BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 9(2019), 245-253
DOI: 10.7251/BIMVI1902245A

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

CONGRUENCE RELATION ON HEYTING ALGEBRAS

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Abstract. In this paper we introduced the concept of congruence relations on Heyting algebra using implicatively as well as multiplicatively closed subsets. Using the definition of homomorphism of Heyting algebras, we characterized and studied some important properties of quotient Heyting algebra by the congruence classes of it. We also give the definitions of ideal (prime ideal) and filters(prime filters) of Heyting algebra. Based on the concept of implicatively closed subset S of a Heyting algebra H, special congruence relation ψ^S which seems similar but quite different from [**3**] was introduced on a Heyting algebra H. Some properties of ψ^S, analogous to that for a distributive lattice proved in [**3**] are furnished. Further, we proved for any prime ideal P and a filter F of a Heyting algebra H, there exists an order preserving onto map between the set of all prime ideals of H/ψ^S and the set of all prime ideals of H disjoint with S.

1. Introduction

Birkhoff [**2**] introduced the concept of Brouwerian lattice as a distributive lattice or Heyting algebra as a bounded distributive lattice in which for any two elements a, b there exists a largest element $a \to b$ such that $a \wedge (a \to b) \leqslant b$. Heyting Algebra (H) is a relatively pseudo complemented distributive lattice. It arises from non classical logic and was first investigated by Skolem [**4**]. It is named as Heyting Algebra after the Dutch Mathematician Arend Heyting [**4**].

In this paper we introduced the concept of congruence relations on Heyting algebra using implicatively as well as multiplicatively closed subsets [**3**] of H. Using the definition of homomorphism of Heyting algebras we characterized and studied some important properties of quotient Heyting algebra [**1**] by the congruence classes of it. We also give the definitions of ideal (prime) ideal and filters (prime) filters

2010 *Mathematics Subject Classification.* 06D20, 06B10.
Key words and phrases. Heyting algebra, implicatively closed, multiplicatively closed, congruence relation, prime ideal, prime filter.

of Heyting algebra. Based on the concept of implicatively as well as multicatively closed subset S of a Heyting algebra H, special congruence relation ψ^S which seems similar to [**3**] but quite different from [**3**] was introduced on a Heyting algebra H. Some properties of ψ^S, analogous to that for a distributive lattice proved in [**3**] are furnished. Further, we proved for any prime ideal P and a filter F of a Heyting algebra H, there exists an order preserving onto map between the set of all prime ideals of H/ψ^S and the set of all prime ideals of H disjoint with S.

2. Preliminaries

Definition 2.1. An algebra $(H, \vee, \wedge \rightarrow, 0, 1)$ is called a Heyting algebra if it satisfies the following

(1) $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice;
(2) $a \rightarrow a = 1$;
(3) $b \leqslant a \rightarrow b$;
(4) $a \wedge (a \rightarrow b) = a \wedge b$;
(5) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$;
(6) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$, for all $a, b, c \in H$.

Theorem 2.1. *A bounded distributive lattice* $(H, \vee, \wedge, 0, 1)$ *is a Heyting Algebra if there exist a binary operation* $\rightarrow$ *on* H *such that, for any* $a, b, c \in H$,

$$a \wedge c \leqslant b \Leftrightarrow c \leqslant a \rightarrow b.$$

Theorem 2.2. *Let* H *be a Heyting algebra. Then for any* $a, b, c \in H$, *the following holds*

(i) $a \wedge c \leqslant b \Leftrightarrow c \leqslant a \rightarrow b$;
(ii) $a \leqslant b \Leftrightarrow a \rightarrow b = 1$

Lemma 2.1. *In any Heyting algebra* H, *the following holds:*

(a) $a \rightarrow (b \wedge a) = a \rightarrow b$;
(b) $a \leqslant b \Rightarrow x \rightarrow a \leqslant x \rightarrow b$;
(c) $a \leqslant b \Rightarrow b \rightarrow x \leqslant a \rightarrow x$, *for all* $a, b, c, x \in H$.

Theorem 2.3. *If* $(H, \vee, \wedge, \rightarrow, 0, 1)$ *is a Heyting Algebra and* $a, b \in H$, *then* $a \rightarrow b$ *is the largest element* c *of* H *such that* $a \wedge c \leqslant b$.

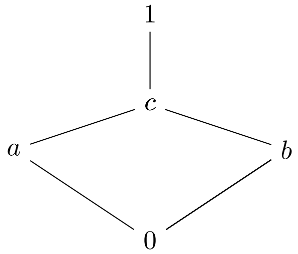
Theorem 2.4. *The following are equivalent:*

(1) H *is Heyting algebra;*
(2) *For any* $a, b, c \in H$, $a \wedge c \leqslant b \Leftrightarrow c \leqslant a \rightarrow b$;
(3) $b \leqslant a \rightarrow b$, *for all* $a, b \in H$.

Definition 2.2. Let H be a Heyting algebra and I be a nonempty subset of H. Then I is said to be an ideal of H if it satisfies the following conditions:

(1) $a, b \in I \Rightarrow a \vee b \in H$;
(2) $a \in I, x \in H \Rightarrow a \wedge x \in I$;
(3) $a \in I, x \in H, a \rightarrow x \neq 1 \Rightarrow x \rightarrow a \in I$.

EXAMPLE 2.1. Let $H = \{0, a, b, c, 1\}$ and $I = \{0, a, b, c\}$ defined by the figure depicted below. Clearly $(H, \vee, \wedge, \rightarrow, 0, 1)$ is Heyting algebra, where $\wedge$, $\vee$, and $\rightarrow$ operators are defined by the following tables.



∧	0	a	b	c	1
0	0	0	0	0	1
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	1
1	0	a	b	c	1

∨	0	a	b	c	1
0	0	a	b	c	1
a	a	a	c	c	1
b	b	c	b	c	1
c	c	c	c	c	1
1	0	a	b	c	1

→	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then I is an ideal of H.

PROOF. Let we see the following

∧	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

∨	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

Define the binary operation $' \rightarrow'$ on I as follows

$$a \rightarrow 0 = 0,\ b \rightarrow o,\ c \rightarrow 0 = 0,\ a \rightarrow b = b,\ b \rightarrow a = a,\ c \rightarrow a = a,\ a \rightarrow c = c,$$
$$b \rightarrow c = c,\ c \rightarrow b = b,$$

and

→	0	a	b	c
1	0	a	b	c

So all the criteria of the definition are satisfied. Then I is an ideal of H. □

LEMMA 2.2. *Let I be an ideal of H and S be a nonempty subset of I, defined by*

$$(S) = \{(\vee_{i=1}^{n} s_i) \wedge x : s_i \in S, x \in H, n \in Z^+\}.$$

Then

$$(\{(\vee_{i=1}^{n} s_i)\} \wedge y) \rightarrow x = (\{\wedge_{i=1}^{n} s_i\} \wedge y) \rightarrow x.$$

DEFINITION 2.3. A nonempty subset F of H is called a filter of H, if it satisfies the following:

(1) $a, b \in F \Rightarrow a \wedge b \in F$;
(2) $a \in F, x \in H \Rightarrow a \vee x \in F$;
(3) $a \in F, x \in H \backslash \{0\} \Rightarrow a \rightarrow x \in F$;

DEFINITION 2.4. A proper ideal (filter) P of H is said to be maximal if there is no proper ideal (filter) Q of H such that $P \subseteq Q$.

An element $m \in H$ is called maximal if it is a maximal element in the partially ordered set $(H, \leqslant)$. That is for any $a \in H$, $m \leqslant a \Rightarrow m = a$.

DEFINITION 2.5. Let Y be a proper ideal of H. We say that Y is a prime ideal of H if $(x \wedge y) \rightarrow z = 1$ then either $x \in Y$ or $y \in Y$, for all $x, y \in H$ and $z \in Y$.

DEFINITION 2.6. Let Y be a proper filter of H. We say that Y is a prime filter of H if $z \rightarrow (x \vee y) = 1$ then either $x \in Y$ or $y \in Y$, for all $x, y \in H$ and $z \in Y$.

LEMMA 2.3. *For any $a, b, c \in H$, we have $(a \wedge b) \rightarrow c = a \rightarrow (b \rightarrow c)$.*

DEFINITION 2.7. Let H be a Heyting algebra and $x \in H$. Then the set $\downarrow x$, defined by $\downarrow x = \{y \in H : A(y, x) > 0\}$, is called principal ideal of H generated by x.

Dually, we can define principal filter.

DEFINITION 2.8. Let H be a Heyting algebra and $x \in$ H. Then, the set $\uparrow x$, defined by $\uparrow x = \{y \in H : A(x, y) > 0\}$, is called principal filter of H generated by x.

DEFINITION 2.9. Let H and $H^{'}$ be any two Hilbert algebras. A mapping $f : H \rightarrow H^{'}$ is called a homomorphism if it satisfies the following:

1. $f(a \vee b) = f(a) \vee f(b)$;
2. $f(a \wedge b) = f(a) \wedge f(b)$;
3. $f(a \rightarrow b) = f(a) \rightarrow f(b)$;
4. $f(0) = 0^{'}; f(1) = 1^{'}$ for all $a, b \in H$.

From now onwards by H we mean Heyting algebra unless otherwise stated.

3. Congruence relation on Heyting algebra

DEFINITION 3.1. An equivalence relation A on H is called a congruence relation if for all $a, b, c, d \in H$

$$a \equiv b(A),\ c \equiv d(A) \Rightarrow a \wedge c \equiv b \wedge d(A),\ a \vee c \equiv b \vee d(A) \text{ and } a \rightarrow c \equiv b \rightarrow d(A).$$

For any congruence relation A on H, we denote the congruence class containing $x \in H$ by $[x]^A$ and the set of all congruence classes of H is denoted by H/A. The set H/A forms a Heyting algebra called quotient Heyting algebra under the binary operations $\wedge, \vee$ and $\rightarrow$ defined by

$$[x]^A \vee [y]^A = [x \vee y]^A, [x]^A \wedge [y]^A = [x \wedge y]^A, [x]^A \to [y]^A = [x \to y]^A$$

for all $[x]^A$ and $[y]^A \in H/A$.

Definition 3.2. A subset S of H is said to be multiplicatively closed subset of H if $S \neq \emptyset$ and for any $a, b \in S \Rightarrow a \wedge b \in S$.

Definition 3.3. A subset S of H is said to be implicatively closed subset of H if $S \neq \emptyset$ and for any $a, b \in S \Rightarrow a \to b \in S$.

Lemma 3.1. *Let I be an ideal and S be a multiplicatively closed subset of H such that $I \cap S = \emptyset$. Then there is a prime ideal M of H such that $I \subseteq M$ and $M \cap S = \emptyset$.*

Proof. Define a relation on H by

$$a \equiv b(\psi^S) \Leftrightarrow a \to t = b \to t,\ t \in S,\ a, b \in H.$$

Clearly, ψ^S is reflexive and symmetric. To show transitive property, let $a \equiv b(\psi^S)$ and $b \equiv c(\psi^S)$. Then $a \to s = b \to s$ and $b \to t = c \to t$ $s, t \in S$. Now consider,

$$\begin{aligned} a \to s \to t = b \to s \to t &= (b \wedge s) \to t = (s \wedge b) \to t \\ &= s \to b \to t = s \to c \to t = c \to s \to t. \end{aligned}$$

Since S is implicatively closed subset of H, we have $a \equiv c(\psi^S)$. Therefore, ψ^S is transitive.

To show ψ^S is a congruence relation, we will show that the three operations hold for the given relation.

Suppose $a \equiv b(\psi^S)$ and $c \equiv d(\psi^S)$. Then $a \to s = b \to s$ and $c \to t = b \to t$, $t, s \in S$. Consider

$$\begin{aligned} a \to c \to s \to t &= (a \wedge c) \to s \to t \text{ [Lemma 2.3]} \\ &= (c \wedge a) \to s \to t = c \to a \to s \to t \\ &= c \to s \to b \to t = b \to s \to d \to t = b \to d \to s \to t. \end{aligned}$$

Hence, $a \to c \equiv b \to d(\psi^S)$.

From this result it follows that

$$(a \wedge c) \to s \to t = (b \wedge d) \to s \to t,\ \ s, t \in S.$$

Hence $a \wedge c \equiv b \wedge d(\psi^S)$.

Finally,

$$\begin{aligned} (a \vee c) \to s \to t &= (a \to s \to t) \wedge (c \to s \to t) \\ &= (b \to s \to t) \wedge (s \to c \to t) = (b \to s \to t) \wedge (s \to d \to t) \\ &= (b \to s \to t) \wedge (d \to s \to t) = (b \vee d) \to s \to t. \end{aligned}$$

Hence, $a \vee c \equiv b \vee d(\psi^S)$.

Therefore, ψ^S is a congruence relation on H. □

Theorem 3.1. *H/ψ^S is a Heyting algebra and the operation " $\to$ " is commutative.*

PROOF. Let $x, y \in H$. Since $S \neq \emptyset$, we can choose $a \in S$. But then $x \to y \to a = y \to x \to a$ implies $x \to y \equiv y \to x(\psi^S)$. Hence

$$[x]^{\psi^S} \to [y]^{\psi^S} = [x \to y]^{\psi^S} = [y \to x]^{\psi^S} = [y]^{\psi^S} \to [x]^{\psi^S}.$$

Thus the operation " $\to$ "is commutative on H/ψ^S. □

THEOREM 3.2. *Let S and T be two any multiplicatively and implicatively closed subsets of H_1 and H_2 respectively. Then for any homomorphism ϕ:$H_1 \to H_2$ such that $\phi(S) \subseteq T$, there exists a homomorphism $f : H_1/\psi^S \to H_2/\psi^T$ such that $f \circ h = k \circ \phi$, where $h : H_1 \to H_1/\psi^S$ and $k : H_2 \to H_2/\psi^T$ denote the canonical epimorphisms. Further, if ϕ is a monomorphism and if $\phi(S) = T$, then f is a monomorphism. If ϕ is an epimorphism, then f is an epimorphism*

PROOF. Define $f : H_1/\psi^S \to H_2/\psi^T$ by $f([x]^{\psi^S}) = [\phi(x)]^{\psi^T}$. Let $[x]^{\psi^S} = [y]^{\psi S}, x, y \in H_1$. Then

$$\begin{aligned} x \equiv y(\psi^S) &\Rightarrow x \to s = y \to s, s \in S \\ &\Rightarrow \phi(x \to s) = \phi(y \to s) \\ &\Rightarrow \phi(x) \to \phi(s) = \phi(y) \to \phi(s) \\ &\Rightarrow \phi(x) = \phi(y)(\psi^T), \text{ as } \phi(s) \in T \\ &\Rightarrow [\phi(x)]^{\psi^T} = [\phi(y)]^{\psi^T}. \\ &\Rightarrow f([x]^{\psi^S}) = f([y]^{\psi^S}). \end{aligned}$$

Hence, f is well defined.

Let $x, y \in S$. Then

$$\begin{aligned} f([x]^{\psi^S} \to [y]^{\psi^S}) &= f([x \to y]^{\psi^S}) = [\phi(x \to y)]^{\psi^T} \\ &= [\phi(x) \to \phi(y)]\psi^T = [\phi(x)]^{\psi^T} \to [\phi(y)]^{\psi^T} \\ &= f([x]^{\psi^S}) \to f([y]^{\psi^S}). \end{aligned}$$

Similarly, we can prove congruence relations.

$$f([x]^{\psi^S}) \vee [y]^{\psi S}) = f([x]^{\psi^S}) \vee f([y]^{\psi^S}) \text{ for all } x, y \in H_1.$$

Hence f is a homomorphism. Now

$$f \circ h : H_1 \to H_2/\psi^T \text{ and for any } x \in H$$

we have

$$[f \circ h](x) = f(h(x)) = f([x]^{\psi^S}) = [\phi(x)]^{\psi^T}.$$

Again,

$$k \circ \phi : H_1 \to H_2/\psi^T \text{ and for any } x \in H_1,$$

we have

$$[k \circ \phi](x) = k(\phi(x)) = [\phi(x)]^{\phi^T}.$$

Hence $[f \circ h](x) = [k \circ \phi](x)$, for all $x \in H_1$. This shows that $f \circ h = k \circ \phi$.

(I) Let ϕ be a monomorphism and let $\phi^S = T$. Let $f([x]^{\psi^S}) = f([y]^{\psi^S})$ for some $x, y \in H_1$. Then $[\phi(x)]^{\psi^T} = [\phi(y)]^{\psi^T} \Rightarrow \phi(x) \equiv \phi(y)(\psi^T) \Rightarrow \phi(x) \to t = \phi(y) \to t$, for some $t \in T$ implies $\phi(x) \to \phi(s) = \phi(y) \to \phi(s)$, for some $s \in S$ (since

$\phi(S) = T)$. Thus $\Rightarrow \phi(x \to s) = \phi(y \to s)$ (since ϕ is a monomorphism) and $\Rightarrow x \to s = y \to s$. $\Rightarrow x \equiv y(\psi^S) \Rightarrow [x]^{\psi S} = [y]^{\psi^S}$. This shows that f is one-one.

(II) Let ϕ be an epimorphism. Let $[y]^{\psi^T} \in H_2/\psi^T$.As $\phi : H_1 \to H_2$ is onto and $y \in H_2, \phi(x) = y$ for some $x \in H_1$. Thus $[x]^{\psi^S} \in H_1/\ \psi^S$ and $f([x]^{\psi^S}) = [\phi(x)]^{\psi^T}$ $= [y]^{\psi T}$. This shows that f is an epimorphism. □

For any two congruence relation ψ^S and ψ^T induced by two implicativelly closed subsets S and T of H with $S \subseteq T$. we have the following theorem.

Theorem 3.3. *Let H be Heyting algebra and let S, T be any two implicatively closed subsets of H with $S \subseteq T$. Then following are equivalent:*

(i) *The mapping $f : H/\psi^S \to H/\psi^T$ defined by $f([x]^{\psi S}) = [x]^{\psi^T}$ for each $x \in H, \psi^T$ is an isomorphism.*

(ii) *For each $t \in T$, there exists $s \in S$ such that $t \to s \in S$.*

(iii) *For any prime ideal P of H, $P \cap T \neq \emptyset \Rightarrow P \cap S \neq \emptyset$.*

Proof. $(i) \Rightarrow (ii)$ Obviously f is a well defined map. Let $x, y \in H$. Then $f([x]^{\psi^S}) = f([y]^{\psi^S}) \Rightarrow [x]^{\psi^S} = [y]^{\psi^T}$ (since f is one-one). Thus $x \equiv y(\psi^S)$. Again, $f([x]^{\psi^S}) = f([y]^{\psi^S})[x]^{\psi T} = [y]^{\psi^T} \Rightarrow x \equiv y(\psi^T)$. Therefore, $x \equiv y(\psi^T) \Rightarrow x \equiv y(\psi^S) \Rightarrow \psi^T \subseteq \psi^S$. As $S \subseteq T, \psi^S \subseteq \psi^T$. Hence $\psi^S = \psi^T$. Hence any $t \in T$ must be congruent to some $s_1 \in S$. i.e. $t \equiv s_1 \in S$. Therefore $t \to s = s_1 \to s$ for some $s \in S$. As $s_1 \to s \in S$, we get $t \to s \in S$.

$(ii) \Rightarrow (iii)$ Let P be a prime ideal in H such that $P \cap T \neq \emptyset$. Select any $t \in P \cap T$. As $t \in T$ there exists $s \in S$ such that $t \to s \in S$. As $t \in P, t \to s \in P$. Thus $t \to s \in P \cap S$. This shows that $P \cap S \neq \emptyset$.

$(iii) \Rightarrow (i)$ Claim: $\psi^S = \psi^T$. As $S \subseteq T \Rightarrow \psi^S \subseteq \psi^T$. To prove that $\psi^T \subseteq \psi^S$. Let $a \equiv b(\psi^T)$. Hence $a \to t = b \to t$ for any $t \in T$. Suppose $S \cap \downarrow x = \emptyset$. Then there is a prime ideal P such that $\downarrow x \subseteq P$ and $P \cap S = \emptyset$ (by Lemma 0.20)???? which contradicts the assumption. (iii) as $t \in P \cap T \Rightarrow P \cap S \neq \emptyset$. Hence $S \cap \downarrow x \neq \emptyset$. Therefore $\exists s \in S \cap \downarrow x$. Hence $s = t \to x$ for some $x \in H$. Now $a \to s = a \to (t \to x) = (a \to t) \to x = (b \to t) \to x = b \to (t \to x) = b \to s$. But this shows that $a \equiv b(\psi^S)$. Thus $\psi^T \subseteq \psi^S$. Combining both the inclusions we get $\psi^T = \psi^S$ and the implication follows. □

Theorem 3.4. *Let H be a Heyting algebra with maximal elements and F be filter of H and let $h : H \to H/\psi^F$ be the canonical epimorphism. Then we have*

(I) *If $P^{'}$ is a prime ideal in H/ψ^S , then $h^{-1}(P^{'})$ is a prime ideal in H disjoint with F.*

(II) *Let $\theta : P(H/\psi^F) \to \{Q \in P(H) | Q \cap F = \emptyset\}$ be defined by $\theta(P^{'}) = h^{-1}(P^{'})$. Then θ is an order preserving onto map, where $P(H)$ and $P(H/\psi^F)$ denote the set of all prime ideals of H and H/ψ^F respectively.*

Proof. (I) As $h : H \to H/\psi^F$ is an epimorphism, we get $h^{-1}(P^{'})$ is a prime ideal in H. Only to prove that $h^{-1}(P^{'}) \cap F = \emptyset$. Let $s \in h^{-1}(P^{'}) \cap F$. If m is a maximal element, then $m, s \in F$ and hence $m \equiv s(\psi^F)$. Therefore $h(m) = h(s) \in P^{'}$ is a contradiction since $h(m)$ is a maximal element in H/ψ^F . Hence,

$h^{-1}(P^{'}) \cap F = \emptyset$. Thus $h^{-1}(P^{'}) \in \{Q \in P(H) | Q \cap F = \emptyset\}$.Let $P^{'}, Q^{'} \in P[H/\psi F]$ such that $P^{'} \subseteq Q^{'}$. Let $\theta(P^{'}) = P$ and $\theta(Q^{'}) = Q$. If $P^{'} \subseteq Q^{'}$ then $h^{-1}(P^{'}) \subseteq h^{-1}(Q^{'})$ and hence $\theta(P^{'}) \subseteq \theta(Q^{'})$. Then θ is order preserving.

(II) Let $P \in P(H)$ be such that $P \cap F = \emptyset$. $P \subseteq h^{-1}(h(P))$ always. To prove that $h^{-1}(h(P)) \subseteq P$. Let $x \in h^{-1}(h(P))$. Then as $h(x) \in h(P), [x]^{\psi^F} = [p]^{\psi^F}$ for some $p \in P$. This means $x \equiv p(\psi^F)$. Therefore $x \to s = p \to s$ for some $s \in F$. As $P \cap F = \emptyset, s \notin P$. Again $p \to s \in P$, but then $x \to s \in P$ implies $x \in P$ as $s \notin P$. This shows that $h^{-1}(h(P)) \subseteq P$. Combining both the inclusions we get $h^{-1}(h(P)) = P$. Hence θ is onto. □

Now we prove the following theorem.

THEOREM 3.5. *Let S denote a implicatively closed subset of a Heyting algebra H with maximal elements. Let $P^{'}$ be a prime ideal in $P[H/\psi^S]$. Define $h^{-1}(P^{'}) = P$, where $h : H \to H/\psi^S$ is the canonical epimorphism. Then the mapping $\alpha : H/\psi^T \to H/\psi^S/\psi^{T^{'}}$ defined by $\alpha([x]^{\psi^T}) = [[x]^{\psi^S}]^{\psi^{T^{'}}}$ is an isomorphism, where $T = H \backslash P$ and $T = [H/\psi^S] \backslash P^{'}$ are the filters in the Heyting algebras H and H/ψ^S respectively.*

PROOF. Let $[x]^{\psi^T} = [y]^{\psi T}$. Then $x \to t = y \to t$ for some $t \in T$ as $x \equiv y(\psi^T)$. But $t \notin P$ implies $h(t) = [t]^{\psi^S} \in P^{'}$ and hence $[t]^{\psi^S} \in [H/\psi^S] \backslash P^{'} = T^{'}$ Further $[x]^{\psi^S} \to [t]^{\psi^S} = [y]^{\psi^S} \to [t]^{\psi^S}$ implies $[x]^{\psi^S} \equiv [y]^{\psi^S}(\psi^{T^{'}})$. Therefore, $[[x]^{\psi^S}]^{\psi^{T^{'}}} = [[y]^{\psi^S}]^{\psi^{T^{'}}}$. Hence $\alpha([x]^{\psi^T}) = \alpha([y]^{\psi^T})$. This shows that α is well defined.

To prove that α is one-one, we will prove the following: Claim : $P \cap S = \emptyset$. As $P^{'}$ is a prime ideal in $[H/\psi^S].P^{'}$ is a proper ideal in $[H/\psi^S]$. Hence $[m]^{\psi^S} \notin P^{'}$ for any maximal element in H. But $[m]^{\psi^S} = S$ for all maximal elements m in H. Hence $S \notin P$. Let $s_1 \in P \cap S$. Then $s_1 \in P$ implies $s_1 \in h^{-1}(P^{'})$ and $(s_1) \in P^{'}$. Thus $[s_1]^{\psi^S}$ and $[s_1]^{\psi^S}$. Therefore $\in P \Rightarrow S \in P^{'}$, which is a contradiction. Hence $P \cap S = \emptyset$. Let $\alpha([x]^{\psi^T}) = ([y]^{\psi^T})$. Then $[[x]^{\psi^S}]^{\psi^{T^{'}}} = [[y]^{\psi^S}]^{\psi^T}$ implies $[x]^{\psi^S} \equiv [y]^{\psi^S}(\psi^{T^{'}})$. Hence $[x]^{\psi^S} \to [t]^{\psi^S} = [y]^{\psi^S} \to [t]^{\psi^S}$ for some $[t]^{\psi^S} \in T^{'} = [H/\psi^S] \backslash P^{'}$. Hence $[x \to t]^{\psi^S} = [y \to t]^{\psi^S}$ for some $[t]^{\psi^S} \in T^{'} = [H/\psi^S] \backslash P^{'}$. Therefore, $(x \to t \to s) = (y \to t \to s)$ for some $s \in S$. As $P^{'}$ is prime ideal in H/ψ^S,by claim $P \cap S = \emptyset$. Hence $t \in T$ and $s \in T$ imply $t \to s \in T$. But then $x \to (t \to s) = y \to (t \to s)$ for $t \to s \in T \Rightarrow x \equiv y(\psi T) \Rightarrow [x]^{\psi^T} = [y]^{\psi^T}$. But this shows that α is one-one.

Now we will prove that α is a homomorphism. For any $x, y \in H$, we have

$$\alpha([x]^{\psi^T} \to [y]^{\psi^T}) = [[x \to y]^{\psi^S}]^{\psi^{T^{'}}} = [[x]^{\psi^S} \to [y]^{\psi^S}]^{\psi^{T'}}$$
$$= [[x]^{\psi^S}]^{\psi^{T^{'}}} \to [[y]^{\psi^S}]^{\psi T^{'}} = \alpha([x]^{\psi^T}) \to \alpha([y]^{\psi^T}).$$

Similarly, we can prove that

$$\alpha([x]^{\psi^T} \wedge [y]^{\psi^T}) = \alpha([x]^{\psi^T}) \wedge \alpha([y]^{\psi^T}) \text{ and } \alpha([x]^{\psi^T} \vee [y]^{\psi^T}) = \alpha([x]^{\psi^T}) \vee \alpha([y]^{\psi^T}).$$

Obviously α being an onto map, we get α is an isomorphism and hence the result. □

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Received by editors 12.02.2018.; Revised version 30.11.2018; Available online 10.12.2018.

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