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# SKEW HEYTING ALMOST DISTRIBUTIVE LATTICES

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Abstract. Using the non commutative nature of an ADL we extend the concept of HADLs to skew HADLs. We also characterize skew HADLs in terms of a congruence relation defined on it and some set of conditions related to HADL together with an induced binary operation defined on it. Further we show that each congruence class is a maximal rectangular subalgebras of the skew HADL.

## 1. Introduction

The foundation of modern theory of skew lattices can be found in Jonathan Leechs 1989 paper [**3**]. Leech [**4, 5**] showed that each right handed skew Boolean algebra can be embedded in to a generic skew Boolean algebra of partial functions from a given set to the co-domain $\{0,1\}$. Heyting algebra is a relatively pseudo-complemented distributive lattice which arises from non-classical logic, and it was first investigated by T. Skolem about 1920 [**8**]. A Heyting algebra named after a Dutch mathematician Arend Heyting was introduced by G. Birkhoff [**1**] and is developed by H. B. Curry about 1963. While Boolean algebras provide algebraic models of classical logic, Heyting algebras provide algebraic models of intuitionistic logic. The notion of skew Heyting algebra was introduced by Karin Cvetko-vah [**2**]. In this paper it is proved that a Heyting algebras form a variety and that the maximal lattice image of a skew Heyting algebra is a generalized Heyting algebra. The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and G. C. Rao [**9**] as common abstraction to most of the existing ring

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theoretic generalization of a Boolean algebra and distributive lattices. G. C. Rao, B. Assaye and M. V. Ratnamani in [**7**] introduced Heyting Almost Distributive Lattices (HADL) as a generalization of Heyting algebra in the class of ADLs and they characterize an HADL in terms of the set of all of its principal ideals.

This paper consists of two sections. The first section describes preliminary concepts which can be used in proving lemmas, theorems and corollaries in the second section. In this paper we introduce the concept of skew Heyting Almost Distributive Lattices(skew HADLs) and we characterize it as a skew Heyting algebras in terms of a congruence relation defined on it.

## 2. Preliminaries

In this section, we give the necessary definitions and results on ADLs, HADLs and skew Heyting algebras which will be used in the subsequent sections.

Definition 2.1. ([**6**]) An algebra $(H, \vee, \wedge)$ of type $(2, 2)$ is called an Almost Distributive Lattice if it satisfies the following axioms:

(1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(3) $(x \vee y) \wedge y = y$
(4) $(x \vee y) \wedge x = x$
(5) $x \vee (x \wedge y) = x$

for all $x, y, z \in H$.

Theorem 2.1 ([**6, 9**]). *Let $H$ be an ADL with 0. Then for any $w, x, y, z \in H$, we have the following.*

(1) $x \vee y = x \Leftrightarrow x \wedge y = y$
(2) $x \vee y = y \Leftrightarrow x \wedge y = x$
(3) $x \wedge y = y \wedge x = x$ *whenever* $x \leqslant y$
(4) $\wedge$ *is associative*
(5) $x \wedge y \wedge z = y \wedge x \wedge z$
(6) $(x \vee y) \wedge z = (y \vee x) \wedge z$
(7) $x \wedge y \leqslant y$ *and* $x \leqslant x \vee y$
(8) $x \wedge (y \wedge x) = y \wedge x$ *and* $x \vee (x \vee y) = x \vee y = (x \vee y) \vee y$
(9) $x \wedge x = x$ *and* $x \vee x = x$
(10) $x \wedge 0 = 0$ *and* $0 \vee x = x$
(11) $\{w \vee (x \vee y)\} \wedge z = \{(w \vee x) \vee y\} \wedge z$
(12) *If* $x \leqslant z$ *and* $y \leqslant z$, *then* $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$

An element $m$ in an ADL $H$ is said to be maximal if for each $a \in H, m \leqslant a$ implies that $a = m$.

Theorem 2.2 ([**6**]). *Let $H$ be an ADL. For any $m \in H$ the following are equivalent.*

(1) *$m$ is maximal element*
(2) $m \vee x = m$ *for all* $x \in H$
(3) $m \wedge x = x$ *for all* $x \in H$

DEFINITION 2.2. ([**7**]) An algebra $(H, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a Heyting algebra if it satisfies the following conditions.

(1) $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice
(2) $x \rightarrow x = 1$
(3) $y \leqslant x \rightarrow y$
(4) $x \wedge (x \rightarrow y) = x \wedge y$
(5) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
(6) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

for all $x, y, z \in H$.

DEFINITION 2.3. ([**7**]) Let $(H, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element $m$. Suppose $\rightarrow$ be a binary operation on $H$ satisfying the following conditions.

(1) $x \rightarrow x = m$
(2) $(x \rightarrow y) \wedge y = y$
(3) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$
(4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
(5) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

for all $x, y, z \in H$. Then $(H, \vee, \wedge, \rightarrow, 0, m)$ is called a Heyting Almost Distributive Lattice(HADL).

Let $H$ be an HADL and $\rightarrow$ be a binary operation on $H$ such that $x \rightarrow y \in H$ for any $x, y \in H$. Then $y \leqslant x \rightarrow y$ implies that $(x \rightarrow y) \wedge y = y$, but the converse is not always true. The converse becomes true whenever $H$ is a lattice, and therefore an HADL becomes a Heyting algebra.

THEOREM 2.3 ([**7**]). *Let $(H; \vee, \wedge, \rightarrow, 0, m)$ be an HADL. Then the following are equivalent.*

(1) *$H$ is a Heyting algebra*
(2) *For any $a, b, c \in H, a \wedge c \leqslant b \Leftrightarrow c \leqslant a \rightarrow b$*
(3) *$b \leqslant a \rightarrow b$ for all $a, b \in H$.*

THEOREM 2.4 ([**3**]). *Let $H$ be an ADL with 0 and a maximal element $m$, then the following are equivalent.*

(1) *$H$ is an HADL*
(2) *$[0, a]$ is a Heyting algebra for all $a \in H$*
(3) *$[0, m]$ is a Heyting algebra.*

LEMMA 2.1 ([**7**]). *Let $H$ be a Heyting algebra, then an equivalence relation $\theta$ on $H$ is a congruence relation if and only if for any $(a, b) \in \theta, d \in H$,*

(1) $(a \wedge d, b \wedge d) \in \theta$
(2) $(a \vee d, b \vee d) \in \theta$
(3) $(a \rightarrow d, b \rightarrow d) \in \theta$
(4) $(d \rightarrow a, d \rightarrow b) \in \theta$.

Definition 2.4. ([**4**]) A skew lattice is an algebra $\mathbf{S} = (S; \wedge, \vee)$ of type $(2, 2)$ such that $\wedge$ and $\vee$ are both idempotent and associative, and they satisfy the following absorption laws:
$x \wedge (x \vee y) = x = x \vee (x \wedge y)$ and $(x \wedge y) \vee y = y = (x \vee y) \wedge y$ for all $x, y \in S$

The natural partial order can be defined on a skew lattice $\mathbf{S}$ by stating that $x \leqslant y$ if and only if $x \vee y = y = y \vee x$, or equivalently $x \wedge y = x = y \wedge x$ for $x, y \in S$ and the natural preorder can be defined by $x \preceq y$ if and only if $y \vee x \vee y = y$, or equivalently $x \wedge y \wedge x = x$ for $x, y \in S$. Green's equivalence relation $D$ is then defined by $xDy$ if and only if $x \preceq y$ and $y \preceq x$ (see [7]).

Definition 2.5. ([**8**]) A skew lattice is called strongly distributive if for all $x, y, z \in S$ it satisfies the following identities: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$; and it is called co-strongly distributive if it satisfies the identities: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.6. ([**8**]) An algebra $\mathbf{S} = (S; \vee, \wedge, \rightarrow, 1)$ of type $(2, 2, 2, 0)$ is said to be a skew Heyting algebra whenever the following conditions are satisfied:

(1) $(S; \vee, \wedge, 1)$ is a co-strongly distributive skew lattice with top 1.
(2) For any $a \in S$, an operation $\rightarrow_a$ can be defined on $a\uparrow = \{a \vee x \vee a | x \in S\} = \{x \in S | a \leqslant x\}$ such that $(u\uparrow, \vee, \wedge, \rightarrow_a, 1, a)$ is a Heyting algebra with top 1 and bottom $a$.
(3) An induced binary operation $\rightarrow$ from $\rightarrow_a$ is defined on $S$ by $x \rightarrow y = (y \vee x \vee y) \rightarrow_y y$.

Lemma 2.2 ([**8**]). *An algebra* $\boldsymbol{S} = (S; \vee, \wedge, \rightarrow, 1)$ *of type* $(2, 2, 2, 0)$ *is a skew Heyting algebra if the following conditions hold:*

(1) *The reduct* $(S; \vee, \wedge, 1)$ *is co-strongly distributive skew lattice with top 1.*
(2) $y \leqslant x \rightarrow y$ *holds for all* $x, y \in S$
(3) $\boldsymbol{S}$ *satisfies* $x \preceq y \rightarrow z$ *if and only if* $x \wedge y \preceq z$.

## 3. Skew Heyting Almost Distributive Lattices

In this section, we introduce the concept of skew HADLs and, characterize it interms of skew Heyting algebras and congruence relations defined on it. More over we investigate some of its algebraic properties.

Through out this section $H$ stands for an ADL with a maximal element $m$ but with out 0, and

(i) For any $a \in H$, $H_a = \{x \wedge a | x \in H\}$
(ii) For any $a \in H$, $\rightarrow_a$ is the binary operation defined on $H_a$
(iii) For any $b, c \in H$, ${}_b\!\rightarrow$ is the binary operation defined on $[b, c]$.

Definition 3.1. Let $H$ be an ADL with a maximal element $m$ and with out 0. Then $H$ is said to be a skew HADL if to each $a \in H$ the algebra $(H_a, \vee, \wedge, \rightarrow_a, a)$ is a skew Heyting algebra.

Since every interval in a Heyting algebra is again a Heyting algebra one can simply observe that every Heyting algebra is a skew HADL. Here under, we give an example of skew HADLs.

EXAMPLE 3.1. Let $H$ be an ADL with a maximal element $m$. Let $b \in H$. For any $a \in H$ such that $a \leqslant b$ define a binary operation ${}_a\!\rightarrow$ on $[a, b]$ by

$$x \ {}_a\!\rightarrow y = \begin{cases} b & \text{if } x \leqslant y \\ y & \text{otherwise.} \end{cases}$$

Let a binary operation $\rightarrow_b$ on $H_b$ induced from ${}_a\!\rightarrow$ be defined by $x \rightarrow_b y = (x \vee y) \ {}_y\!\rightarrow y$. Hence $(H_b, \vee, \wedge, \rightarrow_b, b)$ is a skew Heyting algebra and therefore $H$ is a skew HADL.

Let $H$ be a skew HADL. If we include 0 to $H$, then for each $a \in H, H_a = [0, a]$ is a skew Heyting algebra. Particularly if $a = m$, then $[0, m]$ is a skew Heyting algebra. Thus for each $b \in [0, m]$, $([b, m], \vee, \wedge, {}_b\!\rightarrow, b, m)$ is a Heyting algebra. Now, taking $b = 0$ makes $([0, m], \vee, \wedge, {}_b\!\rightarrow, b, m)$ is a Heyting algebra and Hence $H$ is an HADL.

THEOREM 3.1. *Let $(H, \vee, \wedge, m)$ be an ADL with a maximal element $m$. Then $(H, \vee, \wedge, \rightarrow, m)$ is a skew HADL if and only if the following conditions hold:*

(1) *for any $b \in H, ([b, m], \vee, \wedge, {}_b\!\rightarrow, b, m)$ is a HADL*
(2) *a binary operation $\rightarrow$ on $H$ can be defined by*

$$x \rightarrow y = ((x \vee y) \wedge m)_{(y \wedge m)}\!\rightarrow (y \wedge m).$$

PROOF. Let $H$ be a skew HADL. Since $H$ has maximal element(s), for each $b \in H$ there exist a maximal element m such that $b \leqslant m$ and hence $b \in H_m$. From the definition of skew HADL we have for any $a \in H, H_a$ is a skew Heyting algebra. In particular for $a = m$ we have $H_m$ is a skew Heyting algebra. Consequently, from the definition of skew Heyting algebra we have seen that for any $b \in H_m, ([b, m], \vee, \wedge, {}_b\!\rightarrow, b, m)$ is a Heyting algebra so that $([b, m], \vee, \wedge, {}_b\!\rightarrow, b, m)$ is a HADL .

Since $H_m$ is a skew Heyting algebra, the induced operation $\rightarrow_m$ on $L_m$ from ${}_b\!\rightarrow$ on $[b, m]$ is given by $x \rightarrow_m y = (y \vee x \vee y) \ {}_y\!\rightarrow y$. Thus it is possible to define a binary operation $\rightarrow$ on $H$ by $x \rightarrow y = (x \wedge m) \rightarrow_m (y \wedge m)$. But

$$\begin{aligned} (x \wedge m) \rightarrow_m (y \wedge m) &= ((y \wedge m) \vee (x \wedge m) \vee (y \wedge m))_{(y \wedge m)}\!\rightarrow (y \wedge m) \\ &= ((y \vee x \vee y) \wedge m)_{(y \wedge m)}\!\rightarrow (y \wedge m) \\ &= ((x \vee y) \wedge m)_{(y \wedge m)}\!\rightarrow (y \wedge m), \end{aligned}$$

and hence $x \rightarrow y = ((x \vee y) \wedge m)_{(y \wedge m)}\!\rightarrow (y \wedge m)$ in such a way that $(H, \vee, \wedge, \rightarrow, m)$ is a skew HADL.

Conversely, suppose conditions (1) and (2) hold and let $a \in H$. Then $H_a$ is a co-strongly distributive skew lattice. By (1) for any $b \in H_a, [b, m]$ is a HADL. Since $H$ has maximal element for some maximal element $m$ in $H$ we have $a \leqslant m$ so that $[b, a] \subseteq [b, m]$. Theorem 2.4 asserts that $([b, a], \vee, \wedge, {}_b\!\rightarrow, b, a)$ is a Heyting algebra. The maximal element in $H_a$ is $a$, thus using (2) it is possible to define $\rightarrow_a$ on $H_a$ by $x \rightarrow_a y = ((x \vee y) \wedge a)_{(y \wedge a)}\!\rightarrow (y \wedge a)$. But

$$\begin{aligned} ((x \vee y) \wedge a)_{(y \wedge a)}\!\rightarrow (y \wedge a) &= ((y \vee x \vee y) \wedge a)_{(y \wedge a)}\!\rightarrow (y \wedge a) \\ &= (y \vee x \vee y)_y\!\rightarrow y, \end{aligned}$$

and hence $x \rightarrow_a y = (y \vee x \vee y)_y{\rightarrow}y$. Therefore $(H_a, \vee, \wedge, \rightarrow_a, a)$ is a skew Heyting algebra so that $H$ is a skew HADL. □

Consequently, by a skew HADL we mean an algebra $(H, \vee, \wedge, \rightarrow, m)$ of type $(2, 2, 2, 0)$ satisfying conditions (1) and (2) of Theorem 3.1.

On the rest of this section, H stands for a skew HADL $(H, \vee, \wedge, \rightarrow, m)$ unless otherwise specified.

Corollary 3.1. *For any $a \in H, [a, m]$ is a Heyting algebra.*

Proof. Clear by Theorem 3.1.(i). □

The following lemma is analogous with the statement, any interval on a Heyting algebra is again a Heyting algebra.

Lemma 3.1. *For any $b \in H, [b, m]$ is a skew HADL.*

Proof. From Corollary 3.1 for any $b \in H$ we have $[b, m]$ is a Heyting algebra. Following this for any $c \in [b, m]$, $[b, c]$ is a Heyting algebra so that it is a skew Heyting algebra with top element $c$. Therefore $[b, m]$ is a skew HADL. □

Corollary 3.2. *Let $H$ be a skew HADL. If $x, y \in H$ such that $x \leqslant y$ and $a, b \in [y, m]$, then $a\ {}_x{\rightarrow}\ b = a\ {}_y{\rightarrow}\ b$.*

Proof. Let $x, y \in H$ such that $x \leqslant y$. Then $[y, m] \subseteq [x, m]$. If $a, b \in [y, m]$, then $a\ {}_y{\rightarrow}\ b \in [y, m]$ and hence $a\ {}_y{\rightarrow}\ b \in [x, m]$. By Corollary 3.1, $[x, m]$ and $[y, m]$ are Heyting algebras. Since $a, b \in [x, m]$, $a\ {}_x{\rightarrow}\ b$ also belongs to $[x, m]$. The maximal element characterization of $a\ {}_x{\rightarrow}\ b$ and $a\ {}_y{\rightarrow}\ b$ on the Heyting algebra $[x, m]$ forces the two elements are equal. □

Lemma 3.2. *Let $H$ be a skew HADL. Then the following conditions hold:*

(1) $x \rightarrow y = (x\ {}_{(y\wedge m)}{\rightarrow}\ y) \wedge m$
(2) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$
(3) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$
(4) $x \rightarrow y = (x \rightarrow y) \wedge m$

*for all $x, y, z \in H$.*

Proof. Let $H$ be a skew HADL and $x, y, z \in H$. Clearly $[y \wedge m, m]$ is a HADL and hence we have

$$\begin{aligned}
(1)\ (x_{(y\wedge m)}{\rightarrow}y) \wedge m &= (m_{(y\wedge m)}{\rightarrow}(x_{(y\wedge m)}{\rightarrow}y)) \wedge m \\
&= ((x \wedge m)_{(y\wedge m)}{\rightarrow}y) \wedge m \wedge m \\
&= ((x \wedge m)_{(y\wedge m)}{\rightarrow}y) \wedge ((x \wedge m)_{(y\wedge m)}{\rightarrow}m) \wedge m \\
&= ((x \wedge m)_{(y\wedge m)}{\rightarrow}(y \wedge m)) \wedge ((y \wedge m)_{(y\wedge m)}{\rightarrow}(y \wedge m)) \\
&= ((x \vee y) \wedge m)_{(y\wedge m)}{\rightarrow}(y \wedge m) \\
&= x \rightarrow y.
\end{aligned}$$

$$
\begin{aligned}
(2)\ x \wedge (x \to y) &= x \wedge \{((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)\} \\
&= m \wedge x \wedge \{((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)\} \\
&= x \wedge m \wedge \{((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)\} \\
&= x \wedge (x \vee y) \wedge m \wedge \{((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)\} \\
&= x \wedge \{((x \vee y) \wedge m) \wedge \{((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)\}\} \\
&= x \wedge \{((x \vee y) \wedge m) \wedge (y \wedge m)\} \\
&= x \wedge (y \wedge m) \\
&= x \wedge y \wedge m.
\end{aligned}
$$

$$
\begin{aligned}
(3)\ (x \vee y) \to z &= \{((x \vee y) \vee z) \wedge m\}_{(z\wedge m)}{\to}(z \wedge m) \\
&= \{((x \vee y) \vee z \vee z) \wedge m\}_{(z\wedge m)}{\to}(z \wedge m) \\
&= \{((z \vee x) \vee (y \vee z)) \wedge m\}_{(z\wedge m)}{\to}(z \wedge m) \\
&= \{((x \vee z) \wedge m)_{(z\wedge m)}{\to}(z \wedge m)\} \wedge \{((y \vee z) \wedge m)_{(z\wedge m)}{\to}(z \wedge m)\} \\
&= (x \to z) \wedge (y \to z).
\end{aligned}
$$

$$
\begin{aligned}
(4)\ (x \to y) \wedge m &= ((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)) \wedge m \\
&= \{((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)\} \wedge ((y \wedge m)_{(y\wedge m)}{\to}(y \wedge m)) \\
&= \{((x \vee y) \vee y) \wedge m\}_{(y\wedge m)}{\to}(y \wedge m) \\
&= ((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m) \\
&= x \to y.
\end{aligned}
$$

□

Corollary 3.3. *Let $x, y \in H$. Then $(x\ {}_{y}{\to}\ y) \wedge m = (x \to y) \wedge m$*

Proof. Let $H$ be a skew HADL and $x, y \in H$. Then

$$
\begin{aligned}
(x \to y) \wedge m &= ((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)) \wedge m \\
&= \{((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m)\} \wedge ((y \wedge m)_{(y\wedge m)}{\to}(y \wedge m)) \\
&= \{((x \vee y) \vee y) \wedge m\}_{(y\wedge m)}{\to}(y \wedge m) \\
&= ((x \vee y) \wedge m)_{(y\wedge m)}{\to}(y \wedge m) \\
&= ((x \wedge m)_{(y\wedge m)}{\to}(y \wedge m)) \wedge ((y \wedge m)_{(y\wedge m)}{\to}(y \wedge m)) \\
&= ((x \wedge m)_{(y\wedge m)}{\to}y) \wedge m \wedge m \\
&= (m_{(y\wedge m)}{\to}(x_{(y\wedge m)}{\to}y)) \wedge m .... \lceil \text{since } [y \wedge m, m] \text{ is HADL} \rfloor \\
&= (x_{(y\wedge m)}{\to}y) \wedge m \\
&= (x\ {}_{y}{\to}\ y) \wedge m.
\end{aligned}
$$

□

The next theorem characterizes a skew HADL interms of a congruence relation $\theta$ defined on it. First observe the following lemma which can be verified routinely.

Lemma 3.3. *Let $x, y, z \in H$ such that $x \wedge m = y \wedge m$. Then the following statements hold:*

(1) $x \to y = m$
(2) $x \to z = y \to z$ *and* $z \to x = z \to y$

THEOREM 3.2. *Let $H$ be an ADL with a maximal element $m$. If $H$ is a skew HADL and $\theta$ defined by*

$$\theta = \{(x, y) \in H \times H | x \wedge y = y \text{ and } y \wedge x = x\}$$

*is a relation on $H$, then the following conditions hold:*

(1) *$\theta$ is a congruence relation on $H$*
(2) *The congruence classes are the maximal rectangular subalgebras of $H$.*

PROOF. Suppose $H$ be a skew HADL. (1) Let $x, y, z, a \in H$. It is easly seen that $\theta$ is reflexive and symmetric. Assume that $x\theta y$ and $y\theta z$. Then $x \wedge y = y, y \wedge x = x, y \wedge z = z$ and $z \wedge y = y$. As $z \wedge x = z \wedge y \wedge x = y \wedge x = x$ and $x \wedge z = y \wedge x \wedge z = x \wedge y \wedge z = y \wedge z = z$, it follows that $x\theta z$. Consequently $\theta$ is transitive and hence it is an equivalence relation. To show that $\theta$ is a congruence relation it suffices to show $\theta$ satisfies Lemma (2.1). Given that $x\theta y$ and $a \in H$. Then $(y \wedge a) \wedge (x \wedge a) = y \wedge x \wedge a = x \wedge a$ and $(x \wedge a) \wedge (y \wedge a) = x \wedge y \wedge a = y \wedge a$, and hence $(x \wedge a)\theta(y \wedge a)$. Also $(x \vee a) \wedge (y \vee a) = ((x \vee a) \wedge y) \vee ((x \vee a) \wedge a) = ((x \wedge y) \vee (a \wedge y)) \vee a = y \vee a$. and $(y \vee a) \wedge (x \vee a) = ((y \vee a) \wedge x) \vee ((y \vee a) \wedge a) = ((y \wedge x) \vee (a \wedge x)) \vee a = x \vee a$. Hence $(x \vee a)\theta(y \vee a)$. Finally we show that $(x \to a)\theta(y \to a)$ and $(a \to x)\theta(a \to y)$ hold. From the property of ADLs given by Theorem (2.1) and the given conditions $x \wedge y = y$ and $y \wedge x = x$, one can simply observe that $x \wedge m = y \wedge x \wedge m = x \wedge y \wedge m = y \wedge m$. Indeed, (2) of Lemma (3.3) assures that $(x \to a)\theta(y \to a)$ and $(a \to x)\theta(a \to y)$. Hence $\theta$ is a congruence relation. Suppose $x, y \in [z]\theta$. Then $x\theta y$. Hence $y \vee x = y \vee (y \wedge x) = y = x \wedge y$. Therefore each congruence class is a rectangular subalgebra of $L$. Let $\mathfrak{R}$ be the set of all rectangular subalgebras of $L$. Now take an arbitrary congruence class $[x]\theta$ for some $x \in L$. Let $T$ be a rectangular subalgebra of $L$ such that $[x]\theta \subseteq T$ and let $r \in T$. Since $x \in T$ and $T$ is a rectangular subalgebra of $L$ we have $r \vee x = x \wedge r$ and $x \vee r = r \wedge x$. Thus $x \wedge r = (r \wedge x) \wedge r = (x \vee r) \wedge r = r$ which implies that $r \vee x = x \wedge r = r$ and $x \vee r = x \vee (x \wedge r) = x$. Hence $r \in [x]\theta$. Therefore $T \subseteq [x]\theta$ and we conclude that $[x]\theta = T$. Hence $[x]\theta$ is a maximal element of $\mathfrak{R}$, i.e. each congruence class is a maximal rectangular subalgebra of $L$. □

Skew Heyting algebra is a generalization of Heyting algebra. It has a top element and need not to contain bottom element. A skew Heyting algebra becomes an ADL if it is strongly distributive skew lattice.

THEOREM 3.3. *Let $H$ be an ADL with a maximal element $m$ and $0 \notin H$. Then $H$ is a skew HADL if and only if for any $a, z \in H$ such that $a \in H_z$ and $w, x, y \in [a, z]$, the following conditions hold:*

(1) $x \leqslant w \;{}_a\!\!\to x$
(2) $(x \;{}_a\!\!\to y) \wedge w = w$ *if and only if* $y \wedge w \wedge x = w \wedge x$.

PROOF. Let $H$ be a skew HADL. Indeed for any $z \in H$, $H_z$ is a skew Heyting algebra. Let $a \in H_z$ and $w, x, y \in [a, z]$. Since $[a, z]$ is a Heyting algebra (1) holds directly.

Now, assume that $(x\ {}_a{\rightarrow}\ y) \wedge w = w$. Then

$$w \wedge x = ((x_a{\rightarrow}y) \wedge w) \wedge x = x \wedge ((x_a{\rightarrow}y) \wedge w) = (x \wedge y) \wedge w = y \wedge w \wedge x.$$

On the other hand given that $y \wedge w \wedge x = w \wedge x$, we obtain $x\ {}_a{\rightarrow}\ (w \wedge x) = x\ {}_a{\rightarrow}\ (y \wedge w \wedge x)$. Hence $(x\ {}_a{\rightarrow}\ w) \wedge z = (x\ {}_a{\rightarrow}\ y) \wedge (x\ {}_a{\rightarrow}\ w) \wedge z$.
Therefore,

$$\begin{aligned}
(x\ {}_a{\rightarrow}\ y) \wedge w &= (x\ {}_a{\rightarrow}\ y) \wedge (x\ {}_a{\rightarrow}\ w) \wedge w \\
&= (x\ {}_a{\rightarrow}\ y) \wedge (x\ {}_a{\rightarrow}\ w) \wedge z \wedge w \\
&= (x\ {}_a{\rightarrow}\ w) \wedge z \wedge w \\
&= z \wedge (x\ {}_a{\rightarrow}\ w) \wedge w \\
&= z \wedge w \\
&= w.
\end{aligned}$$

Conversely, let $z \in H$ and assume that (1) and (2) hold. Now for any $a \in H_z$, take $c, d, e \in [a, z]$. By (1), $c \leqslant e\ {}_a{\rightarrow}\ c$ such that $e\ {}_a{\rightarrow}\ c \in [a, z]$ and then by (2) we get

$$\begin{aligned}
d \leqslant \text{ e }{}_a{\rightarrow}\text{ c } &\Leftrightarrow (\text{ e }{}_a{\rightarrow}\text{ c }) \wedge d = d \\
&\Leftrightarrow c \wedge d \wedge e = d \wedge e \\
&\Leftrightarrow d \wedge e \wedge c = d \wedge e \\
&\Leftrightarrow d \wedge e \leqslant c.
\end{aligned}$$

Hence $[a, z]$ is a Heyting algebra. One can define an induced binary operation $\rightarrow_z$ on $H_z$ by $x \rightarrow_z y = (y \vee x \vee y)_y{\rightarrow}y$. Hence $H_z$ is a skew Heyting algebra and therefore $H$ is a skew HADL. □

Corollary 3.4. *For any* $x, y \in H$, $((x \vee y) \wedge m)_{(y \wedge m)}{\rightarrow}(y \wedge m) = m$ *if and only if* $y \wedge x = x$.

Proof. Assume $((x \vee y) \wedge m)_{(y \wedge m)}{\rightarrow}(y \wedge m) = m$. Then

$$\begin{aligned}
x &= m \wedge x \\
&= \{((x \vee y) \wedge m)_{(y \wedge m)}{\rightarrow}(y \wedge m)\} \wedge x \\
&= x \wedge \{((x \vee y) \wedge m)_{(y \wedge m)}{\rightarrow}(y \wedge m)\} \wedge x \\
&= x \wedge ((x \wedge m)_{(y \wedge m)}{\rightarrow}(y \wedge m)) \wedge m \wedge x \\
&= x \wedge m \wedge ((x \wedge m)_{(y \wedge m)}{\rightarrow}(y \wedge m)) \wedge x \\
&= x \wedge m \wedge y \wedge m \wedge x \\
&= m \wedge y \wedge x \\
&= y \wedge x.
\end{aligned}$$

Hence $y \wedge x = x$. The converse is straight forward. □

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