PRINCIPAL RESIDUATED ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of a simple element in a residuated ADL and the concept of Principal residuated Almost Distributive Lattice (P-ADL). We prove important results in a P-ADL.

1. Introduction

Swamy, U.M. and Rao, G.C. [6] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, $p$-rings, biregular rings, associate rings, $P_1$-rings etc.) on one hand and distributive lattices on the other.

In [1], Dilworth, R.P., has introduced the concept of a residuation in lattices and in [7, 8], Ward, M. and Dilworth, R.P., have studied residuated lattices. In [9], Ward, M. has introduced the concept of a principal residuated lattice (or simply a P-Lattice) and studied its properties. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [3]. We have proved some important properties of residuation $' : ' $ and multiplication $' . ' $ in a residuated ADL $L$ in [4]. In [5], we introduced the concept of principal element in a residuated ADL.

In this paper, we introduce the concept of a simple element in a Residuated ADL and the concept of Principal residuated Almost Distributive Lattice. We prove important results in a P-ADL. In Section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL from Swamy, U.M. and Rao, G.C. [6], Rao, G.C. [2] and some important results on a residuated almost distributive lattice from our earlier paper [3]. In section 3,
we introduce the concept of a simple element in a residuated ADL and the concept of Principal residuated Almost Distributive Lattice (P-ADL). We give an example of a P-ADL. We prove that a simple element of a residuated ADL is prime and every multiplicatively irreducible element of a P-ADL is prime. We also prove that any power of a prime element of a P-ADL is a primary element and every primary element of a P-ADL L is a power of a prime element of L.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

We begin with the definition of an ADL :

**Definition 2.1.** (2].) An Almost Distributive Lattice (ADL) is an algebra 
\((L, \vee, \wedge)\) of type \((2, 2)\) satisfying

1. \((a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)\)
2. \(a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\)
3. \((a \vee b) \wedge b = b\)
4. \((a \vee b) \wedge a = a\)
5. \(a \vee (a \wedge b) = a\), for all \(a, b, c \in L\).

It can be seen directly that every distributive lattice is an ADL. If there is an element \(0 \in L\) such that \(0 \wedge a = 0\) for all \(a \in L\), then \((L, \vee, \wedge, 0)\) is called an ADL with 0.

**Example 2.1.** (2].) Let \(X\) be a non-empty set. Fix \(x_0 \in X\). For any \(x, y \in L\), define

\[x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0\end{cases}\]
\[x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0\end{cases}\]

Then \((X, \vee, \wedge, x_0)\) is an ADL, with \(x_0\) as its zero element. This ADL is called a discrete ADL.

For any \(a, b \in L\), we say that \(a\) is less than or equals to \(b\) and write \(a \leq b\), if \(a \wedge b = a\). Then \(\leq\) is a partial ordering on \(L\).

**Theorem 2.1** (2].) Let \((L, \vee, \wedge, 0)\) be an ADL with \(0 \neq 0\). Then, for any \(a, b \in L\), we have

1. \(a \wedge 0 = 0\) and \(0 \vee a = a\)
2. \(a \wedge a = a\)
3. \((a \wedge b) \vee b = b\) and \(a \wedge (b \wedge a) = a\)
4. \(a \wedge b = a \iff a \vee b = b\) and \(a \wedge b = b \iff a \vee b = a\)
5. \(a \wedge b = b \wedge a\) and \(a \vee b = b \vee a\) whenever \(a \leq b\)
6. \(a \wedge b \leq a\) and \(a \leq a \vee b\)
7. \(\wedge\) is associative in \(L\)
8. \(a \wedge b \wedge c = b \wedge a \wedge c\)
9. \((a \vee b) \wedge c = (b \vee a) \wedge c\)
10. \(a \wedge b = 0 \iff b \wedge a = 0\)
11. \(a \vee (b \wedge a) = a \vee b\).
It can be observed that an ADL $L$ satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\lor$ over $\land$, the commutativity of $\lor$, the commutativity of $\land$ and the absorption law $(a \land b) \lor a = a$. Any one of these properties convert $L$ into a distributive lattice.

**Theorem 2.2 ([2]).** Let $(L, \lor, \land, 0)$ be an ADL with $0$. Then the following are equivalent:

1. $(L, \lor, \land, 0)$ is a distributive lattice
2. $a \lor b = b \lor a$, for all $a, b \in L$
3. $a \land b = b \land a$, for all $a, b \in L$
4. $(a \land b) \lor c = (a \lor c) \land (b \lor c)$, for all $a, b, c \in L$.

**Proposition 2.1 ([2]).** Let $(L, \lor, \land)$ be an ADL. Then for any $a, b, c \in L$ with $a \leq b$, we have

1. $a \land c \leq b \land c$
2. $c \land a \leq c \land b$
3. $c \lor a \leq c \lor b$.

**Definition 2.2.** ([2]) An element $m \in L$ is called maximal if it is maximal as in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a$ implies $m = a$.

**Theorem 2.3 ([2]).** Let $L$ be an ADL and $m \in L$. Then the following are equivalent:

1. $m$ is maximal with respect to $\leq$
2. $m \lor a = m$ for all $a \in L$
3. $m \land a = a$ for all $a \in L$.

**Lemma 2.1 ([2]).** Let $L$ be an ADL with a maximal element $m$ and $x, y \in L$. If $x \land y = y$ and $y \lor x = x$ then $x$ is maximal if and only if $y$ is maximal. Also the following conditions are equivalent:

1. $x \land y = y$ and $y \lor x = x$
2. $x \land m = y \land m$.

**Definition 2.3.** ([2]) If $(L, \lor, \land, 0, m)$ is an ADL with $0$ and with a maximal element $m$, then the set $I(L)$ of all ideals of $L$ is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of $I, J$ are given by $I \lor J = \{x \lor y \land m \mid x \in I, y \in J\}$ and $I \land J = I \cap J$. The set $PI(L) = \{[a] \mid a \in L\}$ of all principal ideals of $L$ forms a sublattice of $I(L)$.

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) $L$ and the definition of a residuated almost distributive lattice taken from our earlier paper [3].

**Definition 2.4 ([3]).** Let $L$ be an ADL with a maximal element $m$. A binary operation $: \lor$ on an ADL $L$ is called a residuation over $L$ if, for $a, b, c \in L$ the following conditions are satisfied.

1. $(R1) a \land b = b$ if and only if $a \lor b$ is maximal
2. $(R2) a \land b = b \implies (i) (a : c) \land (b : c) = b \land c$ and (ii) $(c : b) \land (c : a) = c \land a$
operations ‘:’ and ‘.’ on residuated almost distributive lattice (residuated ADL), if there exists two binary operations ‘:’ and ‘.’ in a residuated ADL \( L \). These are taken from our earlier paper [4].

We use the following properties frequently later in the results.

**Lemma 2.2** ([3]). Let \( L \) be an ADL with a maximal element \( m \) and a binary operation on \( L \) satisfying the conditions \( M1 - M4 \). Then for any \( a, b, c, d \in L \),

1. \( a \land (a \cdot b) = a.b \) and \( b \land (a \cdot b) = a.b \)
2. \( a \land b = b \iff (c.a) \land (c.b) = c.b \) and \( (a.c) \land (b.c) = b.c \)
3. \( d \land [(a, b).c] = (a.b).c \) if and only if \( d \land [(a \cdot (b.c)] = (a \cdot b).c \)
4. \( (a.c) \land (b.c) \land [(a \land b).c] = (a \land b).c \)
5. \( d \land (a.c) \land (b.c) = (a.c) \land (b.c) \iff d \land [(a \land b).c] = (a \land b).c \)
6. \( d \land [(a \cdot c) \lor (b \cdot c)] = (a \cdot c) \lor (b \cdot c) \iff d \land [(a \lor b).c] = (a \lor b).c \)

The following result is a direct consequence of \( M1 \) of Definition 2.15.

**Lemma 2.3** ([3]). Let \( L \) be an ADL with a maximal element \( m \) and a binary operation on \( L \) satisfying the condition \( M1 \). For \( a, b, x \in L \), \( a \land (x.b) = x.b \) if and only if \( a \land (b.x) = b.x \).

In the following, we give some important properties of residuation ‘:’ and multiplication ‘.’ in a residuated ADL \( L \). These are taken from our earlier paper [4].

**Lemma 2.4** ([4]). Let \( L \) be a residuated ADL with a maximal element \( m \). For \( a, b, c, d \in L \), the following hold in \( L \).

1. \( (a : b) \land a = a \)
2. \( [a : (a : b)] \land (a \lor b) = a \lor b \)
3. \( [(a : b) : c] \land [a : (b : c)] = a : (b,c) \)
4. \( a : (b,c) \land [a : (b : c)] = (a : b) : c \)
5. \( [(a \land b) : b] \land (a : b) = a : b \)
6. \( (a : b) \land [(a \land b) : b] = (a \land b) : b \)
7. \( a : (b \lor c) \land m = (a : b) \land m \)
8. \( c : (a \land b) \land [(c : a) \lor (c : b)] = (c : a) \lor (c : b) \)
(9) If $a : b = a$ then $a \wedge (b : d) = b : d \implies a \wedge d = d$

(10) $[a : [a : (a : b)]] \wedge (a : b) = a : b$

(11) $[(a \vee b) : c] \wedge [(a : c) \vee (b : c)] = (a : c) \vee (b : c)$

(12) $a \wedge m \geq b \wedge m \implies (a : c) \wedge m \geq (b : c) \wedge m$

(13) $(a : b) \wedge [a : [a : (a : b)]] = a : [a : (a : b)]$

(14) $a \wedge b = b \implies (a : c) \wedge (b : c) = b : c$

(15) $a \wedge b \wedge (a : b) = a : b$

(16) $[(a : b) : a] \wedge b = b$

(17) $(a : b) \wedge [(a \wedge b) : (a \vee b)] = (a \wedge b) : (a \vee b)$

(18) $a \vee b$ is maximal $\implies (a , b) \wedge a \wedge b = a \wedge b$

3. Principal Residuated Almost Distributive Lattices

In this section, we introduce the concept of a simple element in a residuated
ADL and the concept of Principal residuated Almost Distributive Lattice (P-ADL).
We give an example of a P-ADL. We prove important results in a P-ADL.

We recall the following concepts on a residuated ADL $L$ :

**Definition 3.1.** ([5]) An element $p$ of a residuated ADL $L$ is called
(i) irreducible, if for any $f, g \in L$, $f \wedge g = p \implies$ either $f = p$ or $g = p$.
(ii) prime, if for any $a, b \in L$, $p \wedge (a : b) = a : b \implies$ either $p \wedge a = a$ or $p \wedge b = b$.
(iii) primary, if for any $a, b \in L$, $p \wedge (a : b) = a : b$ and $p \wedge a \neq a \implies p \wedge b^s = b^s$, for some $s \in \mathbb{Z}^+$.

**NOTE**: Clearly, every prime element of a residuated ADL is primary.

**Definition 3.2.** ([5]) An ADL $L$ is said to satisfy the ascending chain condition
(a.c.c.), if for every increasing sequence $x_1 \leq x_2 \leq x_3 \leq \ldots \ldots$, in $L$, there exists a positive integer $n$ such that
$x_n = x_{n+1} = x_{n+2} = \ldots \ldots$

**Definition 3.3.** ([5]) Let $L$ be a residuated ADL. An element $a$ of $L$ is called
principal, if $b \in L$ and $a \wedge b = b$, then $a : b = b$, for some $c \in L$.

**Definition 3.4.** ([5]) A residuated ADL $L$ is said to be a Noether ADL, if
(N1) the ascending chain condition (a.c.c.) holds in $L$ and
(N2) every irreducible element of $L$ is primary.

Now, we give the following definitions.

**Definition 3.5.** Let $L$ be an ADL with a maximal element $m$. An element $x$
of $L$ is called an associate of $y$ if $x \wedge m = y \wedge m$ ( or $x$ is equivalent to $y$ ). This is
an equivalence relation on $L$.

**Definition 3.6.** Let $L$ be an ADL and $x, y \in L$.
(i) $y$ is called a divisor of $x$ if $y \wedge x = x$. Observe that every maximal element
$m$ is a divisor of $x$, for any $x \in L$ and every associate of $x$ is a divisor of $x$.
(ii) A divisor $y$ of $x$ other than maximal elements and associates of $x$ is called
a proper divisor of $x$. 
**Definition 3.7.** Let \( L \) be a residuated ADL with a maximal element \( m \) and \( p \in L \). Then

(i) \( p \) is called multiplicatively irreducible if for any \( a, b \in L \),

\[
p \land m = (a \cdot b) \land m \implies \text{either } p \land m = a \land m \text{ or } p \land m = b \land m.
\]

(ii) \( p \) is said to be simple if it has no proper divisors.

Let us recall the Fundamental Theorem (Theorem 3.4) from [5].

**Theorem 3.1** ([5]). Let \( L \) be an ADL with a maximal element \( m \) satisfying the following conditions:

1. \( L \) is residuated.
2. \( L \) satisfies a.c.c.
3. Every element of \( L \) is principal.

Then \( L \) is a Noether ADL.

In view of the above Theorem, we give the following.

**Definition 3.8.** Let \( L \) be a residuated ADL with a.c.c. If every element of \( L \) is principal then \( L \) is called a Principal Residuated Almost Distributive Lattice (or P-ADL).

Thus we get every P-ADL is a Noether ADL.

The following Lemma was proved in our earlier paper [5] and is used frequently later in the results.

**Lemma 3.1** ([5]). Let \( L \) be a residuated ADL with a maximal element \( m \). If \( a, b \in L \) such that \( a \) is principal and \( a \land b = b \) then \([ (b : a) \land m = b \land m \].

Now, we give an example of a residuated ADL and an example of a P-ADL.

**Example 3.1.** If \( L \) is a discrete ADL and \( M = \{0, a, b, 1\} \), where \( 0 \leq b \leq a \leq 1 \), is a P-lattice. Let \( m \) be a maximal element of \( L \). Define residuation ‘ : ’ and multiplication ‘ . ’ on \( L \) by \( x : y = x \) and \( x \cdot y = m \), if \( x \land y \neq 0 \) and \( x \cdot y = 0 \), if \( x = 0 \) or \( y = 0 \). Also define residuation ‘ : ’ and multiplication ‘ . ’ on \( M \) by the following tables:

\[
\begin{array}{c|ccc|c|ccc}
: & 1 & a & b & 0 & 1 & a & b & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & a & b & 0 \\
a & a & 1 & 1 & 1 & a & b & b & 0 \\
b & b & a & 1 & 1 & b & b & b & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Then \( L \times M \) is a P-ADL under pointwise operations ‘ : ’ and ‘ . ’.

**Lemma 3.2.** Let \( L \) be a residuated ADL with a maximal element \( m \). If for every element \( x \) of \( L \), \( p : x \) is an associate of \( p \) or \( p : x \) is maximal then \( p \) is prime.

**Proof.** Let \( x, p \in L \), and suppose \( p : x \) is an associate of \( p \) or \( p : x \) is maximal. Let \( a, c \in L \) be such that \( p \land (a \cdot c) = a \cdot c \) and \( p \land a \neq a \). Now, \( p \land a \neq a \).
\[ \Rightarrow p : a \text{ is not maximal ( By condition R1 of definition 2.4 )} \]
\[ \Rightarrow p : a \text{ is an associate of } p. \]
\[ (p : a) \land m = p \land m \]
\[ ([p \land m] : c) \land m = [(p : a) \land m] : c \land m \]
\[ (p : c) \land (m : c) \land m = [(p : a) : c] \land (m : c) \land m \]
\[ (p : c) \land m = [(p : a) : c] \land m \]
\[ (p : c) \land m = (p : (a.c)) \land m \text{ ( By properties (3) and (4) of Lemma 2.4 )} \]
\[ (p : c) \land m = m \text{ ( Since } p : (a.c) \text{ is maximal) } \]
\[ p : c \text{ is maximal} \]
\[ p \land c = c. \]

Hence \( p \) is prime. \( \square \)

**Theorem 3.2.** In a residuated ADL, every simple element is prime.

**Proof.** Let \( L \) be a residuated ADL and \( p \), a simple element of \( L \). Let \( x \in L \).

Then we have, by Lemma 2.4, \( (p : x) \land p = p \). Since \( p \) is simple, \( p : x \) is maximal or \( p : x \) is an associate of \( p \). Hence by Lemma 3.2, \( p \) is prime. \( \square \)

**Lemma 3.3.** Let \( L \) be a residuated ADL with a maximal element \( m \) and \( a, b \in L \) such that \( a \land m = b \land m \). Then \( (p.a) \land m = (p.b) \land m \), for any \( p \in L \).

**Proof.** Let \( a, b, p \in L \) and suppose \( a \land m = b \land m \). Then \( a \lor b = a \) and \( b \lor a = b \). Now,
\[
(p.a) \land m = [p(a \lor b)] \land m
\]
\[
= [(p.a) \lor (p.b)] \land m \text{ ( By condition M4 of definition 2.4 )}
\]
\[
= [(p.b) \lor (p.a)] \land m
\]
\[
= [p.(b \lor a)] \land m \text{ ( By condition M4 of definition 2.4 )}
\]
\[
= (p.b) \land m.
\]

Hence \( (p.a) \land m = (p.b) \land m \), for any \( p \in L \). \( \square \)

**Lemma 3.4.** Let \( L \) be a \( P \)-ADL with a maximal element \( m \). If \( p \) is a prime element of \( L \) and \( p^{k+1} \land m \neq p^k \land m \), for some positive integer \( k \) then
\[ (p^{k+1} : p^k) \land m = p \land m. \]

**Proof.** Let \( p \in L \) be a prime element of \( L \) such that \( p^{k+1} \land m \neq p^k \land m \). Suppose \( (p^{k+1} : p^k) \land m \neq p \land m \). Since \( p^{k+1} \land (p^k.p) = p^k.p \), we get \( (p^{k+1} : p^k) \land p = p \). Hence \( p \land (p^{k+1} : p^k) \neq p^{k+1} : p^k \). We have \( (p^{k+1} : p^k) \) is principal and \( (p^{k+1} : p^k) \land p = p \).

Now, by Lemma 3.1, we get
\[ p \land m = [(p : (p^{k+1} : p^k)).(p^{k+1} : p^k)] \land m \rightarrow (1) \]

Thus
\[ p \land [(p : (p^{k+1} : p^k)).(p^{k+1} : p^k)] = [p : (p^{k+1} : p^k)].(p^{k+1} : p^k). \]

Since \( p \) is prime, we get either
\[ p \land [p : (p^{k+1} : p^k)] = p : (p^{k+1} : p^k) \text{ or } p \land (p^{k+1} : p^k) = p^{k+1} : p^k. \]
But $p \land (p^{k+1} : p^k) \neq p^{k+1} : p^k$. Thus

$$p \land [p : (p^{k+1} : p^k)] = p : (p^{k+1} : p^k).$$

Also we have $[p : (p^{k+1} : p^k)] \land p = p$ (By property (1) of Lemma 2.4). Thus $[p : (p^{k+1} : p^k)] \land m = p \land m$. So that, by Lemma 3.4, we get

$$[(p^{k+1} : p^k), p] \land m = \{(p^{k+1} : p^k), [p : (p^{k+1} : p^k)]\} \land m.$$ 

Now, by (1), $p \land m = [(p^{k+1} : p^k), p] \land m$. Now,

$$p^k \land m = (p \land m)^{(k+1)} \land m = \{(p^{k+1} : p^k), p, p^{k+1} \land m \} \land m \text{ (By above Lemma 3.3) }$$

$$\implies p^k \land m = [(p^{k+1} : p^k), p^k] \land m \text{ (By M1, M2 of definition 2.5) }$$

$$\implies p^k \land m = (p \land m)^{(k+1)} \land m \text{ (By Lemma 3.1, since } p^k \land p^{k+1} = p^{k+1} \text{) }$$

This is contradiction to hypothesis that $p^k \land m \neq p^{k+1} \land m$. Hence

$$(p^{k+1} : p^k) \land m = p \land m.$$  

\[ \square \]

**Theorem 3.3.** Let $L$ be a P-ADL with a maximal element $m$. If $p$ is a prime element of $L$ and $q$ any proper divisor of $p$, then $p \land m = (p : q) \land m$. 

**Proof.** Suppose $p$ is a prime element of $L$ and $q$ is a proper divisor of $p$ in $L$. Then $q \land p = p$ and $q \land m \neq p \land m$. Since $q$ is principal and $q \land p = p$, by Lemma 3.1, we get

$$p \land m = [(p : q), q] \land m \implies (1)$$

So that $p \land [(p : q), q] = (p : q) \land q$. Since $p$ is prime, we get $p \land [(p : q), q] = (p : q) \land q$ (Since $p \land q = q$). So that $(p : q) \land [(p : q), q] = (p : q) \land q$. 

Also, by Lemma 3.1, we get $p \land [(p : q), q] = (p : q) \land q$. Hence $p \land [(p : q), q] = (p : q) \land [(p : q), q]$. Thus $p \land [(p : q), q] \land m = (p : q) \land [(p : q), q] \land m$. So that $p \land m = (p : q) \land p \land m \text{ (By (1)) }$. Therefore, $p \land m = p \land (p : q) \land m = (p : q) \land m \text{ (By condition (i) of Lemma 2.2) }$. 

In the following result, we prove that every multiplicatively irreducible element of a P-ADL is a prime element.

**Theorem 3.4.** Let $L$ be a P-ADL with a maximal element $m$. If every element of $L$ is multiplicatively irreducible then it is prime. 

**Proof.** Let $p \in L$ be a multiplicatively irreducible element of $L$. Let $a \in L$. Then $[p : (a \land p)] \land m = (p : a) \land (p : p) \land m = (p : a) \land m$. By Lemma 3.3, we get

$$\{(p : (a \land p)),(a \land p)\} \land m = [(p : a),(a \land p)] \land m \implies (1)$$

So that $[p : (a \land p)] \land (p : a) = p : a$. Then $\{(p : (a \land p) \land (a \land p)) \land [(p : a),(a \land p)] = (p : a) \land (a \land p) \land m \geq [(p : a),(a \land p)] \land m$. Since $a \land p$ is principal and $(a \land p) \land p = p$, by Lemma 3.1, we get

$$p \land m = [(p : (a \land p),(a \land p)) \land m = [(p : a),(a \land p)] \land m \text{ (By (1)) }$$
Since \( p \) is multiplicatively irreducible, we get \( p \wedge m = (p : a) \wedge m \) or \( p \wedge m = (a \vee p) \wedge m \).

If \( p \wedge m = (a \vee p) \wedge m \), then \( p \wedge (a \vee p) = a \vee p \) and hence \( p : (a \vee p) \) is maximal.

Now, \( (p : a) \wedge m = [p : (a \vee p)] \wedge m = m \). Hence \( p : a \) is maximal. Thus for any \( a \in L \), \( p : a \) is an associate of \( p \) or \( p : a \) is maximal. Hence, by Lemma 3.2, \( p \) is prime.

In the following Theorem, we give a sufficient condition for a prime element of a P-ADL \( L \) to become a multiplicatively irreducible element of \( L \).

**Theorem 3.5.** Let \( L \) be a P-ADL with a maximal element \( m \) and \( p \in L \). If \( p \)

is prime and for every element \( x \) of \( L \), \( x : p \) is maximal then \( p \) is multiplicatively irreducible.

**Proof.** Suppose \( p \in L \) is a prime element of \( L \) and for every element \( x \) of \( L \),

\( x : p \) is maximal. Let \( a, b \in L \) such that \( p \wedge m = (a.b) \wedge m \). Then \( p \wedge (a.b) = a.b \).

Since \( p \) is prime, we get \( p \wedge a = a \) or \( p \wedge b = b \). Since \( x : p \) is maximal, by condition

R1 of definition 2.4, we get \( x \wedge p = p \), for any \( x \in L \). Hence \( p \wedge m = a \wedge m \) or \( p \wedge m = b \wedge m \). Thus \( p \) is multiplicatively irreducible.

Let \( L \) be a residuated ADL with a maximal element \( m \) and with a.c.c. and \( a \in L \). We define \( a^n \) by induction as follows:

\[ a^1 = a \text{ and } a^{n+1} = a^n.a, \text{ for all } n \in \mathbb{Z}^+. \]

By convention, we take \( a^0 = m \).

**Definition 3.9.** Let \( L \) be a residuated ADL with a.c.c. and \( q \) a primary element of \( L \). A prime element \( p \) of \( L \) is called the prime corresponding to \( q \) if

\[ p \wedge q = q, \text{ for all } k \in \mathbb{Z}^+ \text{ and } q \wedge p^k = p^k \]  

and \( q \wedge p^{k+1} \neq p^{k+1} \), for some \( k \in \mathbb{Z}^+ \).

**Theorem 3.6.** Let \( L \) be a residuated ADL with a maximal element \( m \) which satisfies the a.c.c. If \( q \) is a primary element of \( L \) and \( a \) is any element of \( L \) such that \( q \wedge a \neq a \) then \( q : a \) is primary and corresponds to the same prime as \( q \).

**Proof.** Suppose \( q \) is a primary element of \( L \) and \( a \in L \) such that \( q \wedge a \neq a \).

Let \( b, c \in L \) such that \( (q : a) \wedge (b.c) = b.c \) and \( (q : a) \wedge b \neq b \). Now,

\[ (q : a) \wedge (b.c) = b.c \]

\[ \implies q \wedge [(a.b).c] = (a.b).c \] (By condition (A) of definition 2.6 )

\[ \implies q \wedge (a.b) = (a.b) \text{ ( By condition (iii) of Lemma 2.2 )} \]

\[ \implies q \wedge (a.c) = q \wedge (a.b) = a.b \text{ or } q \wedge c^k = c^k, \text{ for some } k \in \mathbb{Z}^+. \text{ (Since } q \text{ is primary) } \]

If \( q \wedge (a.b) = a.b \) then \( (q : a) \wedge b = b. \text{ Which is not true. Therefore, } q \wedge c^k = c^k, \text{ for some } k \in \mathbb{Z}^+. \text{ So that } (q : a) \wedge (c^k : a) = c^k : a \) and hence

\[ (q : a) \wedge c^k = (q : a) \wedge (c^k : a) \wedge c^k = (c^k : a) \wedge c^k = c^k. \]

Hence \( q : a \) is primary.

Now, we prove that \( q : a \) corresponds to the same prime \( p \) as \( q \). We have

\[ (q : a) \wedge (q : a) = q : a. \text{ So that } q \wedge [a.(q : a)] = a.(q : a). \text{ Since } q \text{ is prime and } q \wedge a \neq a, \text{ we get that } q \wedge (q : a)^s = (q : a)^s, \text{ for some } s \in \mathbb{Z}^+. \text{ Now, } \]

\[ p \wedge (q : a)^s = p \wedge [q \wedge (q : a)^s] = q \wedge (q : a)^s \text{ ( Since } p \wedge q = q \)

\[ \implies p \wedge (q : a)^s = (q : a)^s, \text{ for some } s \in \mathbb{Z}^+. \]
Since \( p \) is prime, we get that \( p \cap (q : a) = q : a \). Since \( p \) is the prime corresponding to \( q \), we get that \( p \cap q = q, q \cap p^k = p^k \) and \( q \cap p^{k-1} \neq p^{k-1} \), for some \( k \in Z^+ \). So that

\[
(q : a) \cap p^k = (q : a) \cap (p^k : a) \cap p^k = (p^k : a) \cap p^k = p^k.
\]

Choose least positive integer \( l \) such that \( (q : a) \cap p^l = p^l \), where \( l \leq k \). Therefore, \((q : a) \cap p^{l-1} \neq p^{l-1}\), for some \( l \in Z^+ \). Thus \( q : a \) corresponds to the same prime \( p \) as \( q \).

\[ \Box \]

**Lemma 3.5.** Let \( L \) be a residuated ADL with a maximal element \( m \) and \( L \) satisfies the a.c.c. If \( q \) is a primary element of \( L \) and \( p \) is the prime corresponding to \( q \). Then, for any \( a \in L \), \((q : a) \cap m = q \cap m \) if and only if \( p \cap a \neq a \).

**Proof.** Suppose \( q \) is a primary element of \( L \) and \( p \) is the prime corresponding to \( q \). Then \( p \cap q = q, p \cap p^k = p^k \) and \( q \cap p^{k-1} \neq p^{k-1} \), for some \( k \in Z^+ \). Suppose \( a \in L \) and \((q : a) \cap m = q \cap m \). If \( p \cap a = a \), then \( q \cap p = (q : a) \cap (q : p) = (q : a) \cap m \cap (q : p) = q \cap m \cap (q : p) = q \cap (q : p) \).

Also we have, \((q : p) \cap q = q \). Hence \((q : p) \cap m = q \cap m \). Now, \( q \cap p^k = p^k \implies q \cap (p \cap p^{k-1}) = p \cap p^{k-1} \Rightarrow (q : p) \cap p^{k-1} = p^{k-1} \) (By condition (A) of definition 2.6).

\[
q \cap m \cap p^{k-1} = p^{k-1} \ (\text{Since } (q : p) \cap m = q \cap m,) \\
\Rightarrow q \cap p^{k-1} = p^{k-1}.
\]

This is a contradiction to \( q \cap p^{k-1} \neq p^{k-1} \), for some \( k \in Z^+ \). Therefore, \( p \cap a \neq a \). Now, suppose that \( p \cap a \neq a \). Then

\[
(q : a) \cap (q : a) = q : a \\
\implies q \cap [a \cap (q : a)] = a \cap (q : a) \ (\text{By condition (A) of definition 2.6}) \\
\Rightarrow q \cap (q : a) = q : a or q \cap a^s = a^s, \text{for some } s \in Z^+. (\text{Since } q \text{ is primary})
\]

If \( q \cap a^s = a^s \), for some \( s \in Z^+ \), then \( a^s = q \cap a^s = p \cap q \cap a^s = p \cap a^s \) and hence \( p \cap a = a \). (Since \( p \) is prime). This is contradiction to \( p \cap a \neq a \). Therefore, \( q \cap (q : a) = q : a \). By Lemma 2.4, we have \((q : a) \cap q = q \) Hence \((q : a) \cap m = q \cap m. \)

**Lemma 3.6.** Let \( L \) be a residuated ADL and \( a, b \in L \) such that \( a \cap b = b \). Then \( a^n \cap b^n = b^n \), for any \( n \in Z^+ \).

**Proof.** Let \( a, b \in L \) be such that \( a \cap b = b \). This result is proved by induction on \( n \). Clearly, the result is true for \( n = 1 \). Assume that \( a^k \cap b^k = b^k \), for some \( k \in Z^+ \). Then

\[
(a^k \cap b^k) \cap b^{k+1} = b^{k+1} \implies (1)
\]

Now,

\[
a^{k+1} \cap b^{k+1} = (a^k \cap a) \cap b^{k+1} = [a^k \cap (a \cap b)] \cap b^{k+1} \ (\text{Since } a = a \cap b) \\
= [a^{k+1} \cap (a \cap b)] \cap b^{k+1} = (a^{k+1} \cap b^{k+1}) \lor [(a^k \cap b) \cap b^{k+1}] \\
= (a^{k+1} \cap b^{k+1}) \lor b^{k+1} \ (\text{By (1)}) = b^{k+1}. \Box
\]

In the following result, we prove that power of a prime element of a P-ADL is primary.
Theorem 3.7. Let $L$ be a $P$-ADL with a maximal element $m$. If $p$ is a prime element of $L$, then $p^n$ is a primary element of $L$, for any $n \in Z^+$. 

Proof. Let $p$ be a prime element of $L$ and $n \in Z^+$. We prove that $p^n$ is primary using induction on $n$. Since every prime element is primary, result is true for $n = 1$. Assume that $p^k$ is primary. If $p^k \land m = p^{k+1} \land m$, then $p^{k+1}$ is primary. Suppose $p^k \land m \neq p^{k+1} \land m$. Suppose $a, b \in L$ such that $p^{k+1} \land (a \cdot b) = a \cdot b$, $p^{k+1} \land a \neq a$. Now, $p^k \land (a \cdot b) = p^k \land p^{k+1} \land (a \cdot b) = p^{k+1} \land (a \cdot b) = a \cdot b$. Since $p^k$ is primary, $p^k \land a = a$ or $p^k \land b = b^*$, for some $s \in Z^+$. If $p^k \land b^* = b^*$, then $p \land b^* = p \land p^k \land b^* = p^k \land b^* = b^*$. Since $p$ is prime, we get $p \land b = b$. Now, suppose $p^k \land a = a$. Since $p^k$ is principal, there exists $q \in L$ such that $p^k \land q = a$. If $p \land q = q$, then $(p^k \cdot p) \land (p^k \cdot q) = p^k \cdot q \implies p^{k+1} \land a = a$. This is a contradiction. Thus $p \land q \neq q$. Now, 

$$
(p^{k+1} : a) \land m = [p^{k+1} : (p^k : q)] \land m = [(p^{k+1} : p^k) : q] \land m = [(p^k : q) \land m : q) \land m = [(p \land m) : q] \land m \quad \text{(By Lemma 3.4)}
$$

Then 

$$
(p \cdot q) \land (m : q) \land m = (p : q) \land m = p \land m \quad \text{(By Lemma 3.5, since $p \land q \neq q$)}
$$

Since $p^{k+1} \land (a \cdot b) = a \cdot b$, then by condition (A) of definition 2.6, we get $b = (p^{k+1} : a) \land b = p \land b$. Thus in both cases, we have $p \land b = b$. So that $p^{k+1} \land b^{k+1} = b^{k+1}$ (By Lemma 3.6). Hence $p^{k+1}$ is primary. 

Lemma 3.7. Let $L$ be a residuated ADL with a maximal element $m$ and suppose $L$ satisfies the a.c.c. Suppose $q$ is a primary element of $L$, $p$ is a prime element of $L$ and $k \in Z^+$ such that $p \land q = q$, $q \land p^k = p^k$, $q \land p^{k-1} \neq p^{k-1}$. If $q \land m = p^r \land m$, for some $r \in Z^+$ then $q \land m = p^k \land m$.

Proof. Suppose $q \land m = p^r \land m$, for some $r \in Z^+$. Then 

$$
r \geq k \implies p^k \land p^r = p^r \implies (1)
$$

Now, 

$$
q \land p^k = p^k \implies q \land m \land p^k \land m = p^k \land m
$$

$$
p^r \land m \land p^k \land m = p^k \land m
$$

$$
p^r \land p^k \land m = p^k \land m
$$

$$
p^k \land p^r \land m = p^k \land m
$$

$$
p^r \land m = p^k \land m \quad \text{(By above (1))}
$$

$$
q \land m = p^k \land m.
$$

Finally, we prove the converse of Theorem 3.7 in the following.

Theorem 3.8. Let $L$ be a $P$-ADL with a maximal element $m$. If $q$ is a primary element of $L$ and $p$ is the prime corresponding to $q$ then $q \land m = p^r \land m$, for some $r \in Z^+$.

Proof. Suppose $q$ is a primary element of $L$ and $p$ is the prime corresponding to $q$. Then $p \land q = q$, $q \land p^k = p^k$ and $q \land p^{k-1} \neq p^{k-1}$, for some $k \in Z^+$. We prove that $q \land m = p^r \land m$. This result is proved by induction on $k$.

If $k = 1$, then we have $p \land q = q$ and $q \land p = p$. Hence $q \land m = p \land m$. 


Assume that the result is true for all \( s \leq k - 1 \). That is, if \( \bar{q} \) is a primary element of \( L \) and \( p \) is the prime corresponding to \( \bar{q} \) such that \( \bar{q} \wedge p^s = p^s \) and \( \bar{q} \wedge p^{s-1} \neq p^{s-1} \), for some \( s \in \mathbb{Z}^+ \) with \( 1 \leq s \leq k - 1 \), then \( q \wedge m = p^s \wedge m \). If \( q \wedge p \neq p \), then, by Theorem 3.6, \( q : p \) is a primary element of \( L \) and \( p \) is the prime corresponding to \( q : p \). So that \( p \wedge (q : p) = q : p \). Also, by condition (A) of definition 2.6, we get that \( q \wedge p^k = p^k \Rightarrow (q : p) \wedge p^{k-1} = p^{k-1} \) and \( q \wedge p^{k-1} \neq p^{k-1} \Rightarrow (q : p) \wedge p^{k-2} \neq p^{k-2} \). Then by induction hypothesis, we get \((q : p) \wedge m = p^{k-1} \wedge m \). Since \( p \wedge q = q \), by Lemma 3.1, we get \( q \wedge m = [(q : p) \cdot p] \wedge m = (p^{k-1} \cdot p) \wedge m \) (By Lemma 3.3) Hence \( q \wedge m = p^k \wedge m \).

References


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