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PRINCIPAL RESIDUATED ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of a simple element in a residuated ADL and the concept of Principal residuated Almost Distributive Lattice (P-ADL). We prove important results in a P-ADL.

1. Introduction

Swamy, U.M. and Rao, G.C. [**6**] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, $p-$rings, biregular rings, associate rings, P_1-rings etc.) on one hand and distributive lattices on the other.

In [**1**], Dilworth, R.P., has introduced the concept of a residuation in lattices and in [**7, 8**], Ward, M. and Dilworth, R.P., have studied residuated lattices. In [**9**], Ward, M. has introduced the concept of a principal residuated lattice (or simply a P-Lattice) and studied its properties. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [**3**]. We have proved some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L in [**4**]. In [**5**], we introduced the concept of principal element in a residuated ADL.

In this paper, we introduce the concept of a simple element in a Residuated ADL and the concept of Principal residuated Almost Distributive Lattice. We prove important results in a P-ADL. In Section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL from Swamy, U.M. and Rao, G.C. [**6**], Rao, G.C. [**2**] and some important results on a residuated almost distributive lattice from our earlier paper [**3**]. In section 3,

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we introduce the concept of a simple element in a residuated ADL and the concept of Principal residuated Almost Distributive Lattice (P-ADL). We give an example of a P-ADL. We prove that a simple element of a residuated ADL is prime and every multiplicatively irreducible element of a P-ADL is prime. We also prove that any power of a prime element of a P-ADL is a primary element and every primary element of a P-ADL L is a power of a prime element of L.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

We begin with the definition of an ADL :

Definition 2.1. ([**2**]). An Almost Distributive Lattice (ADL) is an algebra $(L, \vee, \wedge)$ of type $(2,2)$ satisfying
(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$, for all $a, b, c \in L$.

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

Example 2.1. ([**2**]). Let X be a non-empty set. Fix $x_0 \in X$. For any $x, y \in L$, define

$$x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \quad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a discrete ADL.

For any $a, b \in L$, we say that a is less than or equals to b and write $a \leqslant b$, if $a \wedge b = a$. Then "$\leqslant$" is a partial ordering on L.

Theorem 2.1 ([**2**]). *Let $(L, \vee, \wedge, 0)$ be an ADL with ' 0 '. Then, for any $a, b \in L$, we have*
(1) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b$, $a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$
(4) $a \wedge b = a \Longleftrightarrow a \vee b = b$ *and* $a \wedge b = b \Longleftrightarrow a \vee b = a$
(5) $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *whenever* $a \leqslant b$
(6) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(7) $\wedge$ *is associative in* L
(8) $a \wedge b \wedge c = b \wedge a \wedge c$
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \wedge b = 0 \Longleftrightarrow b \wedge a = 0$
(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert L into a distributive lattice.

THEOREM 2.2 ([**2**]). *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*
(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice*
(2) $a \vee b = b \vee a$, *for all* $a, b \in L$
(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$
(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

PROPOSITION 2.1 ([**2**]). *Let $(L, \vee, \wedge)$ be an ADL. Then for any $a, b, c \in L$ with $a \leqslant b$, we have*
(1) $a \wedge c \leqslant b \wedge c$
(2) $c \wedge a \leqslant c \wedge b$
(3) $c \vee a \leqslant c \vee b$.

DEFINITION 2.2. ([**2**]) An element $m \in L$ is called maximal if it is maximal as in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a$ implies $m = a$.

THEOREM 2.3 ([**2**]). *Let L be an ADL and $m \in L$. Then the following are equivalent:*
(1) m *is maximal with respect to* $\leqslant$
(2) $m \vee a = m$ *for all* $a \in L$
(3) $m \wedge a = a$ *for all* $a \in L$.

LEMMA 2.1 ([**2**]). *Let L be an ADL with a maximal element m and $x, y \in L$. If $x \wedge y = y$ and $y \wedge x = x$ then x is maximal if and only if y is maximal.Also the following conditions are equivalent:*
(i) $x \wedge y = y$ *and* $y \wedge x = x$
(ii) $x \wedge m = y \wedge m$.

DEFINITION 2.3. ([**2**]) If $(L, \vee, \wedge, 0, m)$ is an ADL with 0 and with a maximal element m, then the set $I(L)$ of all ideals of L is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of I, J are given by $I \vee J = \{(x \vee y) \wedge m \mid x \in I, y \in J\}$ and $I \wedge J = I \cap J$.
The set $PI(L) = \{(a] \mid a \in L\}$ of all principal ideals of L forms a sublattice of $I(L)$. (Since $(a] \vee (b] = (a \vee b]$ *and* $(a] \cap (b] = (a \wedge b]$)

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) L and the definition of a residuated almost distributive lattice taken from our earlier paper [**3**].

DEFINITION 2.4 ([**3**]). Let L be an ADL with a maximal element m. A binary operation : on an ADL L is called a residuation over L if, for $a, b, c \in L$ the following conditions are satisfied.
($R1$) $a \wedge b = b$ if and only if $a : b$ is maximal
($R2$) $a \wedge b = b \implies$ (i) $(a : c) \wedge (b : c) = b : c$ and (ii) $(c : b) \wedge (c : a) = c : a$

$(R3)$ $[(a:b):c]\wedge m = [(a:c):b]\wedge m$
$(R4)$ $[(a\wedge b):c]\wedge m = (a:c)\wedge(b:c)\wedge m$
$(R5)$ $[c:(a\vee b)]\wedge m = (c:a)\wedge(c:b)\wedge m$

Definition 2.5 ([**3**]). Let L be an ADL with a maximal element m. A binary operation . on an ADL L is called a multiplication over L if, for $a,b,c\in L$ the following conditions are satisfied.
$(M1)$ $(a.b)\wedge m = (b.a)\wedge m$
$(M2)$ $[(a.b).c]\wedge m = [a.(b.c)]\wedge m$
$(M3)$ $(a.m)\wedge m = a\wedge m$
$(M4)$ $[a.(b\vee c)]\wedge m = [(a.b)\vee(a.c)]\wedge m$

Definition 2.6. ([**3**]) An ADL L with a maximal element m is said to be a residuated almost distributive lattice (residuated ADL), if there exists two binary operations ' : ' and ' . ' on L satisfying conditions R1 to R5, M1 to M4 and the following condition (A).
(A) $(x:a)\wedge b = b$ if and only if $x\wedge(a.b) = a.b$, for any $x,a,b\in L$.

We use the following properties frequently later in the results.

Lemma 2.2 ([**3**]). *Let L be an ADL with a maximal element m and . a binary operation on L satisfying the conditions $M1-M4$. Then for any $a,b,c,d\in L$,*
(i) $a\wedge(a.b) = a.b$ *and* $b\wedge(a.b) = a.b$
(ii) $a\wedge b = b \implies (c.a)\wedge(c.b) = c.b$ *and* $(a.c)\wedge(b.c) = b.c$
(iii) $d\wedge[(a.b).c] = (a.b).c$ *if and only if* $d\wedge[a.(b.c)] = a(b.c)$
(iv) $(a.c)\wedge(b.c)\wedge[(a\wedge b).c] = (a\wedge b).c$
(v) $d\wedge(a.c)\wedge(b.c) = (a.c)\wedge(b.c) \implies d\wedge[(a\wedge b).c] = (a\wedge b).c$
(vi) $d\wedge[(a.c)\vee(b.c)] = (a.c)\vee(b.c) \Leftrightarrow d\wedge[(a\vee b).c] = (a\vee b).c$

The following result is a direct consequence of M1 of Definition 2.15.

Lemma 2.3 ([**3**]). *Let L be an ADL with a maximal element m and . a binary operation on L satisfying the condition $M1$. For $a,b,x\in L$, $a\wedge(x.b) = x.b$ if and only if $a\wedge(b.x) = b.x$*

In the following, we give some important properties of residuation ' : ' and multiplication ' . ' in a residuated ADL L. These are taken from our earlier paper [**4**].

Lemma 2.4 ([**4**]). *Let L be a residuated ADL with a maximal element m. For $a,b,c,d\in L$, the following hold in L.*
(1) $(a:b)\wedge a = a$
(2) $[a:(a:b)]\wedge(a\vee b) = a\vee b$
(3) $[(a:b):c]\wedge[a:(b.c)] = a:(b.c)$
(4) $[a:(b.c)]\wedge[(a:b):c] = (a:b):c$
(5) $[(a\wedge b):b]\wedge(a:b) = a:b$
(6) $(a:b)\wedge[(a\wedge b):b] = (a\wedge b):b$
(7) $[a:(a\vee b)]\wedge m = (a:b)\wedge m$
(8) $[c:(a\wedge b)]\wedge[(c:a)\vee(c:b)] = (c:a)\vee(c:b)$

(9) *If* $a : b = a$ *then* $a \wedge (b.d) = b.d \Longrightarrow a \wedge d = d$
(10) $\{a : [a : (a : b)]\} \wedge (a : b) = a : b$
(11) $[(a \vee b) : c] \wedge [(a : c) \vee (b : c)] = (a : c) \vee (b : c)$
(12) $a \wedge m \geqslant b \wedge m \Longrightarrow (a : c) \wedge m \geqslant (b : c) \wedge m$
(13) $(a : b) \wedge \{a : [a : (a : b)]\} = a : [a : (a : b)]$
(14) $a \wedge b = b \Longrightarrow (a.c) \wedge (b.c) = b.c$
(15) $a \wedge b \wedge (a.b) = a.b$
(16) $[(a.b) : a] \wedge b = b$
(17) $(a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b)$
(18) $a \vee b$ *is maximal* $\Longrightarrow (a.b) \wedge a \wedge b = a \wedge b$

3. Principal Residuated Almost Distributive Lattices

In this section, we introduce the concept of a simple element in a residuated ADL and the concept of Principal residuated Almost Distributive Lattice (P-ADL). We give an example of a P-ADL. We prove important results in a P-ADL.

We recall the following concepts on a residuated ADL L :

Definition 3.1. ([**5**]) An element p of a residuated ADL L is called
(i) irreducible, if for any $f, g \in L$, $f \wedge g = p \Longrightarrow$ either f = p or g = p.
(ii) prime, if for any $a, b \in L$, $p \wedge (a.b) = a.b \Longrightarrow$ either $p \wedge a = a$ or $p \wedge b = b$.
(iii) primary, if for any $a, b \in L$, $p \wedge (a.b) = a.b$ and $p \wedge a \neq a \Longrightarrow p \wedge b^s = b^s$, for some $s \in Z^+$.

NOTE : Clearly, every prime element of a residuated ADL is primary.

Definition 3.2. ([**5**]) An ADL L is said to satisfy the ascending chain condition(a.c.c.), if for every increasing sequence $x_1 \leqslant x_2 \leqslant x_3 \leqslant$ ...., in L, there exists a positive integer n such that $x_n = x_{n+1} = x_{n+2} =$ ....

Definition 3.3. ([**5**]) Let L be a residuated ADL. An element a of L is called principal, if $b \in L$ and $a \wedge b = b$, then $a.c = b$, for some $c \in L$.

Definition 3.4. ([**5**]) A residuated ADL L is said to be a Noether ADL, if
(N1) the ascending chain condition(a.c.c.) holds in L and
(N2) every irreducible element of L is primary.

Now, we give the following definitions.

Definition 3.5. Let L be an ADL with a maximal element m. An element x of L is called an associate of y if $x \wedge m = y \wedge m$ (or x is equivalent to y). This is an equivalence relation on L.

Definition 3.6. Let L be an ADL and $x, y \in L$.
(i) y is called a divisor of x if $y \wedge x = x$. Observe that every maximal element m is a divisor of x, for any $x \in L$ and every associate of x is a divisor of x.
(ii) A divisor y of x other than maximal elements and associates of x is called a proper divisor of x.

DEFINITION 3.7. Let L be a residuated ADL with a maximal element m and $p \in L$. Then

(i) p is called multiplicatively irreducible if for any $a, b \in L$,

$$p \wedge m = (a.b) \wedge m \Longrightarrow \text{either } p \wedge m = a \wedge m \text{ or } p \wedge m = b \wedge m.$$

(ii) p is said to be simple if it has no proper divisors.

Let us recall the Fundamental Theorem (Theorem 3.4) from [**5**].

THEOREM 3.1 (([**5**]). *Let L be an ADL with a maximal element m satisfying the following conditions :*

(1) *L is residuated.*
(2) *L satisfies a.c.c.*
(3) *Every element of L is principal.*

Then L is a Noether ADL.

In view of the above Theorem, we give the following.

DEFINITION 3.8. Let L be a residuated ADL with a.c.c. If every element of L is principal then L is called a Principal Residuated Almost Distributive Lattice (or P-ADL).

Thus we get every P-ADL is a Noether ADL.

The following Lemma was proved in our earlier paper [**5**] and is used frequently later in the results.

LEMMA 3.1 ([**5**]). *Let L be a residuated ADL with a maximal element m. If $a, b \in L$ such that a is principal and $a \wedge b = b$ then $[(b : a).a] \wedge m = b \wedge m$.*

Now, we give an example of a residuated ADL and an example of a P-ADL.

EXAMPLE 3.1. If L is a discrete ADL and M = $\{0, a, b, 1\}$, where $0 \leqslant b \leqslant a \leqslant 1$, is a P-lattice. Let m be a maximal element of L. Define residuation ' : ' and multiplication ' . ' on L by $x : y = x$ and $x.y = m$, if $x \wedge y \neq 0$ and $x.y = 0$, if $x = 0$ or $y = 0$. Also define residuation ' : ' and multiplication ' . ' on M by the following tables :

:	1	a	b	0
1	1	1	1	1
a	a	1	1	1
b	b	a	1	1
0	0	0	0	1

.	1	a	b	0
1	1	a	b	0
a	a	b	b	0
b	b	b	b	0
0	0	0	0	0

Then $L \times M$ is a P-ADL under pointwise operations ' : ' and ' . '.

LEMMA 3.2. *Let L be a residuated ADL with a maximal element m. If for every element x of L, $p : x$ is an associate of p or $p : x$ is maximal then p is prime.*

PROOF. Let $x, p \in L$. and suppose $p : x$ is an associate of p or $p : x$ is maximal. Let $a, c \in L$ be such that $p \wedge (a.c) = a.c$ and $p \wedge a \neq a$. Now,

$p \wedge a \neq a$

$\Longrightarrow p : a$ is not maximal (By condition R1 of definition 2.4)
$\Longrightarrow p : a$ is an associate of p.
$\Longrightarrow (p : a) \wedge m = p \wedge m$
$\Longrightarrow [(p \wedge m) : c] \wedge m = \{[(p : a) \wedge m] : c\} \wedge m$
$\Longrightarrow (p : c) \wedge (m : c) \wedge m = [(p : a) : c] \wedge (m : c) \wedge m$
$\Longrightarrow (p : c) \wedge m = [(p : a) : c] \wedge m$
$\Longrightarrow (p : c) \wedge m = [p : (a.c)] \wedge m$ (By properties (3) and (4) of Lemma 2.4)
$\Longrightarrow (p : c) \wedge m = m$ (Since $p : (a.c)$ is maximal)
$\Longrightarrow p : c$ is maximal
$\Longrightarrow p \wedge c = c$.
Hence p is prime. □

THEOREM 3.2. *In a residuated ADL, every simple element is prime.*

PROOF. Let L be a residuated ADL and p, a simple element of L. Let $x \in L$. Then we have, by Lemma 2.4, $(p : x) \wedge p = p$. Since p is simple, $p : x$ is maximal or $p : x$ is an associate of p. Hence by Lemma 3.2, p is prime. □

LEMMA 3.3. *Let L be a residuated ADL with a maximal element m and $a, b \in L$ such that $a \wedge m = b \wedge m$. Then $(p.a) \wedge m = (p.b) \wedge m$, for any $p \in L$.*

PROOF. Let $a, b, p \in L$ and suppose $a \wedge m = b \wedge m$. Then $a \vee b = a$ and $b \vee a = b$. Now,
$(p.a) \wedge m = [p.(a \vee b)] \wedge m$
$= [(p.a) \vee (p.b)] \wedge m$ (By condition M4 of definition 2.4)
$= [(p.b) \vee (p.a)] \wedge m$
$= [p.(b \vee a)] \wedge m$ (By condition M4 of definition 2.4)
$= (p.b) \wedge m$.
Hence $(p.a) \wedge m = (p.b) \wedge m$, for any $p \in L$. □

LEMMA 3.4. *Let L be a P-ADL with a maximal element m. If p is a prime element of L and $p^{k+1} \wedge m \neq p^k \wedge m$, for some positive integer k then*

$$(p^{k+1} : p^k) \wedge m = p \wedge m.$$

PROOF. Let $p \in L$ be a prime element of L such that $p^{k+1} \wedge m \neq p^k \wedge m$. Suppose $(p^{k+1} : p^k) \wedge m \neq p \wedge m$. Since $p^{k+1} \wedge (p^k.p) = p^k.p$, we get $(p^{k+1} : p^k) \wedge p = p$. Hence $p \wedge (p^{k+1} : p^k) \neq p^{k+1} : p^k$. We have $(p^{k+1} : p^k)$ is principal and $(p^{k+1} : p^k) \wedge p = p$. Now, by Lemma 3.1, we get

$$p \wedge m = \{[p : (p^{k+1} : p^k)].(p^{k+1} : p^k)\} \wedge m \longrightarrow (1)$$

Thus

$$p \wedge \{[p : (p^{k+1} : p^k)].(p^{k+1} : p^k)\} = [p : (p^{k+1} : p^k)].(p^{k+1} : p^k).$$

Since p is prime, we get either

$$p \wedge [p : (p^{k+1} : p^k)] = p : (p^{k+1} : p^k) \text{ or } p \wedge (p^{k+1} : p^k) = p^{k+1} : p^k.$$

But $p \wedge (p^{k+1} : p^k) \neq p^{k+1} : p^k$. Thus

$$p \wedge [p : (p^{k+1} : p^k)] = p : (p^{k+1} : p^k).$$

Also we have $[p : (p^{k+1} : p^k)] \wedge p = p$ (By property (1) of Lemma 2.4). Thus $[p : (p^{k+1} : p^k)] \wedge m = p \wedge m$. So that, by Lemma 3.4, we get

$$[(p^{k+1} : p^k).p] \wedge m = \{(p^{k+1} : p^k).[p : (p^{k+1} : p^k)]\} \wedge m.$$

Now, by (1), $p \wedge m = [(p^{k+1} : p^k).p] \wedge m$. Now,

$$p^k \wedge m = (p.p^{k-1}) \wedge m = \{[(p^{k+1} : p^k).p].p^{k-1}\} \wedge m \text{ (By above Lemma 3.3)}$$

$$\Longrightarrow p^k \wedge m = [(p^{k+1} : p^k).p^k] \wedge m \text{ (By M1, M2 of definition 2.5)}$$
$$\Longrightarrow p^k \wedge m = p^{k+1} \wedge m \text{ (By Lemma 3.1, since } p^k \wedge p^{k+1} = p^{k+1} \text{)}$$

This is contradiction to hypothesis that $p^k \wedge m \neq p^{k+1} \wedge m$. Hence

$$(p^{k+1} : p^k) \wedge m = p \wedge m.$$

□

THEOREM 3.3. *Let L be a P-ADL with a maximal element m. If p is a prime element of L and q any proper divisor of p, then* $p \wedge m = (p.q) \wedge m$.

PROOF. Suppose p is a prime element of L and q is a proper divisor of p in L. Then $q \wedge p = p$ and $q \wedge m \neq p \wedge m$. Since q is principal and $q \wedge p = p$, by Lemma 3.1, we get

$$p \wedge m = [(p : q).q] \wedge m \longrightarrow (1)$$

So that $p \wedge [(p : q).q] = (p : q).q$ Since p is prime, we get $p \wedge (p : q) = p : q$ (Since $p \wedge q = q$) So that $(p.q) \wedge [(p : q).q] = (p : q).q$

Also, by Lemma 3.1, we get $p \wedge [(p : q).q] = (p : q).q$ Hence $p \wedge [(p : q).q] = (p.q) \wedge [(p : q).q]$. Thus $p \wedge [(p : q).q] \wedge m = (p.q) \wedge [(p : q).q] \wedge m$. So that $p \wedge m = (p.q) \wedge p \wedge m$ (By (1)) Therefore, $p \wedge m = p \wedge (p.q) \wedge m = (p.q) \wedge m$ (By condition (i) of Lemma 2.2). □

In the following result, we prove that every multiplicatively irreducible element of a P-ADL is a prime element.

THEOREM 3.4. *Let L be a P-ADL with a maximal element m. If every element of L is multiplicatively irreducible then it is prime.*

PROOF. Let $p \in L$ be a multiplicatively irreducible element of L. Let $a \in L$. Then $[p : (a \vee p)] \wedge m = (p : a) \wedge (p : p) \wedge m = (p : a) \wedge m$. By Lemma 3.3, we get

$$\{[p : (a \vee p)].(a \vee p)\} \wedge m = [(p : a).(a \vee p)] \wedge m \longrightarrow (1)$$

So that $[p : (a \vee p)] \wedge (p : a) = p : a$ Then $\{[p : (a \vee p)].(a \vee p)\} \wedge [(p : a).(a \vee p)] = (p : a).(a \vee p)$ Thus $\{[p : (a \vee p)].(a \vee p)\} \wedge m \geqslant [(p : a).(a \vee p)] \wedge m$. Since $a \vee p$ is principal and $(a \vee p) \wedge p = p$, by Lemma 3.1, we get

$$p \wedge m = \{[p : (a \vee p)].(a \vee p)\} \wedge m = [(p : a).(a \vee p)] \wedge m \text{ (By (1))}$$

Since p is multiplicatively irreducible, we get $p \wedge m = (p : a) \wedge m$ or $p \wedge m = (a \vee p) \wedge m$. If $p \wedge m = (a \vee p) \wedge m$, then $p \wedge (a \vee p) = a \vee p$ and hence $p : (a \vee p)$ is maximal. Now, $(p : a) \wedge m = [p : (a \vee p)] \wedge m = m$. Hence $p : a$ is maximal. Thus for any $a \in L$, $p : a$ is an associate of p or $p : a$ is maximal. Hence, by Lemma 3.2, p is prime. □

In the following Theorem, we give a sufficient condition for a prime element of a P-ADL L to become a multiplicatively irreducible element of L.

THEOREM 3.5. *Let L be a P-ADL with a maximal element m and $p \in L$. If p is prime and for every element x of L, $x : p$ is maximal then p is multiplicatively irreducible.*

PROOF. Suppose $p \in L$ is a prime element of L and for every element x of L, $x : p$ is maximal. Let $a, b \in L$ such that $p \wedge m = (a.b) \wedge m$. Then $p \wedge (a.b) = a.b$. Since p is prime, we get $p \wedge a = a$ or $p \wedge b = b$. Since $x : p$ is maximal, by condition R1 of definition 2.4, we get $x \wedge p = p$, for any $x \in L$. Hence $p \wedge m = a \wedge m$ or $p \wedge m = b \wedge m$. Thus p is multiplicatively irreducible. □

Let L be a residuated ADL with a maximal element m and with a.c.c. and $a \in L$. We define a^n by induction as follows :

$$a^1 = a \text{ and } a^{n+1} = a^n.a, \text{ for all } n \in Z^+.$$

By convention, we take $a^0 = m$.

DEFINITION 3.9. Let L be a residuated ADL with a.c.c. and q a primary element of L. A prime element p of L is called the prime corresponding to q if

$$p \wedge q = q,\ q \wedge p^k = p^k \text{ and } q \wedge p^{k-1} \neq p^{k-1}, \text{ for some } k \in Z^+.$$

THEOREM 3.6. *Let L be a residuated ADL with a maximal element m which satisfies the a.c.c. If q is a primary element of L and a is any element of L such that $q \wedge a \neq a$ then $q : a$ is primary and corresponds to the same prime as q.*

PROOF. Suppose q is a primary element of L and $a \in L$ such that $q \wedge a \neq a$. Let $b, c \in L$ such that $(q : a) \wedge (b.c) = b.c$ and $(q : a) \wedge b \neq b$. Now,

$(q : a) \wedge (b.c) = b.c$

$\implies q \wedge [a(b.c)] = a.(b.c)$ (By condition (A) of definition 2.6)

$\implies q \wedge [(a.b).c] = (a.b).c$ (By condition (iii) of Lemma 2.2)

$\implies q \wedge (a.b) = a.b$ or $q \wedge c^k = c^k$, for some $k \in Z^+$(Since q is primary)

If $q \wedge (a.b) = a.b$ then $(q : a) \wedge b = b$. Which is not true. Therefore, $q \wedge c^k = c^k$, for some $k \in Z^+$. So that $(q : a) \wedge (c^k : a) = c^k : a$ and hence

$$(q : a) \wedge c^k = (q : a) \wedge (c^k : a) \wedge c^k = (c^k : a) \wedge c^k = c^k.$$

Hence $q : a$ is primary.

Now, we prove that $q : a$ corresponds to the same prime p as q. We have $(q : a) \wedge (q : a) = q : a$. So that $q \wedge [a.(q : a)] = a.(q : a)$. Since q is primary and $q \wedge a \neq a$, we get that $q \wedge (q : a)^s = (q : a)^s$, forsome $s \in Z^+$. Now,

$p \wedge (q : a)^s = p \wedge [q \wedge (q : a)^s] = q \wedge (q : a)^s$ (Since $p \wedge q = q$)

$\implies p \wedge (q : a)^s = (q : a)^s$, forsome $s \in Z^+$.

Since p is prime, we get that $p \wedge (q : a) = q : a$. Since p is the prime corresponding to q, we get that $p \wedge q = q$, $q \wedge p^k = p^k$ and $q \wedge p^{k-1} \neq p^{k-1}$, for some $k \in Z^+$. So that
$(q : a) \wedge p^k = (q : a) \wedge (p^k : a) \wedge p^k = (p^k : a) \wedge p^k = p^k$.
$\Longrightarrow (q : a) \wedge p^k = p^k$, for some $k \in Z^+$.
Choose least positive integer l such that $(q : a) \wedge p^l = p^l$, where $l \leqslant k$. Therefore, $(q : a) \wedge p^{l-1} \neq p^{l-1}$, for some $l \in Z^+$. Thus $q : a$ corresponds to the same prime p as q. □

Lemma 3.5. *Let L be a residuated ADL with a maximal element m and L satisfies the a.c.c. If q is a primary element of L and p is the prime corresponding to q. Then, for any $a \in L$, $(q : a) \wedge m = q \wedge m$ if and only if $p \wedge a \neq a$.*

Proof. Suppose q is a primary element of L and p is the prime corresponding to q. Then $p \wedge q = q$, $q \wedge p^k = p^k$ and $q \wedge p^{k-1} \neq p^{k-1}$, for some $k \in Z^+$. Suppose $a \in L$ and $(q : a) \wedge m = q \wedge m$. If $p \wedge a = a$, then $q : p = (q : a) \wedge (q : p) = (q : a) \wedge m \wedge (q : p) = q \wedge m \wedge (q : p) = q \wedge (q : p)$.

Also we have, $(q : p) \wedge q = q$. Hence $(q : p) \wedge m = q \wedge m$. Now,
$q \wedge p^k = p^k \Longrightarrow q \wedge (p.p^{k-1}) = p.p^{k-1}$
$\Longrightarrow (q : p) \wedge p^{k-1} = p^{k-1}$ (By condition (A) of definition 2.6)
$\Longrightarrow (q : p) \wedge m \wedge p^{k-1} = p^{k-1}$
$\Longrightarrow q \wedge m \wedge p^{k-1} = p^{k-1}$ (Since $(q : p) \wedge m = q \wedge m$.)
$\Longrightarrow q \wedge p^{k-1} = p^{k-1}$.
This is a contradiction to $q \wedge p^{k-1} \neq p^{k-1}$, for some $k \in Z^+$. Therefore, $p \wedge a \neq a$.
Now, suppose that $p \wedge a \neq a$. Then
$(q : a) \wedge (q : a) = q : a$
$\Longrightarrow q \wedge [a.(q : a)] = a.(q : a)$ (By condition (A) of definition 2.6)
$\Longrightarrow q \wedge (q : a) = q : a$ or $q \wedge a^s = a^s$, for some $s \in Z^+$ (Since q is primary)
If $q \wedge a^s = a^s$, for some $s \in Z^+$, then $a^s = q \wedge a^s = p \wedge q \wedge a^s = p \wedge a^s$ and hence $p \wedge a = a$ (Since p is prime). This is contradiction to $p \wedge a \neq a$. Therefore, $q \wedge (q : a) = q : a$ By Lemma 2.4, we have $(q : a) \wedge q = q$ Hence $(q : a) \wedge m = q \wedge m$. □

Lemma 3.6. *Let L be a residuated ADL and $a, b \in L$ such that $a \wedge b = b$. Then $a^n \wedge b^n = b^n$, for any $n \in Z^+$.*

Proof. Let $a, b \in L$ be such that $a \wedge b = b$. This result is proved by induction on n. Clearly, the result is true for $n = 1$. Assume that $a^k \wedge b^k = b^k$, for some $k \in Z^+$. Then

$$(a^k.b) \wedge b^{k+1} = b^{k+1} \longrightarrow (1)$$

Now,
$a^{k+1} \wedge b^{k+1} = (a^k.a) \wedge b^{k+1} = [a^k.(a \vee b)] \wedge b^{k+1}$ (Since $a = a \vee b$)
$= [a^{k+1} \vee (a^k.b)] \wedge b^{k+1} = (a^{k+1} \wedge b^{k+1}) \vee [(a^k.b) \wedge b^{k+1}]$
$= (a^{k+1} \wedge b^{k+1}) \vee b^{k+1}$ (By (1)) $= b^{k+1}$. □

In the following result, we prove that power of a prime element of a P-ADL is primary.

THEOREM 3.7. *Let L be a P-ADL with a maximal element m. If p is a prime element of L, then p^n is a primary element of L, for any $n \in Z^+$.*

PROOF. Let p be a prime element of L and $n \in Z^+$. We prove that p^n is primary using induction on n. Since every prime element is primary, result is true for $n = 1$. Assume that p^k is primary. If $p^k \wedge m = p^{k+1} \wedge m$, then p^{k+1} is primary. Suppose $p^k \wedge m \neq p^{k+1} \wedge m$. Suppose $a, b \in L$ such that $p^{k+1} \wedge (a.b) = a.b$, $p^{k+1} \wedge a \neq a$. Now, $p^k \wedge (a.b) = p^k \wedge p^{k+1} \wedge (a.b) = p^{k+1} \wedge (a.b) = a.b$. Since p^k is primary, $p^k \wedge a = a$ or $p^k \wedge b^s = b^s$, for some $s \in Z^+$. If $p^k \wedge b^s = b^s$, then $p \wedge b^s = p \wedge p^k \wedge b^s = p^k \wedge b^s = b^s$. Since p is prime, we get $p \wedge b = b$. Now, suppose $p^k \wedge a = a$. Since p^k is principal, there exists $q \in L$ such that $p^k.q = a$. If $p \wedge q = q$, then $(p^k.p) \wedge (p^k.q) = p^k.q \Longrightarrow p^{k+1} \wedge a = a$ This is a contradiction. Thus $p \wedge q \neq q$. Now,

$$\begin{aligned}(p^{k+1} : a) \wedge m &= [p^{k+1} : (p^k.q)] \wedge m \\ &= [(p^{k+1} : p^k) : q] \wedge m = [(p^{k+1} : p^k) : q] \wedge (m : q) \wedge m \\ &= \{[(p^{k+1} : p^k) \wedge m] : q\} \wedge m = [(p \wedge m) : q] \wedge m \text{ (By Lemma 3.4)} \\ &= (p : q) \wedge (m : q) \wedge m = (p : q) \wedge m \\ &= p \wedge m \text{ (By Lemma 3.5, since } p \wedge q \neq q \text{)}\end{aligned}$$

Since $p^{k+1} \wedge (a.b) = a.b$, then by condition (A) of definition 2.6, we get $b = (p^{k+1} : a) \wedge b = p \wedge b$. Thus in both cases, we have $p \wedge b = b$. So that $p^{k+1} \wedge b^{k+1} = b^{k+1}$ (By Lemma 3.6). Hence p^{k+1} is primary. □

LEMMA 3.7. *Let L be a residuated ADL with a maximal element m and suppose L satisfies the a.c.c. Suppose q is a primary element of L, p is a prime element of L and $k \in Z^+$ such that $p \wedge q = q$, $q \wedge p^k = p^k$, $q \wedge p^{k-1} \neq p^{k-1}$. If $q \wedge m = p^r \wedge m$, for some $r \in Z^+$ then $q \wedge m = p^k \wedge m$.*

PROOF. Suppose $q \wedge m = p^r \wedge m$, for some $r \in Z^+$. Then

$$r \geqslant k \text{ implies } p^k \wedge p^r = p^r \longrightarrow (1)$$

Now,

$$\begin{aligned}q \wedge p^k = p^k &\Longrightarrow q \wedge m \wedge p^k \wedge m = p^k \wedge m \\ &\Longrightarrow p^r \wedge m \wedge p^k \wedge m = p^k \wedge m \\ &\Longrightarrow p^r \wedge p^k \wedge m = p^k \wedge m \\ &\Longrightarrow p^k \wedge p^r \wedge m = p^k \wedge m \\ &\Longrightarrow p^r \wedge m = p^k \wedge m \text{ (By above (1))} \\ &\Longrightarrow q \wedge m = p^k \wedge m.\end{aligned}$$

□

Finally, we prove the converse of Theorem 3.7 in the following.

THEOREM 3.8. *Let L be a P-ADL with a maximal element m. If q is a primary element of L and p is the prime corresponding to q then $q \wedge m = p^r \wedge m$, for some $r \in Z^+$.*

PROOF. Suppose q is a primary element of L and p is the prime corresponding to q. Then $p \wedge q = q$, $q \wedge p^k = p^k$ and $q \wedge p^{k-1} \neq p^{k-1}$, for some $k \in Z^+$. We prove that $q \wedge m = p^k \wedge m$. This result is proved by induction on k.

If $k = 1$, then we have $p \wedge q = q$ and $q \wedge p = p$. Hence $q \wedge m = p \wedge m$.

Assume that the result is true for all $s \leqslant k-1$. That is, if $\bar{q}$ is a primary element of L and p is the prime corresponding to $\bar{q}$ such that $\bar{q} \wedge p^s = p^s$ and $\bar{q} \wedge p^{s-1} \neq p^{s-1}$, for some $s \in Z^+$ with $1 \leqslant s \leqslant k-1$, then $\bar{q} \wedge m = p^s \wedge m$. If $q \wedge p \neq p$, then, by Theorem 3.6, $q : p$ is a primary element of L and p is the prime corresponding to $q : p$. So that $p \wedge (q : p) = q : p$. Also, by condition (A) of definition 2.6, we get that $q \wedge p^k = p^k \Longrightarrow (q : p) \wedge p^{k-1} = p^{k-1}$ and $q \wedge p^{k-1} \neq p^{k-1} \Longrightarrow (q : p) \wedge p^{k-2} \neq p^{k-2}$. Then by induction hypothesis, we get $(q : p) \wedge m = p^{k-1} \wedge m$. Since $p \wedge q = q$, by Lemma 3.1, we get $q \wedge m = [(q : p).p] \wedge m = (p^{k-1}.p) \wedge m$ (By Lemma 3.3) Hence $q \wedge m = p^k \wedge m$. □

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