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DOMINATION AND TOTAL DOMINATION NUMBER OF BLOCK-TRANSFORMATION GRAPH G^{101}

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Abstract. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or adjacent to the element of S i.e. $N[S] = V(G)$. The domination number γ(G) of a graph is the minimum cardinality of a dominating set in G.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a total dominating set if every vertex $v \in V$ there exists a vertex $u \in S$, $u \neq v$; u is adjacent to v. i.e. $N(S) = V(G)$. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set. This concept was introduced by Cockayne, Dawes and Hedetniemi in [**5**]. In this paper, we determine domination as well as total domination number for block-transformation of some class of graphs.

1. Introduction

The graphs considered here are finite, undirected and without multiple loops or multiple edges. We follow [**1, 3**] for unexplained terminology and notation. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. A block is connected, nontrivial graph having no cut vertices. If B is a block of G with its vertex set $V(B) = \{u_1, u_2, \ldots u_r; r \geqslant 2\}$ then we say that u_i $(1 \leqslant i \leqslant r)$, and B are incident with each other. If two blocks B_i and B_j of G, have a common cutvertex, then we say that B_i and B_j are adjacent blocks of G. Let $U(G)$ denote the block set of G and is the set $\{B_i : B_i \text{ is a block of} G\}$. The general concept of the block-transformation graph G was introduced in [**1**]. Let $G = (V(G), E(G))$ be a graph with block set $U(G)$, and α, β, γ be three variables having the values 0 or 1. The block-transformation graph G is the graph having $V(G) \cup U(G)$, as

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its vertex set. For any two vertices x and y in $V(G) \cup U(G)$ we define α, β, γ as follows:

(i) Suppose x, y are in $V(G)$. $\alpha = 1$ if x and y are adjacent in G. $\alpha = 0$ if x and y are not adjacent in G.

(ii) Suppose x, y are in $U(G)$. $\beta = 1$ if x and y are adjacent in G. $\beta = 0$ if x and y are not adjacent in G.

(iii) Suppose $x \in V(G)$ and $y \in U(G)$. $\gamma = 1$ if x and y are incident in G. $\gamma = 0$ if x and y are not incident in G.

Definition 1.1. The block-transformation graph G^{101} of a graph G is the graph with vertex set $V(G) \cup U(G)$ in which the vertices x and y are joined by an edge if one of the following conditions hold

(i) $x, y \in V(G)$, and x and y are adjacent in G.

(ii) $x, y \in U(G)$, and x and y are not adjacent in G.

(iii) One of x and y is in $V(G)$ and the other is in $U(G)$, and they are incident in G (see Figure 1).

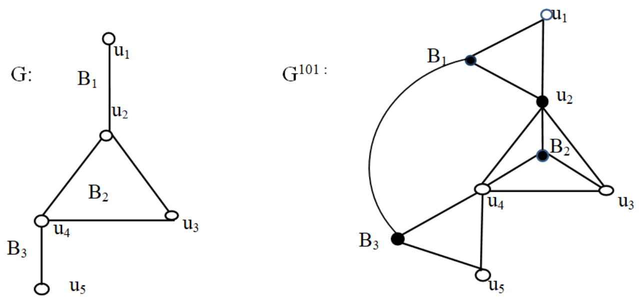


Figure 1

In G^{101}, the light vertices correspond to the vertices of G and dark vertices correspond to the blocks of G. Without loss of generality, the vertices corresponding to a block B_i of G will be referred as the block-vertex B_i in G^{101}.

Definition 1.2. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or adjacent to the element of S i.e. $N[S] = V(G)$. The domination number γ(G) of a graph is the minimum cardinality of a dominating set in G.

Definition 1.3. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a total dominating set if every vertex $v \in V$ there exists a vertex $u \in S$, $u \neq v$; u is adjacent to v. i.e. $N(S) = V(G)$. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set.

Definition 1.4. A full binary tree T is a tree in which every node other than the leaves has two children.

DEFINITION 1.5. A thorny ring C_n^+ consists of $2n$ vertices where n vertices on the cycle are of degree three and remaining n vertices are pendant vertices.

DEFINITION 1.6. A thorny ring C_n^v consists of $n(t-1)$ vertices of which n vertices are in cycle (each of degree t) and the remaining $n(t-2)$ are pendent vertices.

DEFINITION 1.7. Thorn star $S_{k,t}$ is obtained from a k-arm star by attaching $t-1$ terminal vertices to each of the star arms.

THEOREM 1.1 ([**5**]). *A dominating set S is a minimal dominating set if and only if for each $u \in S$, one of the following two conditions hold:*

(a) *u is an isolate of S,*
(b) *there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.*

THEOREM 1.2 ([**7**]). *A dominating set S is a minimal total dominating set if and only if for each $v \in S$, one of the following two conditions hold:*

(a) There exists a vertex $w \in V - S$ such that $N(w) \cap S = \{v\}$

(b) $< S - \{v\} >$ contains an isolated vertex.

2. Domination number of block-transformation of some class of graphs

The following observations are immediate consequences of the definition of G^{101}

PROPOSITION 2.1. *The following holds*

(i) *For any cycle $C_n, n \geqslant 3$, we have $\gamma\left(C_n^{101}\right) = 1$.*
(ii) *For any wheel $W_n, n > 3$, we have $\gamma\left(W_n^{101}\right) = 1$.*
(iii) *For any complete graph $K_n, n \geqslant 2$, we have $\gamma\left(K_n^{101}\right) = 1$.*
(iv) *For any complete bipartite graph $K_{m,n}$ with m, $n \geqslant 1$, we have $\gamma\left(K_{m,n}^{101}\right) = 1$.*
(v) *For any star graph $K_{1,n}$ we have $\gamma\left(K_{1,n}^{101}\right) = 1$.*

PROPOSITION 2.2. *The following holds*

(i) $\gamma(P_2^{101}) = 1$.
(ii) $\gamma(P_3^{101}) = 1$.
(iii) $\gamma(P_4^{101}) = 2$.
(iv) $\gamma(P_5^{101}) = 2$.

THEOREM 2.1. *For $G = (P_n^{101})$, $n \geqslant 6$ we have*

$$\gamma(G) = \begin{cases} \left\lceil \dfrac{n}{3} \right\rceil, n \equiv 1\,(mod\;3) \\ \left\lceil \dfrac{n}{3} \right\rceil + 1, n \not\equiv 1\,(mod\;3) \end{cases}$$

PROOF. Let $V(Pn) = \{u_1, u_2, u_3....., u_n\}$ and let B_i joins u_i and u_{i+1} in P_n^{101}, $1 \leqslant i \leqslant n-1$. Hence $V(P_n^{101}) = \{u_1, u_2, u_3....., u_n\} \cup \{B_1, B_2, B_3....., B_{n-1}\}$. Let S be the dominating set of G.

To prove the results we consider the following cases.

Case 1: If $n \equiv 1\,(mod\,3)$
We construct a set of vertices S as follows:

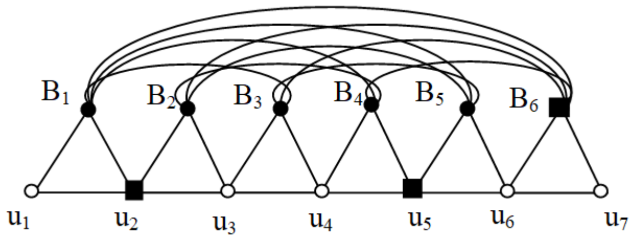
$$S \;=\; \{u_2, u_{3i+5}\;/0 \leqslant i \leqslant \left\lfloor \frac{n}{3} \right\rfloor - 1\} \cup \{B_{n-1}\}$$

Then $|S| = \lceil \frac{n}{3} \rceil$. Clearly S is a dominating set of G as $N\,[S] = V\,(G)$. Now for each $u \in S$ there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.1, S is also the minimal dominating set. Hence

$$\gamma\,(G) = |S| = \left\lceil \frac{n}{3} \right\rceil.$$

Case 2: Assume that $n \not\equiv 1\,(\text{mod}3)$. We construct a set of vertices S as follows: $S = \{u_2, u_{3i+5}\;/0 \leqslant i \leqslant \lfloor \frac{n}{3} \rfloor - 1\} \cup \{B_1\}$ Then $|S| = \lceil \frac{n}{3} \rceil$. . Clearly S is a dominating set of G as $N\,[S] = V\,(G)$. Now for each $u \in S$ there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then by Theorem 1.1, S is also the minimal dominating set. Hence $\gamma\,(G) = |S| = \lceil \frac{n}{3} \rceil + 1$ □

Illustration: In figure 2, the graph P_7^{101} is shown in which the set of solid square vertices form its dominating set of minimum cardinality, for case 1.



$\gamma(P_7^{101}) = 3.$

FIGURE 2

THEOREM 2.2. *For* $G = C_n^v$, *we have* $\gamma\,(C_n^v) = n$.

PROOF. Consider the set $S = \{u_1, u_2, u_3....., u_n\}$ of the vertices on the cycle. Then $|S| = n$. Clearly S is a dominating set of G. Now for each $u \in S$ there exists a vertex v which is a pendent vertex of G such that $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.1, S is also the minimal dominating set. Hence $\gamma\,(C_n^v) = n$. □

PROPOSITION 2.3. *For* $G = C_n^+$, *we have* $\gamma\,(C_n^+) = n$.

PROOF. Consider the set $S = \{u_1, u_2, u_3....., u_n\}$ of the vertices on the cycle. Then $|S| = n$. Clearly S is a dominating set of G. Now for each $u \in S$ there exists a vertex v which is a pendent vertex of G such that $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.1, S is also the minimal dominating set. Hence $\gamma\,(C_n^+) = n$. □

THEOREM 2.3. *For $G = S_{k,t}$, we have $\gamma(S_{k,t}) = k$.*

PROOF. Consider the set $S = \{k_1, k_2, k_3 ....., k_n\}$ of the vertices on the cycle. Then $|S| = n$. Clearly S is a dominating set of G. Now for each $k \in S$ there exists a vertex v which is a pendent vertex of G such that $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.1, S is also the minimal dominating set. Hence $\gamma(S_{k,t}) = k$. □

THEOREM 2.4. *For any full binary tree T, we have*

$$\gamma\left(T^{101}\right) = \begin{cases} \dfrac{4^{\left\lceil \frac{n}{2} \right\rceil} - 1}{3}, n = 2k - 1 \\ \dfrac{2\left(2^n - 1\right)}{3}, n = 2k \end{cases}$$

PROOF. Consider the levels $L_0, L_1, L_2, ..., L_n$ each level L_i $(0 \leqslant i \leqslant n)$ has 2^i vertices. We discuss the following two cases:

Case 1: Let $n = 2k - 1,\ k \in N$.

Vertices in L_n will be labeled as 0, vertices in L_{n-1} will be labeled as 1, vertices in L_{n-2} will be labeled 0, vertices in L_{n-3} will be labeled as 1. Continuing in this way the root vertex is labeled 1. Let S be the set of vertices labeled 1. Clearly S is a dominating set. The vertices at levels $L_0, L_2, L_4, ..., L_{n-1}$ have been labeled 1. We see that for each vertex $u \in S$ there exists a vertex v which is a leaf node of T such that $N(v) \cap S = \{u\}$. Then from Theorem 1.1, S is also the minimal dominating set. Therefore,

$$|S| = |L_0| + |L_2| + |L_4| +, ..., +|L_{(n-1)}| = 2^0 + 2^2 + 2^4 +, ..., +2^{(n-1)}.$$

which is a geometric series with $a = 1$, $r = 4$ and number of terms being $\left\lceil \frac{n}{2} \right\rceil$. Hence $|S| = \frac{1\left(4^{\left\lceil \frac{n}{2} \right\rceil} - 1\right)}{3}$ and $\gamma\left(T^{101}\right) = \frac{4^{\left\lceil \frac{n}{2} \right\rceil} - 1}{3}$

Cade 2: Let $n = 2k,\ k \in N$.

Vertices in L_n will be labeled as 0, vertices in L_{n-1} will be labeled as 1, vertices in L_{n-2} will be labeled 0, vertices in L_{n-3} will be labeled as 1. Continuing in this way the root vertex is labeled 0. Let S be the set of vertices labeled 1. Clearly S is a dominating set. The vertices at levels $L_1, L_3, L_5, ..., L_{n-1}$ have been labeled 1. We see that for each vertex $u \in S$ there exists a vertex v which is a leaf node of T such that $N(v) \cap S = \{u\}$. Then from Theorem 1.1, S is also the minimal dominating set. Therefore,

$$|S| = |L_1| + |L_3| + |L_5| +, ..., +|L_{(n-1)}| = 2^1 + 2^3 + 2^5 +, ..., +2^{(n-1)}$$

which is a geometric series with $a = 2$, $r = 4$ and number of terms being $\frac{n}{2}$. Hence $|S| = \frac{2\left(4^{\left(\frac{n}{2}\right)} - 1\right)}{3} = \frac{2(2^n - 1)}{3}$ and $\gamma\left(T^{101}\right) = \frac{2(2^n - 1)}{3}$. □

3. Total domination number of block-transformation of some class of graphs.

PROPOSITION 3.1. *The following holds*

(i) *For any cycle* $C_n, n \geqslant 3$, *we have* $\gamma_t\left(C_n^{101}\right) = 2$.

(ii) *For any wheel* $W_n, n > 3$, *we have* $\gamma_t\left(W_n^{101}\right) = 2$.

(iii) *For any complete graph* $K_n, n \geqslant 2$, *we have* $\gamma_t\left(K_n^{101}\right) = 2$.

(iv) *For any complete bipartite graph* $K_{m,n}$*with* $m,\ n\ \geqslant\ 1$, *we have*

$$\gamma_t\left(K_{m,n}^{101}\right) = 2.$$

(v) *For any star graph* $K_{1,n}$ *we have* $\gamma_t\left(K_{1,n}^{101}\right) = 2$.

PROPOSITION 3.2. *The following holds:* (i) $\gamma_t(P_2^{101}) = 2$. (ii) $\gamma_t(P_3^{101}) = 2$. (iii) $\gamma_t(P_4^{101}) = 2$. (iv) $\gamma_t(P_5^{101}) = 3$. (v) $\gamma_t(P_6^{101}) = 4$. (iv) $\gamma_t(P_7^{101}) = 4$.

THEOREM 3.1. *For* $G = (P_n^{101})$, $n \geqslant 8$, *we have*

$$\gamma_t(P_n^{101}) = \begin{cases} 2\left\lfloor \dfrac{n}{2} \right\rfloor + 2, n \equiv 1(\text{mod})4 \\ \dfrac{n}{2} + 1, n = 4k, k = 2, 3, 4, ... \\ \left\lfloor \dfrac{n}{2} \right\rfloor + 1,\ otherwise \end{cases}$$

PROOF. Let $V(Pn) = \{u_1, u_2, u_3....., u_n\}$ and let B_i joins u_i and u_{i+1} in P_n^{101}, $1 \leqslant i \leqslant n-1$. Hence $V(P_n^{101}) = \{u_1, u_2, u_3....., u_n\} \cup \{B_1, B_2, B_3....., B_{n-1}\}$. Let S is total dominating set of G.

To prove the results we consider the following cases.

Case 1: Let $n \equiv 1\,(mod\ 4)$. We construct a set of vertices S as follows:

$$S\ =\ \{u_{4i+2},\ u_{4i+3}\,|\,0 \leqslant\ i < \left\lfloor \frac{n}{4} \right\rfloor\} \cup \{v_{n-1}\} \cup \{B_1\}$$

Then $|S| = 2\left\lfloor \frac{n}{4} \right\rfloor\ + 2$. Clearly S is a total dominating set of G, as $N(S) = V(G)$. Now for each $u\ \in\ S$ there exists a vertex $v\ \in\ V -\ S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.2 S is also the minimal total dominating set. Hence, $\gamma_t\,(G) = 2\left\lfloor \frac{n}{4} \right\rfloor\ + 2$.

Case 2: Let $n = 4k, k = 2, 3, 4,$ We construct a set of vertices S as follows:

$$S\ =\ \{u_{4i+2},\ u_{4i+3}/0 \leqslant\ i < \frac{n}{4}\} \cup \{B_1\}.$$

Then |S| $= \frac{n}{2} +\ 1$. Clearly S is a total dominating set of G, as $N(S) = V(G)$. Now for each $u\ \in\ S$ there exists a vertex $v\ \in\ V -\ S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.2 S is also the minimal total dominating set. Hence, $\gamma_t(P_n^{101}) = \frac{n}{2} + 1$

Case 3: Let other than Case 1 and Case 2. We construct a set of vertices S as follows:

$$S\ =\ \{u_{4i+2},\ u_{4i+3}/0 \leqslant\ i < \left\lfloor \frac{n}{4} \right\rfloor\} \cup \{v_{n-1}\} \cup \{B_{n-1}\}.$$

Then |S| = $\lfloor \frac{n}{2} \rfloor + 1$. Clearly S is a total dominating set of G, as N(S)= V(G). Now for each $u \in S$ there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.2 S is also the minimal total dominating set. Hence, $\gamma_t(P_n^{101}) = \lfloor \frac{n}{2} \rfloor + 1$ □

THEOREM 3.2. *For* $G = C_n^v$, *we have* $\gamma_t\,(C_n^v) = n$.

PROOF. Consider the set $S = \{u_1, u_2, u_3....., u_n\}$ of the vertices on the cycle. Then $|S| = n$. Clearly S is a total dominating set of G. Now for each $u \in S$ there exists a vertex v which is a pendent vertex of G such that $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.2, S is also the minimal total dominating set. Hence, $\gamma_t\,(C_n^v) = n$. □

PROPOSITION 3.3. *For* $G = C_n^+$, *we have* $\gamma_t\,(C_n^+) = n$.

PROOF. Consider the set $S = \{u_1, u_2, u_3....., u_n\}$ of the vertices on the cycle. Then $|S| = n$. Clearly S is a total dominating set of G. Now for each $u \in S$ there exists a vertex v which is a pendent vertex of G such that $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.2, S is also the minimal total dominating set. Hence, $\gamma_t\,(C_n^+) = n$. □

THEOREM 3.3. *For* $G = S_{k,t}$, *we have* $\gamma_t\,(S_{k,t}) = k$.

PROOF. Consider the set $S = \{k_1, k_2, k_3....., k_n\}$ of the vertices on the cycle. Then $|S| = n$. Clearly S is a total dominating set of G. Now for each $k \in S$ there exists a vertex v which is a pendent vertex of G such that $v \in V - S$ for which $N(v) \cap S = \{u\}$. Then from Theorem 1.2, S is also the minimal total dominating set. Hence, $\gamma_t\,(S_{k,t}) = k$. □

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