FUNCTION

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Abstract. By using some analytical techniques, we present some properties of the Nielsen's β-function. The results established are analogous to some known works involving the gamma and digamma functions.

1. Introduction

In 1974, Gautschi [**3**] presented an interesting inequality involving the classical Euler's Gamma function, $\Gamma(x)$. He proved that, for $x > 0$, the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is always greater than or equal to 1. That is,

$$1 \leqslant \frac{2\Gamma(x)\Gamma(1/x)}{\Gamma(x)+\Gamma(1/x)}, \quad x > 0, \tag{1.1}$$

with equality if $x = 1$. As a direct consequence of (1.1), the inequalities

$$2 \leqslant \Gamma(x) + \Gamma(1/x), \quad x > 0, \tag{1.2}$$

and

$$1 \leqslant \Gamma(x)\Gamma(1/x), \quad x > 0, \tag{1.3}$$

are obtained. Then recently, Alzer and Jameson [**1**] established a striking companion of (1.1) which involves the digamma function, $\psi(x)$. They proved that the inequality

$$-\gamma \leqslant \frac{2\psi(x)\psi(1/x)}{\psi(x)+\psi(1/x)}, \quad x > 0, \tag{1.4}$$

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holds, with equality if $x = 1$, where $\gamma = 0.57721, ...$ is the Euler-Mascheroni constant. In addition, they proved that

$$P(x) = \psi(x) + \psi(1/x), \tag{1.5}$$

is strictly concave on $(0, \infty)$ and that

$$\psi(x) + \psi(1/x) < -2\gamma, \quad x > 0, x \neq 1. \tag{1.6}$$

$$\psi(1 + y)\psi(1 - y) < \gamma^2, \quad y \in (0, 1). \tag{1.7}$$

$$\psi(x)\psi(1/x) < \gamma^2, \quad x > 0, x \neq 1. \tag{1.8}$$

Also, in [**11**], it was established among other things that the function

$$h_1 = \psi\left(x + \frac{1}{2}\right) - \psi(x) - \frac{1}{2x}, \tag{1.9}$$

is strictly decreasing and convex on $(0, \infty)$. Motivated by the result (1.9), Mortici [**6**] proved that the generalized function

$$f_a = \psi(x + a) - \psi(x) - \frac{a}{x}, \quad a \in (0, 1), \tag{1.10}$$

is strictly completely monotonic on $(0, \infty)$.

Inspired by the above results, the purpose of this paper is to establish analogous results for the Nielsen's β-function.

2. Preliminary Definitions

The Nielsen's β-function may be defined by any of the following equivalent forms (see [**2**], [**4**], [**7**], [**10**]).

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt, \quad x > 0, \tag{2.1}$$

$$= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt, \quad x > 0, \tag{2.2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad x > 0, \tag{2.3}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \quad x > 0, \tag{2.4}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the digamma or psi function and $\Gamma(x)$ is the Euler's Gamma function. It is known to satisfy the properties:

$$\beta(x + 1) = \frac{1}{x} - \beta(x), \tag{2.5}$$

$$\beta(x) + \beta(1 - x) = \frac{\pi}{\sin \pi x}. \tag{2.6}$$

Some particular values of the function are $\beta(1) = \ln 2$, $\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}$, $\beta\left(\frac{3}{2}\right) = 2 - \frac{\pi}{2}$ and $\beta(2) = 1 - \ln 2$.

By differentiating n-times of (2.1), (2.2), (2.3), (2.4) and (2.5), one obtains

$$\beta^{(n)}(x) = \int_0^1 \frac{(\ln t)^n t^{x-1}}{1+t}\, dt, \quad x > 0 \tag{2.7}$$

$$= (-1)^n \int_0^\infty \frac{t^n e^{-xt}}{1+e^{-t}}\, dt, \quad x > 0 \tag{2.8}$$

$$= (-1)^n n! \sum_{k=0}^\infty \frac{(-1)^k}{(k+x)^{n+1}}, \quad x > 0 \tag{2.9}$$

$$= \frac{1}{2^{n+1}} \left\{ \psi^{(n)}\left(\frac{x+1}{2}\right) - \psi^{(n)}\left(\frac{x}{2}\right) \right\}, \quad x > 0 \tag{2.10}$$

$$\beta^{(n)}(x+1) = \frac{(-1)^n n!}{x^{n+1}} - \beta^{(n)}(x), \quad x > 0 \tag{2.11}$$

where $n \in \mathbb{N}_0$ and $\beta^{(0)}(x) = \beta(x)$.
For additional information on this special function, one may refer to [**7**], [**8**], [**9**] and the related references therein.

3. Main Results

Lemma 3.1. *The function $x\beta(x)$ is decreasing and convex on $(0,\infty)$. Consequently, the inequalities*

$$\beta(x) + x\beta'(x) < 0, \quad x > 0, \tag{3.1}$$

and

$$2\beta'(x) + x\beta''(x) > 0, \quad x > 0, \tag{3.2}$$

are satisfied.

Proof. In Theorem 3 of [**9**], the function $x\left|\beta^{(m)}(x)\right|$, $x > 0$, $m \in \mathbb{N}_0$ was proved to be completely monotonic. Thus, $x\beta(x)$ (i.e. the case where $m = 0$) is completely monotonic. Since every completely monotonic function is decreasing and convex [**5**], we conclude that $x\beta(x)$ is decreasing and convex. These give rise to inequalities (3.1) and (3.2). □

Theorem 3.2. *The function*

$$Q(x) = \beta(x) + \beta(1/x), \tag{3.3}$$

is strictly convex on $(0,\infty)$.

Proof. By direct differentiation, and by applying (3.2), we obtain

$$Q'(x) = \beta'(x) - \frac{1}{x^2}\beta'(1/x),$$

$$Q''(x) = \beta''(x) + \frac{2}{x^3}\beta'(1/x) + \frac{1}{x^4}\beta''(1/x)$$

$$= \beta''(x) + \frac{1}{x^3}\left[2\beta'(1/x) + \frac{1}{x}\beta''(1/x)\right] > 0,$$

which completes the proof. □

Theorem 3.3. *The inequality*

$$\beta(x) + \beta(1/x) \geqslant 2\ln 2, \tag{3.4}$$

holds for $x > 0$.

Proof. Let $Q(x)$ be defined as in (3.3). Since $Q''(x) > 0$, then $(Q'(x))' > 0$ which implies that $Q'(x)$ is increasing. Then $Q'(x) \leqslant Q'(1) = 0$ for $x \in (0, 1]$ and $Q'(x) \geqslant Q'(1) = 0$ for $x \in [1, \infty)$. These imply that $Q(x)$ is decreasing on $(0, 1]$ and increasing on $[1, \infty)$. Therefore, in either case, we have $Q(x) \geqslant Q(1) = 2\ln 2$ which gives the desired result. □

Theorem 3.4. *The inequality*

$$\beta(1+s)\beta(1-s) \geqslant (\ln 2)^2, \tag{3.5}$$

holds for $s \in [0, 1)$.

Proof. Since $\beta(x)$ is logarithmically convex (see [**7**]), then we have

$$\beta\left(\frac{x+y}{2}\right) \leqslant \sqrt{\beta(x)\beta(y)}, \tag{3.6}$$

for $x > 0$ and $y > 0$. Now, by letting $x = 1 + s$ and $y = 1 - s$ in (3.6), we obtain the desired result (3.5). □

Theorem 3.5. *The inequality*

$$\beta(x)\beta(1/x) \geqslant (\ln 2)^2, \tag{3.7}$$

holds for $x > 0$.

Proof. If $x \geqslant 1$, then $0 < 1/x \leqslant 1$. Also, if $0 < x \leqslant 1$, then $1/x \geqslant 1$. Hence it suffices to prove (3.7) for $x \geqslant 1$. For $x \geqslant 1$ and $s \in [0, 1)$, let $x = 1 + s$ and $1/x = 1 - s$. Then by (3.5), we obtain

$$\beta(x)\beta(1/x) = \beta(1+s)\beta(1-s) \geqslant (\ln 2)^2,$$

which concludes the proof. □

Theorem 3.6. *For* $x, y \in [1, \infty)$*, the inequality*

$$\frac{2\beta(x)\beta(y)}{\beta(x)+\beta(y)} \leqslant \ln 2, \tag{3.8}$$

is satisfied. In other words, for $x, y \in [1, \infty)$*, the harmonic mean of* $\beta(x)$ *and* $\beta(y)$ *is at most* $\ln 2$.

Proof. Note that for $v \in [1, \infty)$, we have $\beta(v) \leqslant \beta(1) = \ln 2$, since $\beta(v)$ is decreasing. Thus, $[\beta(v)]^2 \leqslant (\ln 2)\beta(v)$ for all $v \in [1, \infty)$. Now, let $x, y \in [1, \infty)$. Then, we have

$$2\beta(x)\beta(y) \leqslant [\beta(x)]^2 + [\beta(y)]^2 \leqslant (\ln 2)\left[\beta(x) + \beta(y)\right],$$

which gives the desired result. □

In view of the harmonic mean inequalities (1.1) and (1.4), we give the following conjecture.

CONJECTURE 3.7. For $x \in (0, \infty)$, the inequality

$$\frac{2\beta(x)\beta(1/x)}{\beta(x) + \beta(1/x)} \leqslant \ln 2, \tag{3.9}$$

is satisfied, with equality if $x = 1$.

THEOREM 3.8. *The double inequality*

$$\frac{1}{x} - \ln 2 < \beta(x) < \frac{1}{x}, \tag{3.10}$$

holds for $x \in (0, \infty)$.

PROOF. As a a direct consequence of (2.5), we obtain

$$\beta(x) < \frac{1}{x}, \tag{3.11}$$

for $x \in (0, \infty)$. Also, by (2.5), we obtain the limit

$$\lim_{x \to 0^+} \left\{ \frac{1}{x} - \beta(x) \right\} = \ln 2. \tag{3.12}$$

Now, let $\theta(x) = \frac{1}{x} - \beta(x)$ for $x \in (0, \infty)$. Then by (2.11), we obtain

$$\theta'(x) = -\frac{1}{x^2} - \beta'(x) < 0,$$

which shows that $\theta(x)$ is decreasing. Hence

$$\frac{1}{x} - \beta(x) = \theta(x) < \lim_{x \to 0^+} \theta(x) = \ln 2. \tag{3.13}$$

Then, by combining (3.11) and (3.13), we obtain the result (3.10). □

THEOREM 3.9. *The limit*

$$\lim_{z \to 0^+} \frac{1}{z} \left\{ \frac{1}{\beta(1-z)} - \frac{1}{\beta(1+z)} \right\} = -\frac{\pi^2}{6(\ln 2)^2}, \tag{3.14}$$

is valid for $z \in (0, 1)$.

PROOF. It can be shown from relation (2.4) that $\beta'(1) = -\frac{\pi^2}{12}$. Then by L'Hopital's rule, we obtain

$$\begin{aligned} \lim_{z \to 0^+} \frac{1}{z} \left\{ \frac{1}{\beta(1-z)} - \frac{1}{\beta(1+z)} \right\} &= \lim_{z \to 0^+} \left\{ \frac{\beta'(1-z)}{[\beta(1-z)]^2} + \frac{\beta'(1+z)}{[\beta(1+z)]^2} \right\} \\ &= -\frac{\pi^2}{6(\ln 2)^2}. \end{aligned}$$

□

THEOREM 3.10. *For $a > 0$ and $x \in (0, \infty)$, let f_a be defined as*

$$f_a(x) = \beta(x + a) - \beta(x) - \frac{a}{x}. \tag{3.15}$$

Then $-f_a$ is strictly completely monotonic.

Proof. Recall that a function $f : (0,\infty) \to \mathbb{R}$ is said to be completely monotonic on $(0,\infty)$ if f has derivatives of all order and $(-1)^n f^{(n)}(x) \geqslant 0$ for all $x \in (0,\infty)$ and $n \in \mathbb{N}$. Let

$$h_a(x) = -f_a(x) = \frac{a}{x} + \beta(x) - \beta(x+a).$$

Then by repeated differentiation and by using (2.8), we obtain

$$\begin{aligned}
h_a^{(n)}(x) &= (-1)^n a \frac{n!}{x^{n+1}} + \beta^{(n)}(x) - \beta^{(n)}(x+a) \\
&= (-1)^n a \int_0^\infty t^n e^{-xt}\,dt + (-1)^n \int_0^\infty \frac{t^n e^{-xt}}{1+e^{-t}}\,dt \\
&\quad - (-1)^n \int_0^\infty \frac{t^n e^{-(x+a)t}}{1+e^{-t}}\,dt, \\
(-1)^n h_a^{(n)}(x) &= a \int_0^\infty t^n e^{-xt}\,dt + \int_0^\infty \frac{t^n e^{-xt}}{1+e^{-t}}\,dt - \int_0^\infty \frac{t^n e^{-(x+a)t}}{1+e^{-t}}\,dt \\
&= \int_0^\infty \left[a + \frac{1-e^{-at}}{1+e^{-t}} \right] t^n e^{-xt}\,dt > 0,
\end{aligned}$$

which completes the proof. □

Corollary 3.11. *The inequality*

$$0 < \beta(x) - \beta(x+a) + \frac{a}{x} \leqslant \ln 2 + a - \frac{1}{a} + \beta(a), \tag{3.16}$$

holds for $a > 0$ and $x \in [1,\infty)$.

Proof. Since $h_a(x)$ is completely monotonic on $(0,\infty)$, then it is decreasing on $(0,\infty)$. Then for $x \in [1,\infty)$, we have

$$\begin{aligned}
0 = \lim_{x\to\infty} h_a(x) < h_a(x) \leqslant h_a(1) &= a + \beta(1) - \beta(1+a) \\
&= \ln 2 + a - \frac{1}{a} + \beta(a),
\end{aligned}$$

yielding the desired result. □

Remark 3.12. In particular, if $a = \frac{1}{2}$, we obtain the sharp inequality

$$0 < \beta(x) - \beta\left(x + \frac{1}{2}\right) + \frac{1}{2x} \leqslant \ln 2 + \frac{\pi - 3}{2}, \tag{3.17}$$

for $x \in [1,\infty)$. If $x \in (0,1]$, then the right-hand sides of (3.16) and (3.17) are reversed.

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