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ON SOLUTION TO THE SET-THEORETICAL YANG-BAXTER EQUATIONS VIA BL-ALGEBRAS

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Abstract. In this study, we introduce BL-algebras (here, the letters BL represent Basic Logic) by giving some definitions and notions about BL-algebras. We then present some solutions to the set-theoretical Yang-Baxter equation by using properties of BL- algebras.

1. Introduction

BL-algebras which have been discovered by Petr Hájek were introduced to provide algebraic proof of the completeness theorem of Hájek's Basic Logic (for details, see [**19, 18**]) as in the fuzzy logic framework. BL-algebras emerge as Lindenbaum algebras from some logical axioms in the same vein that MV-algebras [**3**]-[**22**] (or, Wajsberg algebras which are their implicative versions [**7, 12**]) obtain from the axioms of Lukasiewicz Logic. BL-algebras are also a specific type of resituated lattices [**34, 20, 23**]. Besides, since MV-algebras (or Wajsberg algebras) are BL-algebras such that the converse is generally not true [**13**] and BL-algebras constitute a subclass of BCK-algebras, BL-algebras are closely related to the quantum structures and quantum mechanics [**36**]-[**8**].

On the other side, the Yang-Baxter equation which can be applied to many different parts of science, tehnology and industry was originally used in theoretical physics [**32**] and statical mechanics [**2**]-[**38**] and has been recently used by many researchers from various scientific areas such as quantum groups, quantum mechanics, quantum computing, knot theory, integrable systems, non-commutative geometry,

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C*-algebras, etc.(see, for instance, [**30**] and [**25**]-[**28**]). Particularly, to find (set-teoretical) solutions to this equation has attracted the attention of scientists studying in pure mathematics and other mathematical areas such as quantum binomial algebras [**14, 15**], semigroups of I-type and Bieberbach groups [**16, 33**], bijective 1-cocyles [**10**], semisimple minimal triangular Hopf algebras [**9**], dynamical systems [**35**], and geometric crystals [**11**]. Thus, we wish to examine the Yang-Baxter equation in relation with BL-algebras rather than quantum structures.

The set-theoretical solutions to the Yang-Baxter equation using Wajsberg-algebras were given by [**31**].

In this work, giving required definitions and notions about BL-algebras we investigate some solutions to the set-teoretical Yang-Baxter equation in BL-algebras.

2. Preliminaries

In this part, we start to give certain definitions and notions about BL-algebras.

Definition 2.1. ([**19**]) A *BL-algebra* is a structure $(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ where four binary operations $\wedge, \vee, \odot, \longrightarrow$ and two constants $0, 1$ such that:

$(BL1)$ $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,
$(BL2)$ $(A, \odot, 1)$ is a commutative monoid,
$(BL3)$ $\odot$ and $\longrightarrow$ form an adjoint pair i.e, $c \leqslant a \longrightarrow b$ if and only if $a \odot c \leqslant b$,
$(BL4)$ $a \wedge b = a \odot (a \longrightarrow b)$,
$(BL5)$ $(a \longrightarrow b) \vee (b \longrightarrow a) = 1$, for all $a,\ b,\ c \in A$.

Remark 2.1. ([**27**])

(a) If $a \odot a = a$ for all $a \in A$, then a BL-algebra A is called a Gödel algebra.
(b) If $\neg(\neg a) = a$ or equivalently $(a \longrightarrow b) \longrightarrow b = (b \longrightarrow a) \longrightarrow a$ for all $a,\ b \in A$, where $\neg a = a \longrightarrow 0$, then a BL-algebra A is called an MV-algebra.

Lemma 2.1 ([**21**]). *In each BL-algebra A, the following relations hold for all $a,\ b,\ c \in A$:*

(1) $a \longrightarrow (b \longrightarrow c) = b \longrightarrow (a \longrightarrow c)$,
(2) $1 \longrightarrow a = a,\ a \longrightarrow a = 1,\ a \leqslant b \longrightarrow a,\ a \longrightarrow 1 = 1$,
(3) $(a \longrightarrow b) \longrightarrow (a \longrightarrow c) = (a \wedge b) \longrightarrow c$,
(4) $a \leqslant \neg\neg a,\ \neg 1 = 0,\ \neg 0 = 1,\ \neg\neg\neg a = \neg a,\ \neg\neg a \leqslant \neg a \longrightarrow a$
(5) $\neg\neg(a \odot b) = \neg\neg a \odot \neg\neg b$
(6) $a \longrightarrow \neg b = b \longrightarrow \neg a = \neg\neg a \longrightarrow \neg b = \neg(a \odot b)$.

3. The solutions to the set-theoretical Yang-Baxter equation in BL-algebras

In this part, we examine solution to the set-theoretical Yang-Baxter equation in BL-algebras.

Let k be a field and tensor products will be defined over this field. For a *k-space* V, we denote by $\tau : V \otimes V \longrightarrow V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$ and by $I : V \longrightarrow V$ the identity map over the space V; for a *k-linear* map $R : V \otimes V \longrightarrow V \otimes V$, let $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, and $R^{13} = (I \otimes \tau)(R \otimes I)(\tau \otimes I)$.

Definition 3.1. ([**30**]) A Yang-Baxter operator is *k-linear* map $R : V \otimes V \longrightarrow V \otimes V$, which is invertible, and it satisfies the braid condition called the Yang-Baxter equation:

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \qquad (1)$$

If R satisfies Equation (1), then both $R \circ \tau$ and $\tau \circ R$ satisfy the quantum Yang-Baxter equation (QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \qquad (2)$$

Lemma 3.1 ([**30**]). *Equations (1) and (2) are equivalent to each other.*

A relation between the set-theoretical Yang-Baxter equation and BL-algebras is constituted by the following definition.

Definition 3.2. ([**30**]) Let X be a set and $S : X \times X \longrightarrow X \times X$, $S(u, v) = (u', v')$ be a map. The map S is a solution of the set-theoretical Yang-Baxter equation if it satisfies the following equation:

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \qquad (3)$$

which is also equivalent

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}, \qquad (4)$$

where

$$S^{12} : X \times X \times X \longrightarrow X \times X \times X, S^{12}(u, v, w) = (u', v', w),$$
$$S^{23} : X \times X \times X \longrightarrow X \times X \times X, S^{23}(u, v, w) = (u, v', w'),$$
$$S^{13} : X \times X \times X \longrightarrow X \times X \times X, S^{13}(u, v, w) = (u', v, w').$$

Now, we build solutions of the set-theoretical Yang-Baxter equation by using the properties of BL-algebras.

Theorem 3.1. *Let* $(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ *be a BL-algebra. If the BL-algebra is a Gödel algebra, then* $S(x, y) = (x \odot y, x)$ *is a solution of the set-theoretical Yang-Baxter equation.*

Proof. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x, y, z) = (x \odot y, x, z) \text{ and } S^{23}(x, y, z) = (x, y \odot z, y).$$

We show that the equality $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$ holds for all $(x, y, z) \in A \times A \times A$. By using (BL2) and Remark 1 (a), we obtain

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= (S^{12} \circ S^{23})(S^{12}(x, y, z)) \\ &= (S^{12} \circ S^{23})(x \odot y, x, z) \\ &= S^{12}(S^{23}(x \odot y, x, z)) \\ &= S^{12}(x \odot y, x \odot z, x) \\ &= ((x \odot y) \odot (x \odot z), x \odot y, x) \\ &= ((x \odot x) \odot (y \odot z), x \odot y, x) \\ &= (x \odot (y \odot z), x \odot y, x)\end{aligned}$$

and

$$\begin{aligned}(S^{23}\circ S^{12}\circ S^{23})(x,y,z) &= (S^{23}\circ S^{12})(S^{23}(x,y,z))\\ &= (S^{23}\circ S^{12})(x,y\odot z,y)\\ &= S^{23}(S^{12}(x,y\odot z,y))\\ &= S^{23}(x\odot(y\odot z),x,y)\\ &= (x\odot(y\odot z),x\odot y,x),\end{aligned}$$

that is, $(S^{12}\circ S^{23}\circ S^{12})(x,y,z) = (S^{23}\circ S^{12}\circ S^{23})(x,y,z)$ for every $(x,y,z)\in A\times A\times A$. Thus, $S(x,y)=(x\odot y,x)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A whenever the BL-algebra A is a Gödel algebra. □

COROLLARY 3.1. *Since the operation $\odot$ is commutative from (BL2), and by Theorem 3.1, $S(x,y)=(y\odot x,x)$, $S(x,y)=(y\odot x,y)$ and $S(x,y)=(x\odot y,y)$ are solutions to the set-theoretical Yang-Baxter equation in BL-algebras.*

LEMMA 3.2. *Let $(A,\wedge,\vee,\odot,\longrightarrow,0,1)$ be a BL-algebra. Then $S(x,y)=(x\odot y,1)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x,y,z)=(x\odot y,1,z) \text{ and } S^{23}(x,y,z)=(x,y\odot z,1).$$

We show that the equality $S^{12}\circ S^{23}\circ S^{12}=S^{23}\circ S^{12}\circ S^{23}$ holds for all $(x,y,z)\in A\times A\times A$. By using (BL2), we have

$$\begin{aligned}(S^{12}\circ S^{23}\circ S^{12})(x,y,z) &= (S^{12}\circ S^{23})(S^{12}(x,y,z))\\ &= (S^{12}\circ S^{23})(x\odot y,1,z)\\ &= S^{12}(S^{23}(x\odot y,1,z))\\ &= S^{12}(x\odot y,1\odot z,1)\\ &= S^{12}(x\odot y,z,1)\\ &= ((x\odot y)\odot z,1,1)\\ &= (x\odot(y\odot z),1,1)\\ &= (x\odot(y\odot z),1\odot 1,1)\\ &= S^{23}(x\odot(y\odot z),1,1)\\ &= S^{23}(S^{12}(x,y\odot z,1))\\ &= (S^{23}\circ S^{12})(x,y\odot z,1)\\ &= (S^{23}\circ S^{12})(S^{23}(x,y,z))\\ &= (S^{23}\circ S^{12}\circ S^{23})(x,y,z)),\end{aligned}$$

that is, $(S^{12}\circ S^{23}\circ S^{12})(x,y,z) = (S^{23}\circ S^{12}\circ S^{23})(x,y,z)$ for every $(x,y,z)\in A\times A\times A$. Thus, $S(x,y)=(x\odot y,1)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A. □

Moreover, $S(x,y)=(y\odot x,1)$ is a solution to the set-theoretical Yang-Baxter equation in the BL-algebra A from (BL2).

LEMMA 3.3. *Let $(A,\wedge,\vee,\odot,\longrightarrow,0,1)$ be a BL-algebra. Then $S(x,y)=(\neg y,\neg x)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x,y,z)=(\neg y,\neg x,z) \text{ and } S^{23}(x,y,z)=(x,\neg z,\neg y).$$

We show that the equality $S^{12}\circ S^{23}\circ S^{12}=S^{23}\circ S^{12}\circ S^{23}$ holds for all $(x,y,z)\in A\times A\times A$. We get

$$\begin{aligned}
(S^{12}\circ S^{23}\circ S^{12})(x,y,z) &= (S^{12}\circ S^{23})(S^{12}(x,y,z))\\
&= (S^{12}\circ S^{23})(\neg y,\neg x,z)\\
&= S^{12}(S^{23}(\neg y,\neg x,z))\\
&= S^{12}(\neg y,\neg z,\neg\neg x)\\
&= (\neg\neg z,\neg\neg y,\neg\neg x)\\
&= S^{23}(\neg\neg z,\neg x,\neg y)\\
&= S^{23}(S^{12}(x,\neg z,\neg y))\\
&= (S^{23}\circ S^{12})(x,\neg z,\neg y)\\
&= ((S^{23}\circ S^{12})(S^{23}(x,y,z))\\
&= (S^{23}\circ S^{12}\circ S^{23})(x,y,z)),
\end{aligned}$$

that is, $(S^{12}\circ S^{23}\circ S^{12})(x,y,z)=(S^{23}\circ S^{12}\circ S^{23})(x,y,z)$ for every $(x,y,z)\in A\times A\times A$. Thus, $S(x,y)=(\neg y,\neg x)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A. □

LEMMA 3.4. *Let $(A,\wedge,\vee,\odot,\longrightarrow,0,1)$ be a BL-algebra. Then $S(x,y)=(\neg x\longrightarrow y,0)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x,y,z)=(\neg x\longrightarrow y,0,z)\quad S^{23}(x,y,z)=(x,y\longrightarrow z,0).$$

We show that the equality $S^{12}\circ S^{23}\circ S^{12}=S^{23}\circ S^{12}\circ S^{23}$ holds for all $(x,y,z)\in A\times A\times A$. By using firstly Lemma 2.1 (4) and secondly (2), we have

$$\begin{aligned}
(S^{12}\circ S^{23}\circ S^{12})(x,y,z) &= (S^{12}\circ S^{23})(S^{12}(x,y,z))\\
&= (S^{12}\circ S^{23})(\neg x\longrightarrow y,0,z)\\
&= S^{12}(S^{23}(\neg x\longrightarrow y,0,z))\\
&= S^{12}(\neg x\longrightarrow y,\neg 0\longrightarrow z,0)\\
&= S^{12}(\neg x\longrightarrow y,1\longrightarrow z,0)\\
&= S^{12}(\neg x\longrightarrow y,z,0)\\
&= (\neg(\neg x\longrightarrow y)\longrightarrow z,0,0),
\end{aligned}$$

and by applying Lemma 2.1 (1), (2), (4) and (6) we get

$$\begin{aligned}
(S^{23}\circ S^{12}\circ S^{23})(x,y,z) &= (S^{23}\circ S^{12})(S^{23}(x,y,z))\\
&= (S^{23}\circ S^{12})(x,\neg y\longrightarrow z,0)\\
&= S^{23}(S^{12}(x,\neg y\longrightarrow z,0))\\
&= S^{23}(\neg x\longrightarrow(\neg y\longrightarrow z),0,0)\\
&= (\neg x\longrightarrow(\neg y\longrightarrow z),\neg 0\longrightarrow 0,0)\\
&= (\neg x\longrightarrow(\neg y\longrightarrow z),1\longrightarrow 0,0)\\
&= (\neg x\longrightarrow(\neg y\longrightarrow z),0,0)\\
&= (\neg y\longrightarrow(\neg x\longrightarrow z),0,0)\\
&= (\neg y\longrightarrow(\neg z\longrightarrow x),0,0)\\
&= (\neg z\longrightarrow(\neg y\longrightarrow x),0,0)\\
&= (\neg(\neg x\longrightarrow y)\longrightarrow z,0,0).
\end{aligned}$$

Then $(S^{12}\circ S^{23}\circ S^{12})(x,y,z) = (S^{23}\circ S^{12}\circ S^{23})(x,y,z)$ for every $(x,y,z) \in A\times A\times A$. Thus, $S(x,y) = (\neg x \longrightarrow y, 0)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A. □

LEMMA 3.5. *Let $(A,\wedge,\vee,\odot,\longrightarrow,0,1)$ be a BL-algebra. Then $S(x,y) = (\neg\neg x, x)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x,y,z) = (\neg\neg x, x, z) \text{ and } S^{23}(x,y,z) = (x, \neg\neg y, y).$$

We show that the equality $S^{12}\circ S^{23}\circ S^{12} = S^{23}\circ S^{12}\circ S^{23}$ holds for all $(x,y,z) \in A\times A\times A$. By using Lemma 2.1 (4), we have

$$\begin{aligned}
(S^{12}\circ S^{23}\circ S^{12})(x,y,z) &= (S^{12}\circ S^{23})(S^{12}(x,y,z)) \\
&= (S^{12}\circ S^{23})(\neg\neg x, x, z) \\
&= S^{12}(S^{23}(\neg\neg x, x, z)) \\
&= S^{12}(\neg\neg x, \neg\neg x, x) \\
&= (\neg\neg\neg\neg x, \neg\neg x, x) \\
&= (\neg\neg x, \neg\neg x, x) \\
&= S^{23}(\neg\neg x, x, y) \\
&= S^{23}(S^{12}(x, \neg\neg y, y)) \\
&= (S^{23}\circ S^{12})(x, \neg\neg y, y) \\
&= (S^{23}\circ S^{12})(S^{23}(x,y,z)) \\
&= (S^{23}\circ S^{12}\circ S^{23})(x,y,z),
\end{aligned}$$

that is, $(S^{12}\circ S^{23}\circ S^{12})(x,y,z) = (S^{23}\circ S^{12}\circ S^{23})(x,y,z)$ for every $(x,y,z) \in A\times A\times A$. Thus, $S(x,y) = (\neg\neg x, x)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A. □

LEMMA 3.6. *Let $(A,\wedge,\vee,\odot,\longrightarrow,0,1)$ be a BL-algebra. Then $S(x,y) = (\neg\neg x, y)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x,y,z) = (\neg\neg x, y, z) \text{ and } S^{23}(x,y,z) = (x, \neg\neg y, z).$$

We show that the equality $S^{12}\circ S^{23}\circ S^{12} = S^{23}\circ S^{12}\circ S^{23}$ holds for all $(x,y,z) \in A\times A\times A$. By using Lemma 2.1 (4), we get

$$\begin{aligned}
(S^{12}\circ S^{23}\circ S^{12})(x,y,z) &= (S^{12}\circ S^{23})(S^{12}(x,y,z)) \\
&= (S^{12}\circ S^{23})(\neg\neg x, y, z) \\
&= S^{12}(S^{23}(\neg\neg x, y, z)) \\
&= S^{12}(\neg\neg x, \neg\neg y, z) \\
&= (\neg\neg\neg\neg x, \neg\neg y, z) \\
&= (\neg\neg x, \neg\neg\neg\neg y, z) \\
&= S^{23}(\neg\neg x, \neg\neg y, z) \\
&= S^{23}(S^{12}(x, \neg\neg y, Z)) \\
&= (S^{23}\circ S^{12})(x, \neg\neg y, Z) \\
&= (S^{23}\circ S^{12})(S^{23}(x,y,z)) \\
&= (S^{23}\circ S^{12}\circ S^{23})(x,y,z),
\end{aligned}$$

that is, $(S^{12} \circ S^{23} \circ S^{12})(x, y, z) = (S^{23} \circ S^{12} \circ S^{23})(x, y, z)$ for every $(x, y, z) \in A \times A \times A$. Thus, $S(x, y) = (\neg\neg x, y)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A. □

LEMMA 3.7. *Let $(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ be a BL-algebra. If the condition*

$$x \longrightarrow (y \longrightarrow z) = (x \wedge y) \longrightarrow z \qquad (5)$$

holds for all $x,\ y,\ z \in A$, then $S(x, y) = (x \longrightarrow y, x)$ is a solution of the set-theoretical Yang-Baxter equation.

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x, y, z) = (x \longrightarrow y, x, z) \quad \text{and} \quad S^{23}(x, y, z) = (x, y \longrightarrow z, y).$$

We show that the equality

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$$

holds for all $(x, y, z) \in A \times A \times A$. By using firstly Lemma 2.1 (3), and secondly the above condition (5), we obtain

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= (S^{12} \circ S^{23})(S^{12}(x, y, z)) \\
&= (S^{12} \circ S^{23})(x \longrightarrow y, x, z) \\
&= S^{12}(S^{23}(x \longrightarrow y, x, z)) \\
&= S^{12}(x \longrightarrow y, x \longrightarrow z, x) \\
&= S^{12}((x \longrightarrow y) \longrightarrow (x \longrightarrow z), x \longrightarrow y, x) \\
&= ((x \wedge y) \longrightarrow z, x \longrightarrow y, x) \\
&= (x \longrightarrow (y \longrightarrow z), x \longrightarrow y, x) \\
&= S^{23}(x \longrightarrow (y \longrightarrow z), x, y) \\
&= S^{23}(S^{12}(x, y \longrightarrow z, y)) \\
&= (S^{23} \circ S^{12})(x, y \longrightarrow z, y) \\
&= (S^{23} \circ S^{12})(S^{23}(x, y, z)) \\
&= (S^{23} \circ S^{12} \circ S^{23})(x, y, z),
\end{aligned}$$

that is, $(S^{12} \circ S^{23} \circ S^{12})(x, y, z) = (S^{23} \circ S^{12} \circ S^{23})(x, y, z)$ for every $(x, y, z) \in A \times A \times A$. Thus, $S(x, y) = (\neg\neg x, y)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A under the above condition (5). □

LEMMA 3.8. *Let $(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ be a BL-algebra. If the BL-algebra is a Gödel algebra, then $S(x, y) = (\neg\neg(x \odot y), x)$ is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x, y, z) = (\neg\neg(x \odot y), x, z) \quad \text{and} \quad S^{23}(x, y, z) = (x, \neg\neg(y \odot z), y).$$

We show that the equality

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$$

holds for all $(x, y, z) \in A \times A \times A$. By using Lemma 2.1 (4)-(5), (BL2), and Remark 1 (a), we have

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= (S^{12} \circ S^{23})(S^{12}(x, y, z)) \\
&= (S^{12} \circ S^{23})(\neg\neg(x \odot y), x, z) \\
&= S^{12}(S^{23}(\neg\neg(x \odot y), x, z)) \\
&= S^{12}(\neg\neg(x \odot y), \neg\neg(x \odot z), x) \\
&= (\neg\neg(\neg\neg(x \odot y) \odot \neg\neg(x \odot z)), \neg\neg(x \odot y), x) \\
&= (\neg\neg\neg\neg(x \odot y) \odot \neg\neg\neg\neg(x \odot z), \neg\neg(x \odot y), x) \\
&= (\neg\neg(x \odot y) \odot \neg\neg(x \odot z), \neg\neg(x \odot y), x) \\
&= ((\neg\neg x \odot \neg\neg y) \odot (\neg\neg x \odot \neg\neg z), \neg\neg(x \odot y), x) \\
&= ((\neg\neg x \odot \neg\neg x) \odot (\neg\neg y \odot \neg\neg z), \neg\neg(x \odot y), x) \\
&= (\neg\neg x \odot (\neg\neg y \odot \neg\neg z), \neg\neg(x \odot y), x),
\end{aligned}$$

and by applying only Lemma 2.1 (4)-(5) we obtain

$$\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= (S^{23} \circ S^{12})(S^{23}(x, y, z)) \\
&= (S^{23} \circ S^{12})(x, \neg\neg(y \odot z), y) \\
&= S^{23}(S^{12}(x, \neg\neg(y \odot z), y)) \\
&= S^{23}(\neg\neg(x \odot \neg\neg(y \odot z)), x, y) \\
&= (\neg\neg(x \odot \neg\neg(y \odot z)), \neg\neg(x \odot y), x) \\
&= (\neg\neg x \odot \neg\neg\neg\neg(y \odot z), \neg\neg(x \odot y), x) \\
&= (\neg\neg x \odot \neg\neg(y \odot z), \neg\neg(x \odot y), x) \\
&= (\neg\neg x \odot (\neg\neg y \odot \neg\neg z), \neg\neg(x \odot y), x).
\end{aligned}$$

Then

$$(S^{12} \circ S^{23} \circ S^{12})(x, y, z) = (S^{23} \circ S^{12} \circ S^{23})(x, y, z)$$

for every $(x, y, z) \in A \times A \times A$. Thus, $S(x, y) = (\neg\neg(x \odot y), x)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A whenever the BL-algebra A is a Gödel algebra. □

PROPOSITION 3.1 ([**34**]). *BL-algebras are distributive lattices.*

THEOREM 3.2. *Let* $(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ *be a BL-algebra. Then* $S(x, y) = (x \wedge y, x \vee y)$ *is a solution of the set-theoretical Yang-Baxter equation.*

PROOF. S^{12} and S^{23} are defined in the following forms:

$$S^{12}(x, y, z) = (x \wedge y, x \vee y, z) \text{ and } S^{23}(x, y, z) = (x, y \wedge z, y \vee z).$$

We show that the equality

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$$

holds for all $(x, y, z) \in A \times A \times A$. By applying (BL1) and Proposition 3.1, we obtain

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x, y, z) &= (S^{12} \circ S^{23})(S^{12}(x, y, z)) \\
&= (S^{12} \circ S^{23})(x \wedge y, x \vee y, z) \\
&= S^{12}(S^{23}(x \wedge y, x \vee y, z)) \\
&= S^{12}(x \wedge y, (x \vee y) \wedge z, (x \vee y) \vee z) \\
&= ((x \wedge y) \wedge ((x \vee y) \wedge z), (x \wedge y) \vee ((x \vee y) \wedge z), \\
&\quad (x \vee y) \vee z) \\
&= (((x \wedge y) \wedge (x \vee y)) \wedge z, (x \wedge y) \vee ((x \vee y) \wedge z), \\
&\quad x \vee (y \vee z)) \\
&= ((x \wedge y) \wedge z, (x \wedge y) \vee ((x \vee y) \wedge z), x \vee (y \vee z)) \\
&= ((x \wedge y) \wedge z, (x \wedge y) \vee ((x \wedge z) \vee (y \wedge z)), \\
&\quad x \vee (y \vee z)) \\
&= (x \wedge (y \wedge z), ((x \wedge y) \vee (x \wedge z)) \vee (y \wedge z), \\
&\quad x \vee (y \vee z)) \\
&= (x \wedge (y \wedge z), (x \wedge (y \vee z)) \vee (y \wedge z), x \vee (y \vee z))
\end{aligned}$$

and

$$\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x, y, z) &= (S^{23} \circ S^{12})(S^{23}(x, y, z)) \\
&= (S^{23} \circ S^{12})(x, y \wedge z, y \vee z) \\
&= S^{23}(S^{12}(x, y \wedge z, y \vee z)) \\
&= S^{23}(x \wedge (y \wedge z), x \vee (y \wedge z), y \vee z) \\
&= (x \wedge (y \wedge z), (x \vee (y \wedge z)) \wedge (y \vee z), \\
&\quad (x \vee (y \wedge z)) \vee (y \vee z)) \\
&= (x \wedge (y \wedge z), (x \wedge (y \vee z)) \vee ((y \wedge z) \wedge (y \vee z)), \\
&\quad (x \vee (y \wedge z)) \vee (y \vee z)) \\
&= (x \wedge (y \wedge z), (x \wedge (y \vee z)) \vee (y \wedge z), \\
&\quad (x \vee (y \wedge z)) \vee (y \vee z)) \\
&= (x \wedge (y \wedge z), (x \wedge (y \vee z)) \vee (y \wedge z), \\
&\quad x \vee ((y \wedge z) \vee (y \vee z))) \\
&= (x \wedge (y \wedge z), (x \wedge (y \vee z)) \vee (y \wedge z), x \vee (y \vee z)),
\end{aligned}$$

that is,

$$(S^{12} \circ S^{23} \circ S^{12})(x, y, z) = (S^{23} \circ S^{12} \circ S^{23})(x, y, z)$$

for every $(x, y, z) \in A \times A \times A$. Thus, $S(x, y) = (x \wedge y, x \vee y)$ is a solution of the set-theoretical Yang-Baxter equation in the BL-algebra A. □

Additionally, $S(x, y) = (x \wedge y, x \vee y)$ is mostly not a solution of the set-theoretical Yang-Baxter equation in Wajsberg-algebras (see, for details, [**31**]) while it is a solution of the set-theoretical Yang-Baxter equation in BL-algebras by Theorem 3.2.

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