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ON CHARACTERIZATIONS OF LATTICES USING THE GENERALIZED SYMMETRIC BI-DERIVATIONS

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Abstract. In this paper, we introduced the notion of generalized symmetric bi-derivation on lattices and investigated some related properties.We characterized the distributive and modular lattices by generalized symmetric bi-derivations.

1. Introduction

The lattice algebra plays a significant role in various branches such as information theory, information retrieval, information access controls and cryptanalysis [4, 6, 8, 16]. Recently, the properties of lattices were widely researched. In the theory of rings, the notion of derivation is an important topic to study. After the derivation on a ring was defined by Posner in [15], many researchers studied the derivation theory on various algebraic structures. In [9, 20], authors introduced the notion of derivation on a lattice and discussed some related properties. In [1], Alshehri is introduced the notion of generalized derivation for a lattice and investigated various properties. After the symmetric biderivation on rings was defined in [10, 11] by Maksa, a lot of researchers studied the symmetric biderivation in rings and near-rings [12, 13, 17, 18, 19]. In [7], Çeven applied the notion of symmetric biderivation to lattices and investigated some related properties. The notion of generalized biderivation on rings was introduced by Argaç in [2].

In this paper, we apply the notion of generalized symmetric biderivation to lattices and investigate some related properties which is discussed in [7] and [14]. Also we characterize the distributive and modular lattices by generalized symmetric biderivations.

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2. Preliminaries

Definition 1. ([5]) Let L be a nonempty set endowed with operations "$\wedge$" and "$\vee$". If $(L,\wedge,\vee)$ satisfies the following conditions for all $x,y,z\in L$, then L is called a lattice.

(a) $x\wedge x=x, x\vee x=x$,
(b) $x\wedge y=y\wedge x, x\vee y=y\vee x$,
(c) $(x\wedge y)\wedge z=x\wedge(y\wedge z),(x\vee y)\vee z=x\vee(y\vee z)$,
(d) $(x\wedge y)\vee x=x,(x\vee y)\wedge x=x$.

Definition 2. ([5]) Let $(L,\wedge,\vee)$ be a lattice. A binary relation "$\leq$ " is defined by $x\leq y$ if and only if $x\wedge y=x$ and $x\vee y=y$.

Definition 3. [5] A lattice L is distributive if the following identities hold:
i) $x\wedge(y\vee z)=(x\wedge z)\vee(x\wedge z)$,
ii) $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$.

In any lattice, the above conditions are equivalent.

Definition 4. ([3]) A lattice L is modular if the following identity holds:
If $x\leq z$, then $x\vee(y\wedge z)=(x\vee y)\wedge z$.

Lemma 1 ([20]). *Let $(L,\wedge,\vee)$ be a lattice. Define the binary relation "$\leq$ " as the Definition 2. Then $(L,\leq)$ is a poset and for any $x,y\in L$, $x\wedge y$ is the g.l.b of $\{x,y\}$, and $x\vee y$ is the l.u.b. of $\{x,y\}$.*

Definition 5. Let L be a lattice. A mapping $D(.,.):L\times L\to L$ is called symmetric if $D(x,y)=D(y,x)$ holds for all $x,y\in L$.

Definition 6. Let L be a lattice. A mapping $d:L\to L$ defined by $d(x)=D(x,x)$ is called trace of $D(.,.)$, where $D(.,.):L\times L\to L$ is a symmetric mapping.

Definition 7. ([10, 20]) Let L be a lattice and $d:L\to L$ be a mapping. The mapping d is called a derivation on L, if it satisfies the following condition

$$d(x\wedge y)=(d(x)\wedge y)\vee(x\wedge d(y))$$

for all $x,y,z\in L$.

Definition 8. ([1]) Let L be a lattice. A function $D:L\to L$ is called a generalized derivation on L if there exists a derivation $d:L\to L$ such that

$$D(x\wedge y)=(D(x)\wedge y)\vee(x\wedge d(y))$$

for all $x,y,z\in L$.

Definition 9. ([7]) Let L be a lattice and $D:L\times L\to L$ be a symmetric mapping. We call D a symmetric biderivation on L, if it satisfies the following condition

$$D(x\wedge y,z)=(D(x,z)\wedge y)\vee(x\wedge D(y,z))$$

for all $x,y,z\in L$.

Note that if D is a symmetric biderivation on a lattice L, then the mappings $d_1:L\longrightarrow L$, $d_1(x)=D(x,y)$ and $d_2:L\longrightarrow L$, $d_2(y)=D(x,y)$ are derivations on L.

Proposition 1 ([7])**.** *Let L be a lattice and let d be the trace of symmetric bi-derivation D. Then the following hold:*
i) $D(x,y) \leq x$ *and* $D(x,y) \leq y$,
ii) $D(x,y) \leq x \wedge y$,
iii) $d(x) \leq x$,
iv) $d^2(x) = d(x)$,
for all $x, y \in L$.

3. The Generalized Symmetric bi-derivations on lattices

The following definition introduces the notion of generalized symmetric bi-derivation for a lattice.

Definition 10. Let L be a lattice, $D : L \times L \to L$ be a symmetric biderivation and $\Delta : L \times L \to L$ be a symmetric mapping. We call Δ a generalized symmetric biderivation related to D, if it satisfies the following condition

$$\Delta(x \wedge y, z) = (\Delta(x,z) \wedge y) \vee (x \wedge D(y,z))$$

for all $x, y, z \in L$.The mapping $\delta : L \longrightarrow L$ defined by $\delta(x) = \Delta(x,x)$ is called the trace of generalized symmetric biderivation Δ.

Obviously, a generalized symmetric biderivation Δ on L satisfies the relation $\Delta(x, y \wedge z) = (\Delta(x,y) \wedge z) \vee (y \wedge D(x,z))$ for all $x, y, z \in L$.
Now we give examples and present some properties for a generalized symmetric biderivation on L.

Example 1. Let L be a lattice with a least element 0. The mapping $D(x,y) = 0$ is a symmetric biderivation on L. Define a mapping on L by $\Delta(x,y) = x \wedge y$ for all $x, y \in L$. Then we can see that Δ is a generalized symmetric biderivation related to D on L.

Example 2. Let L be a lattice with a least element 0 and $a \in L$. The mapping on L defined by $\Delta(x,y) = (x \wedge y) \wedge a$ is a generalized symmetric biderivation related to $D(x,y) = 0$ on L.

Proposition 2. *Let Δ is a generalized symmetric biderivation related to a symmetric biderivation D. Then the mappings $f_1 : L \longrightarrow L$, $f_1(x) = \Delta(x,z)$ and $f_2 : L \longrightarrow L$, $f_2(y) = \Delta(x,y)$ are generalized derivations on L.*

Proof. We have
$$\begin{aligned} f_1(x \wedge y) &= \Delta(x \wedge y, z) \\ &= (\Delta(x,z) \wedge y) \vee (x \wedge D(y,z)) \\ &= (f_1(x) \wedge y) \vee (x \wedge g_1(y)). \end{aligned}$$

In this equation, the mapping $g_1 : L \longrightarrow L$, $g_1(y) = D(y,z)$ is a derivation on L where D is the symmetric biderivation. Hence the mapping f_1 is a generalized derivation on L. □

Theorem 1. *Let L be a lattice, Δ be a generalized symmetric biderivation related to a symmetric biderivation D, δ be the trace of Δ and d be the trace of D. Then*
(i) $D(x,y) \leq \Delta(x,y)$ *for all* $x, y \in L$.
If L is distributive lattice, then
(ii) $\Delta(x,y) \leq x$ *and* $\Delta(x,y) \leq y$,
(iii) $\Delta(x,y) \leq x \wedge y$,

(iv) $d(x) \leq \delta(x) \leq x$
(v) $d(x) = x \Longrightarrow \delta(x) = x$
for all $x, y \in L$.

Proof. (i) Since

$$\begin{aligned} \Delta(x,y) &= \Delta(x \wedge x, y) \\ &= (\Delta(x,y) \wedge x) \vee (x \wedge D(x,y)) \\ &= (\Delta(x,y) \wedge x) \vee D(x,y), \quad \text{(by Proposition 1 (i))} \end{aligned}$$

it is seen that $D(x,y) \leq \Delta(x,y)$.

(ii) If L is distributive lattice, then we have

$$\begin{aligned} \Delta(x,y) &= (\Delta(x,y) \wedge x) \vee D(x,y) \\ &= (\Delta(x,y) \vee D(x,y)) \wedge (x \vee D(x,y)) \\ &= \Delta(x,y) \wedge x, \ \text{by (i) and Proposition 1 (i).} \end{aligned}$$

Hence it is seen that $\Delta(x,y) \leq x$. Since Δ is a symmetric, we have also $\Delta(x,y) \leq y$.

(iii) is clear by (ii).
(iv) is clear by (i) and (ii).
(v) is clear by (iv). □

Corollary 1. *Let L be a distributive lattice and Δ be a generalized symmetric biderivation related to a symmetric biderivation D. Let the least element of L be* 0 *and the greatest element of L be* 1*. Then* $\Delta(0,x) = \Delta(x,0) = 0$ *and* $\Delta(1,x) = \Delta(x,1) \leq x$ *for all* $x \in L$.

Proof. It is trivial from the Theorem 1 (ii). □

Theorem 2. *Let L be a distributive lattice, Δ be a generalized symmetric biderivation related to a symmetric biderivation D, δ be the trace of Δ and d be the trace of D. Then*
(i) $\delta^2(x) = \delta(x)$,
(ii) $\delta(x \wedge y) = (\delta(x) \wedge y) \vee (x \wedge d(y)) \vee D(x,y)$
(iii) $D(x,y) \leq \delta(x \wedge y), \delta(x) \wedge y \leq \delta(x \wedge y), x \wedge d(y) \leq \delta(x \wedge y)$
for all $x, y \in L$.

Proof. (i) Using Proposition 1(i) and Theorem 1 (iv), we have

$$\begin{aligned} \delta^2(x) &= \delta(\delta(x)) \\ &= \delta(x \wedge \delta(x)) \\ &= \Delta(x \wedge \delta(x), x \wedge \delta(x)) \\ &= (\Delta(x, x \wedge \delta(x)) \wedge \delta(x)) \vee (x \wedge D(\delta(x), x \wedge \delta(x))) \\ &= \{[(\Delta(x,x) \wedge \delta(x)) \vee (x \wedge D(\delta(x), x))] \wedge \delta(x)\} \\ &\quad \vee \{x \wedge [(D(x, \delta(x)) \wedge \delta(x)) \vee (x \wedge D(\delta(x), \delta(x)))]\} \\ &= \{[\delta(x) \vee D(\delta(x), x)] \wedge \delta(x)\} \vee \{x \wedge [\delta(x) \vee \delta(x)]\} \\ &= \delta(x). \end{aligned}$$

(ii) Using Proposition 1(i) and Theorem 1 (iv), we have

$$\begin{aligned} \delta(x \wedge y) &= \Delta(x \wedge y, x \wedge y) \\ &= (\Delta(x, x \wedge y) \wedge y) \vee (x \wedge D(y, x \wedge y)) \\ &= \{[(\Delta(x,x) \wedge y) \vee (x \wedge D(x,y))] \wedge y\} \\ &\quad \vee \{x \wedge [(D(x,y) \wedge y) \vee (x \wedge D(y,y))]\} \\ &= \{[(\delta(x) \wedge y) \vee D(x,y)] \wedge y\} \vee \{x \wedge [D(x,y) \vee (x \wedge d(y))]\} \\ &= \{(\delta(x) \wedge y) \vee D(x,y)\} \vee \{D(x,y) \vee (x \wedge d(y))\} \\ &= (\delta(x) \wedge y) \vee (x \wedge d(y)) \vee D(x,y) \end{aligned}$$

(iii) It is clear by (ii). □

Theorem 3. *Let L be a distributive lattice and Δ be a generalized symmetric biderivation related to a symmetric biderivation D and δ be the trace of Δ. Then*

$$d(x) \wedge d(y) \leq \delta(x) \wedge \delta(y) \leq \delta(x \wedge y)$$

for all $x, y \in L$.

Proof. By Theorem 2 (iii) and Theorem 1 (iv), since $\delta(x) \wedge y \leq \delta(x \wedge y)$ and $\delta(y) \leq y$, we get $\delta(x) \wedge \delta(y) \leq \delta(x) \wedge y \leq \delta(x \wedge y)$. Using Theorem 1 (iv), we have $d(x) \wedge d(y) \leq \delta(x) \wedge \delta(y)$. □

Theorem 4. *Let L be a distributive lattice and Δ be a generalized symmetric biderivation related to a symmetric biderivation D. Let the least element of L be 0 and the greatest element of L be 1, then*

(i) *if $x \leq \delta(1)$, then $\delta(x) = x$,*
(ii) *if $x \geq \delta(1)$, then $\delta(1) \leq \delta(x)$,*
(iii) *if $x \leq y$ and $d(y) = y$, then $\delta(x) = x$.*

Proof. From Theorem 2 (iii), we have $\delta(1) \wedge x \leq \delta(x)$. Hence

(i) if $x \leq \delta(1)$, then we get $x \leq \delta(x)$. Using Theorem 1 (iv), we get $\delta(x) = x$.
(ii) If $x \geq \delta(1)$, then we have $\delta(1) \leq \delta(x)$.
(iii) Since $\delta(x) \leq x \leq y, d(y) = y, D(x, y) \leq x$ we have, by Theorem 2 (ii),

$$\begin{aligned} \delta(x) &= \delta(x \wedge y) \\ &= (\delta(x) \wedge y) \vee (x \wedge d(y)) \vee D(x, y) \\ &= \delta(x) \vee x \vee D(x, y) \\ &= x. \end{aligned}$$

□

In every lattice L with the least element 0, the mapping $D(x, y) = 0$ is a symmetric biderivation. The mapping $\Delta(x, y) = x \vee y$ in any lattice L is not a generalized symmetric biderivation related to $D(x, y) = 0$. Then we have the following Corollary:

Corollary 2. *In a lattice with the least element 0, if the mapping $\Delta(x, y) = x \vee y$ related to $D(x, y) = 0$ is a generalized symmetric biderivation, then the lattice L is modular lattice.*

Proof. From the equality $\Delta(y \wedge z, x) = (\Delta(y, x) \wedge z) \vee (y \wedge D(z, x))$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in L$, hence L is a modular lattice. □

Definition 11. Let L be a lattice. The mapping Δ satisfying $\Delta(x \vee y, z) = \Delta(x, z) \vee \Delta(y, z)$ for all $x, y \in L$, is called a joinitive mapping.

Theorem 5. *Let L be a lattice and Δ be a joinitive and symmetric mapping with the trace δ on L. Then*

(i) $\delta(x \vee y) = \delta(x) \vee \delta(y) \vee \Delta(x, y)$,
(ii) $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$

for all $x, y \in L$.

Proof. (i) By the definition of joinitive mapping and symmetry, we have

$$\begin{aligned} \delta(x \vee y) &= \Delta(x \vee y, x \vee y) \\ &= \Delta(x, x) \vee \Delta(x, y) \vee \Delta(y, y) \\ &= \delta(x) \vee \delta(y) \vee \Delta(x, y). \end{aligned}$$

(ii) it is clear from (i). □

Proposition 3. *If the mapping $\Delta(x,y) = x \wedge y$ in a lattice L is also a joinitive mapping, then the lattice L is a distributive lattice.*

Proof. Using the equality $\Delta(x \vee y, z) = \Delta(x,z) \vee \Delta(y,z)$, we have $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y \in L$. So L is a distributive lattice. □

Let L be a distributive lattice, Δ be a generalized symmetric biderivation related to a symmetric biderivation D, δ be the trace of Δ and d be the trace of D. Denote $Fix_d(L) = \{x \in L : d(x) = x\}$. By Theorem 1 (v), $x \in Fix_d(L)$ implies that $x \in Fix_\delta(L)$ for all $x \in L$. That is, $Fix_d(L) \subseteq Fix_\delta(L)$. Furthermore, from the Theorem 4 (iii), $x \in Fix_d(L)$ and $y \leq x$ implies that $y \in Fix_\delta(L)$.

Definition 12. Let L be a lattice, Δ be a generalized symmetric biderivation related to a symmetric biderivation D, δ be the trace of Δ. If $x \leq y$ implies $\delta(x) \leq \delta(y)$, then δ is called an isotone mapping.

Proposition 4. *Let L be a distributive lattice, Δ be a generalized symmetric biderivation related to a symmetric biderivation D, δ be the trace of Δ. If δ is isotone and $x, y \in Fix_\delta(L)$, then $\delta(x \vee y) = x \vee y$ for all $x, y \in L$.*

Proof. Since $x \leq x \vee y$ and $y \leq x \vee y$ and δ is isotone, we have $\delta(x) \leq \delta(x \vee y)$ and $\delta(y) \leq \delta(x \vee y)$. So it is seen that $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ and since $x, y \in Fix_\delta(L)$, $x \vee y \leq \delta(x \vee y)$. By Theorem 1 (iv), since $\delta(x \vee y) \leq x \vee y$, we obtain $\delta(x \vee y) = x \vee y$. □

Proposition 5. *Let L be a distributive lattice, Δ be a generalized symmetric biderivation related to a symmetric biderivation D, δ be the trace of Δ and d be the trace of D. Then $1 \in Fix_\delta(L)$ if and only if δ is an identity mapping.*

Proof. If $1 \in Fix_\delta(L)$, since $\delta(1) = 1$, by Theorem 2, we have
$$\delta(x) = \delta(1 \wedge x) = (\delta(1) \wedge x) \vee (1 \wedge d(x)) \vee D(1,x) = x \vee d(x) \vee D(1,x) = x.$$
Converse is trivial. □

Theorem 6. *Let L be a distributive lattice, Δ_1 and Δ_2 be generalized symmetric biderivations related to a same symmetric biderivation D. The mapping $\Delta_1 \wedge \Delta_2$ defined by $(\Delta_1 \wedge \Delta_2)(x,y) = \Delta_1(x,y) \wedge \Delta_2(x,y)$, is a generalized symmetric biderivation related to the symmetric biderivation D.*

Proof. Since
$$\begin{aligned}(\Delta_1 \wedge \Delta_2)(x \wedge y, z) &= \Delta_1(x \wedge y, z) \wedge \Delta_2(x \wedge y, z)\\ &= [(\Delta_1(x,z) \wedge y) \vee (x \wedge D(y,z))]\\ &\wedge [(\Delta_2(x,z) \wedge y) \vee (x \wedge D(y,z))]\\ &= [(\Delta_1(x,z) \wedge y) \wedge (\Delta_2(x,z) \wedge y)] \vee (x \wedge D(y,z))\\ &= (\Delta_1(x,z) \wedge \Delta_2(x,z) \wedge y) \vee (x \wedge D(y,z))\\ &= ((\Delta_1 \wedge \Delta_2)(x,z) \wedge y) \vee (x \wedge D(y,z)),\end{aligned}$$
so the Theorem is true. □

Theorem 7. *Let L be a distributive lattice, Δ_1 and Δ_2 be generalized symmetric biderivations related to a same symmetric biderivation D. The mapping $\Delta_1 \vee \Delta_2$ defined by $(\Delta_1 \vee \Delta_2)(x,y) = \Delta_1(x,y) \vee \Delta_2(x,y)$, is a generalized symmetric biderivation related to the symmetric biderivation D.*

Proof. Since

$$
\begin{aligned}
(\Delta_1 \vee \Delta_2)(x \wedge y, z) &= \Delta_1(x \wedge y, z) \vee \Delta_2(x \wedge y, z) \\
&= (\Delta_1(x, z) \wedge y) \vee (x \wedge D(y, z)) \\
&\vee (\Delta_2(x, z) \wedge y) \vee (x \wedge D(y, z)) \\
&= (\Delta_1(x, z) \wedge y) \vee (\Delta_2(x, z) \wedge y) \vee (x \wedge D(y, z)) \\
&= ((\Delta_1(x, z) \vee \Delta_2(x, z)) \wedge y) \vee (x \wedge D(y, z)) \\
&= ((\Delta_1 \vee \Delta_2)(x, z) \wedge y) \vee (x \wedge D(y, z)),
\end{aligned}
$$

so the Theorem is true. □

Proposition 6. *Let L be a distributive lattice, Δ_1 and Δ_2 be generalized symmetric biderivations, δ_1 be the trace of Δ_1 and δ_2 be the trace of Δ_2. If δ_1 and δ_2 are isotone mapping, then $\delta_1 = \delta_2$ if and only if $Fix_{\delta_1}(L) = Fix_{\delta_2}(L)$.*

Proof. Let $Fix_{\delta_1}(L) = Fix_{\delta_2}(L)$. If $x \in Fix_{\delta_1}(L)$, since $\delta_1(\delta_1(x)) = \delta_1(x)$, we have $\delta_1(x) \in Fix_{\delta_1}(L) = Fix_{\delta_2}(L)$. Hence $\delta_2(\delta_1(x)) = \delta_1(x)$. Similarly, we see that $\delta_1(\delta_2(x)) = \delta_2(x)$. Since δ_1 and δ_2 are isotone mapping and $\delta_1(x) \leq x, \delta_2(x) \leq x$, we get $\delta_2(\delta_1(x)) \leq \delta_2(x) = \delta_1(\delta_2(x))$ and $\delta_1(\delta_2(x)) \leq \delta_1(x) = \delta_2(\delta_1(x))$. So $\delta_1(\delta_2(x)) = \delta_2(\delta_1(x))$. Therefore we obtain $\delta_1(x) = \delta_2(\delta_1(x)) = \delta_1(\delta_2(x)) = \delta_2(x)$, that is, $\delta_1 = \delta_2$. The converse is trivial. □

Proposition 7. *Let L be a distributive lattice and Δ be a generalized symmetric biderivation related to a symmetric biderivation D, δ be the trace of Δ and the greatest element of L be 1. Then the following conditions are equivalent:*
(i) *δ is an isotone mapping*
(ii) $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$
(iii) $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$
(iv) $\delta(x) = x \wedge \delta(1)$ *for all $x, y \in L$.*

Proof. (i)$\Longrightarrow$(ii): Since $x \leq x \vee y$ and $y \leq x \vee y$ and δ is an isotone mapping, we have $\delta(x) \leq \delta(x \vee y)$ and $\delta(y) \leq \delta(x \vee y)$, so $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$.

(ii)$\Longrightarrow$(i): Let $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ and $x \leq y$. Since $x \vee y = y$, we have $\delta(x) \vee \delta(y) \leq \delta(y)$. Also it is known that $\delta(y) \leq \delta(x) \vee \delta(y)$. Hence we obtain $\delta(x) \vee \delta(y) = \delta(y)$, so $\delta(x) \leq \delta(y)$.

(i)$\Longrightarrow$(iii): Since $x \wedge y \leq x$ and $x \wedge y \leq y$ and δ is an isotone mapping, we have $\delta(x \wedge y) \leq \delta(x)$ and $\delta(x \wedge y) \leq \delta(y)$ and so $\delta(x \wedge y) \leq \delta(x) \wedge \delta(y)$. By Theorem 3 (ii), we have $\delta(x) \wedge \delta(y) \leq \delta(x \wedge y)$. Hence $\delta(x) \wedge \delta(y) = \delta(x \wedge y)$.

(iii)$\Longrightarrow$(i): Let $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$ and $x \leq y$. Since $x \wedge y = x$, we get
$\delta(x) = \delta(x \wedge y) = \delta(x) \wedge \delta(y) \leq \delta(y)$.

(i)$\Longrightarrow$(iv): Since $x \leq 1$ and δ is an isotone mapping, we have $\delta(x) \leq \delta(1)$.By Theorem 1 (iv), since $\delta(x) \leq x$, we get $\delta(x) \leq x \wedge \delta(1)$. By Theorem 2 (ii), we have $\delta(x) \wedge y \leq \delta(x \wedge y)$. Taking $x = 1$, we get $\delta(1) \wedge y \leq \delta(y)$ for all $y \in L$. Hence we have $\delta(x) = x \wedge \delta(1)$.

(iv)$\Longrightarrow$(iv): Let $\delta(x) = x \wedge \delta(1)$ and $x \leq y$. Since $x \wedge y = x$, we have
$\delta(x) = \delta(x \wedge y) = (x \wedge y) \wedge 1 = (x \wedge 1) \wedge (y \wedge 1) = \delta(x) \wedge \delta(y)$. Hence $\delta(x) \leq \delta(y)$. □

Conflict of Interests

The author declares that there is no conflict of interest regarding the publication of this paper.

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