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SOME APPLICATIONS OF EXISTENCE OF COMMON FIXED AND COMMON STATIONARY POINT OF A HYBRID PAIR

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Abstract. The common fixed and the common stationary point in a symmetric space using Hausdorff distance and δ-distance respectively are established. Results obtained are utilised to solve the functional equations in dynamic programming and Volterra integral inclusion and are supported by illustrative examples.

1. Introduction

Contractive conditions perform significant role in establishing common fixed and common stationary point of a single valued, set-valued and a hybrid pair of mappings. One of the most advantageous result in the fixed point theory is the Banach contraction principle (1922, [**1**]), which has been generalized in distinctive directions. In particular, set-valued generalization of Banach contraction is given by Nadler [**4**]. It is well known that coincidence point, common fixed point and common stationary point theorems of pairs of mappings are some generalizations of theorems of a single mapping. Over the last decades, fixed point theory for a hybrid pair has been investigated extensively as it provides the techniques for solving a variety of problems emerging in different branches of mathematics, physics, biology economics, engineering and so on. Motivated by the fact that a common fixed point of a hybrid pair of mappings may be viewed as a rest-point of the dynamic system whereas a common stationary point may be observed as an end-point of the system, we establish coincidence point, common fixed point and common stationary point in a symmetric space using CLR-property introduced by Sintunaravat and Kumam [**5**] and later extended it to a hybrid pair of mapping by Imdad et al.

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[**2**]. Also we establish common fixed point and common stationary point using a common limit range property-type I, (CLR-I property) recently introduced by Yamaod and Sintunavarat [**7**]. In the last section, results obtained are applied to find the solutions of functional equations emerging in dynamic programming and Volterra integral inclusion.

2. Preliminaries

A symmetry on a set X is a function $d : X \times X \to [0, \infty)$ satisfying $d(x, y) = 0$ iff $x = y$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. A pair (X, d) is known as a symmetric space. A subset A of X is said to be

(1) Closed if $A = \bar{A}$ where $\bar{A} = \{x \in X : d(x, A) = 0\}$.
(2) Bounded if $\delta(A) < \infty$ where $\delta(A) = \sup\{d(a, b) : a, b \in A\}$.

We shall follow the following notations and definitions.

$CB(X)$ = Class of all nonempty closed and bounded subsets of X.

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\},$$

for every $A, B \in CB(X)$, $x \in X$, where $d(x, A) = \inf\{d(x, a) : a \in A\}$. Such a mapping H is called a Pompeiu-Hausdorff metric (distance) on $CB(X)$. Clearly, $(CB(X), H)$ is a symmetric space.

- (t, S) is known as a hybrid pair if $t : X \to X$ is a single valued and $S : X \to CB(X)$ is a set-valued mapping.
- $u \in X$ is a coincidence point of a hybrid pair (t, S) if $tu \in Su$.
- $u \in X$ is a stationary coincidence point of a hybrid pair (t, S) if $Su = \{tu\}$.
- $u \in X$ is a common fixed point of a hybrid pair (t, S) if $u = tu \in Su$.
- $u \in X$ is a common stationary point of a hybrid pair (t, S) if $Su = \{tu\} = \{u\}$.

By the convergence of Hausdorff metric H we mean that, $\lim_{n\to\infty} H(A_n, A) = 0$, where $A \in CB(X)$ and $\{A_n\} \subset CB(X)$, i.e., for any $\epsilon > 0$, there exists a positive integer N such that $A_n \in N_\delta(A) = \{x \in X : d(x, A) < \epsilon\}$, for all $n \geqslant N$.

Definition 2.1. ([**2**]) A hybrid pair (t, S) satisfies the CLR_t-property (common limit in the range property) with respect to S, if $\exists$ a sequence $\{x_n\} \in X$ and $A \in CB(X)$ satisfying

$$\lim_{n\to\infty} tx_n = tu \in A = \lim_{n\to\infty} Sx_n, \text{for } u \in X.$$

Now we furnish example of CLR_t-property with respect to S in a symmetric space.

Example 2.1. Let $X = [0, 5]$ and $d(x, y) = (x - y)^2$ be given. Let us define t and S as $tx = \frac{3+x}{2}$ and $Sx = [2, x]$, for all $x \in X$. Let $\{u_n\}$ in X be defined as $u_n = 3 + \frac{1}{2n}, n \in \mathbb{N}$. Clearly, we have

$$\lim_{n\to\infty} tu_n = 3 = t3 \in [2, 3] = \lim_{n\to\infty} Su_n.$$

Therefore, the hybrid pair (t, S) satisfies (CLR_t)-property.

Significance of the (CLR_t)-property is that one does not need closedness of range subspaces or completeness of space for the existence of common fixed point as well as common stationary point using it. It is interesting to notice that, if tX is closed, then a non-compatible hybrid pair (t, S) satisfies the (CLR_t)-property with respect to S.

Recently Yamaod and Sintunavarat [**7**] introduced the following modification of the common limit in the range property.

DEFINITION 2.2. [**7**] A hybrid pair (t, S) of a metric space (X, d) satisfies the *CLR_t-I* property with respect to t, (common limit in the range property-type I), if $\exists\ \{x_n\} \in X, u \in X$ and $A \in CB(X)$ satisfying

$$\lim_{n\to\infty} tx_n = tu \in A = \lim_{n\to\infty} Sx_n$$

and $\lim\limits_{n\to\infty} Sx_n \neq Su$ for all $n \in \mathbb{N}$.

Clearly, a hybrid pair (t, S) satisfying *CLR_t-I* property also satisfies (CLR_t)-property however reverse implication need not be true.

3. Main Results

In all that follows $\phi : [0, \infty) \to [0, \infty)$ is a continuous monotonic increasing function satisfying $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$.

THEOREM 3.1. *Let a hybrid pair (t, S) of a symmetric space (X, d) satisfies:*

(1) *(CLR_t)-property with respect to S,*

(2) $\delta^P(Sx, Sy) \leqslant$

$$\phi\Big\{ \max\{d^P(tx, ty), d^P(tx, Sx), d^P(ty, Sy), \tfrac{1}{2}(d^P(Sy, tx) + d^P(Sx, ty))\} \Big\},$$

for each $x, y \in X$, $P > 1$. Then t and S have a stationary coincidence point. Furthermore, they have a unique common stationary point given that $ttu = tu$ for some $u \in C(t, S) \neq \phi$.

PROOF. Since a hybrid pair (t, S) satisfies the (CLR_t)-property with respect to S, $\exists$ a sequence $\{x_n\} \in X$ satisfying $\lim\limits_{n\to\infty} tx_n = tu \in A = \lim\limits_{n\to\infty} Sx_n$, for $u \in X$ and $A \in CB(X)$. Now, we prove $Su = \{tu\}$. If not, taking $x = x_n$ and $y = u$ in condition (2),

$$\delta^P(Sx_n, Su) \leqslant$$

$$\phi\Big\{ \max\{d^P(tx_n, tu), d^P(tx_n, Sx_n), d^P(tu, Su), \frac{1}{2}(d^P(Su, tx_n) + d^P(Sx_n, tu))\} \Big\}.$$

Letting $n \to \infty$,

$$\delta^P(A, Su) \leqslant$$

$$\phi\Big\{ \max\{d^P(tu, tu), d^P(tu, A), d^P(tu, Su), \frac{1}{2}(d^P(Su, tu) + d^P(A, tu))\} \Big\}$$

or $\delta^P(A, Su) \leqslant \phi(d^P(tu, Su))$ or $\delta^P(A, Su) < d^P(tu, Su)$. Since $tu \in A$, it follows from the definition of δ distance

$$d^P(tu, Su) \leqslant \delta^P(A, Su) < d^P(tu, Su), \quad \text{a contradiction.}$$

Hence, $d^P(tu, Su) = 0$, i.e., $Su = \{tu\}$. Thus u is a stationary coincidence point, i.e., $C(t, S) \neq \phi$. Now by the assumption we have $Su = \{ttu\} = \{tu\}$. From condition (2), for $x = tu$ and $y = u$ we get

$$\delta^P(Stu, Su) \leqslant$$

$$\phi\left\{ \max\{d^P(ttu, tu), d^P(ttu, Stu), d^P(tu, Su), \frac{1}{2}(d^P(Su, ttu) + d^P(Stu, tu))\}\right\}$$

or

$$\delta^P(Stu, Su) \leqslant \phi(d^P(tu, Stu)) < (d^P(tu, Stu)).$$

Using the definition of δ distance

$$d^P(tu, Stu) \leqslant \delta^P(Stu, Su) < d^P(tu, Stu), \quad \text{a contradiction.}$$

Hence, $Su = \{ttu\} = \{tu\} \Rightarrow tu$ is a common stationary point. Now the uniqueness follows using condition (2). □

THEOREM 3.2. *Let a hybrid pair (t, S) of a symmetric space (X, d) satisfies:*

(1) *(CLR_t)-property with respect to S.*

(2) $\delta(Sx, Sy) \leqslant$

$$\phi\left\{ \max\{d(tx, ty), d(tx, Sx), d(ty, Sy), \tfrac{1}{2}(d(Sy, tx) + d(Sx, ty))\}\right\},$$

for each $x, y \in X$. Then t and S have a stationary coincidence point. Furthermore, they have a unique common stationary point, given that $ttu = tu$ for some $u \in C(t, S) \neq \phi$.

PROOF. Proof follows on putting $P = 1$ in Theorem 3.1. □

Example furnished exhibits a fascinating feature of the main result that continuity of mappings is not essentially required for the existence of a common stationary point.

EXAMPLE 3.1. $X = [0, 7]$ and d be symmetric on X such that $d(x, y) = (x-y)^2$. Let mappings t and S on X be defined as follows:

$$tx = \begin{cases} 5 + x, & 0 \leqslant x < 2 \\ 2, & 2 \leqslant x \leqslant 7, \end{cases} \qquad Sx = \begin{cases} [0, 1], & 0 \leqslant x < 2 \\ \{2\}, & 2 \leqslant x \leqslant 7. \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \leqslant 1$ such that $x_n = 2 + \log(1 + \frac{1}{n})$. It is clear that

$$\lim_{n\to\infty} tx_n = 2 \in \{2\} = \lim_{n\to\infty} Sx_n.$$

Hence, the hybrid pair (t, S) satisfies (CLR_t)-property with respect to S. The point $t = 2$ is a stationary coincidence point and $tt2 = t2$.

For $x, y \in [0, 2)$, we have $\delta(Sx, Sy) = 1 \leqslant \frac{1}{2}(4 + x)^2 = \frac{d(tx, Sx)}{2}$.

For $x \in [0, 2)$ and $y \in [2, 7]$, we have $\delta(Sx, Sy) = 4 \leqslant \frac{1}{2}(4 + x)^2 = \frac{d(tx, Sx)}{2}$.

For $x \in [2,7]$ and $y \in (0,2]$, we have $\delta(Sx, Sy) = 4 \leqslant \frac{1}{2}(4+y)^2 = \frac{d(ty,Sy)}{2}$.

For $x, y \in [2,7]$, we have $\delta(Sx, Sy) = 0$.

Thus, all the hypotheses of Theorem 3.2 are satisfied for $\phi(t) = \frac{t}{2}$ and 2 is their common stationary point. Furthermore, t and S are both discontinuous mappings and $tx \not\subset Sx$.

Now we establish our next result using Hausdorff distance.

THEOREM 3.3. *Let a hybrid pair (t, S) of a symmetric space satisfies:*

(1) *(CLR_t)-property with respect to S,*

(2) $H^P(Sx, Sy) \leqslant \phi\Big\{ \max\{d^P(tx, ty), d^P(tx, Sx), d^P(ty, Sy), \frac{1}{2}(d^P(Sy, tx) + d^P(Sx, ty))\}\Big\}$,

for each $x, y \in X$, $P > 1$. Then t and S have a coincidence point. Furthermore, they have a unique common fixed point in X given that $ttu = tu$ for some $u \in C(t, S) \neq \phi$.

PROOF. Since a hybrid pair (t, S) satisfies the (CLR_t)-property with respect to S, $\exists$ a sequence $\{x_n\} \in X$ satisfying $\lim\limits_{n\to\infty} tx_n = tu \in A = \lim\limits_{n\to\infty} Sx_n, u \in X$ and $A \in CB(X)$. Now, we prove $tu \in Su$. If not, taking $x = x_n$ and $y = u$ in condition (2),

$$H^P(Sx_n, Su) \leqslant \phi\Big\{ \max\{d^P(tx_n, tu), d^P(tx_n, Sx_n), d^P(tu, Su), \frac{1}{2}(d^P(Su, tx_n) + d^P(Sx_n, tu))\}\Big\}.$$

Letting $n \to \infty$,

$$H^P(A, Su) \leqslant \phi\Big\{ \max\{d^P(tu, tu), d^P(tu, A), d^P(tu, Su), \frac{1}{2}(d^P(Su, tu) + d^P(A, tu))\}\Big\}$$

or

$$H^P(A, Su) \leqslant \phi(d^P(tu, Su))$$

or

$$H^P(A, Su) < d^P(tu, Su).$$

Since $tu \in A$, it follows from the definition of hausdroff distance

$$d^P(tu, Su) \leqslant H^P(A, Su) < d^P(tu, Su), \text{ a contradiction.}$$

Hence, $d^P(tu, Su) = 0$, i.e., $tu \in Su \Rightarrow u$ is a coincidence point, i.e., $C(t, S) \neq \phi$. Now by the assumption we have $ttu = tu \in Su$. Taking $x = tu$, $y = u$ in condition (2),

$$H^P(Stu, Su) \leqslant \phi\Big\{ \max\{d^P(ttu, tu), d^P(ttu, Stu), d^P(tu, Su), \frac{1}{2}(d^P(Su, ttu) +$$

$$d^P(Stu, tu))\}\Bigg\}$$

or

$$H^P(Stu, Su) \leqslant \phi(d^P(tu, Stu)) < d^P(tu, Stu).$$

From the definition of Hausdorff distance

$$d^P(tu, Stu) \leqslant H^P(Stu, Su) < d^P(tu, Stu), \quad \text{a contradiction.}$$

Hence, $tu = ttu \in Stu$, i.e., tu is a common fixed point. Now the uniqueness follows using condition (2). $\square$

Theorem 3.4. *Let a hybrid pair (t, S) of a symmetric space (X, d) satisfies:*

1. *(CLR_t)-property with respect to S,*
2. $H(Sx, Sy) \leqslant \phi\Bigg\{\max\{d(tx, ty), d(tx, Sx), d(ty, Sy), \frac{1}{2}(d(Sy, tx) + d(Sx, ty))\}\Bigg\}$,

for each $x, y \in X$. Then t and S have a coincidence point. Furthermore, t and S have a unique common fixed point in X given that $ttu = tu$ for some $u \in C(t, S) \neq \phi$.

Proof. Proof follows on putting $P = 1$ in Theorem 3.3. $\square$

Example furnished demonstrate the fact that continuity of mappings is no longer required using Hausdorff distance for the existence of common fixed point via CLR_t-property.

Example 3.2. $X = [0, 4]$ and d be the symmetry such that $d(x, y) = (x - y)^2$. Let mappings t and S on X be defined as:

$$tx = \begin{cases} x, & 0 \leqslant x \leqslant 1 \\ 4, & 1 < x \leqslant 4, \end{cases} \qquad Sx = \begin{cases} [1, 2], & 0 \leqslant x \leqslant 1 \\ [\frac{1}{2}, \frac{3}{4}], & 1 < x \leqslant 4. \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geqslant 1$ in such a way that $x_n = 1 - \frac{1}{n}$. So

$$\lim_{n\to\infty} tx_n = 1 \in [1, 2] = \lim_{n\to\infty} Sx_n.$$

Hence, the hybrid pair (t, S) satisfies CLR_t-property with respect to S. The point $t = 1$ is a coincidence point and $tt1 = t1$. Now,

For $x, y \in [0, 1]$, we have $H(Sx, Sy) = 0$.

For $x \in [0, 1]$ and $y \in (1, 4]$, we have $H(Sx, Sy) = \frac{9}{4} \leqslant \frac{1}{2}(4 - x)^2 = \frac{d(tx,ty)}{2}$.

For $x \in (1, 4]$ and $y \in [0, 1]$, we have $H(Sx, Sy) = \frac{25}{16} \leqslant \frac{1}{2}(4 - y)^2 = \frac{d(tx,ty)}{2}$.

For $x, y \in (1, 4]$, we have $H(Sx, Sy) = 0$.

Thus, all the hypotheses of Theorem 3.4 are verified for $\phi(t) = \frac{t}{2}$ and 1 is the common fixed point of t and S. Further, t and S are both discontinuous mappings and $tx \not\subset Sx$.

Now we utilise recently introduced CLR_t-I property to establish unique common fixed point of a hybrid pair using Hausdorff distance.

Theorem 3.5. *Let a hybrid pair (t, S) of a symmetric space (X, d) satisfies:*

(1) *(CLR_t)-I property with respect to S,*

(2) $H^P(Sx, Sy) \leqslant \phi\Big\{ \max\{d^P(tx, ty), d^P(tx, Sx), d^P(ty, Sy), \frac{1}{2}(d^P(Sy, tx) + d^P(Sx, ty))\}\Big\},$

for each $x, y \in X$, $P > 1$. Then t and S have a coincidence point. Furthermore, they have a unique common fixed point in X given that $ttu = tu$ for some $u \in C(t, S) \neq \phi$.

Proof. Since (t, S) satisfies (CLR_t)-I property with respect to the mapping S, $\exists$ a sequence $\{x_n\} \in X$ satisfying

$$\lim_{n\to\infty} tx_n = tu \in A = \lim_{n\to\infty} Sx_n, u \in X, A \in CB(X)$$

and $\lim_{n\to\infty} Sx_n \neq Su$ for all $n \in \mathbb{N}$.
Now, we prove $tu \in Su$. If not, taking $x = x_n$ and $y = u$ in condition (2) we have

$$H^P(Sx_n, Su) \leqslant \phi\Big\{ \max\{d^P(tx_n, tu), d^P(tx_n, Sx_n), d^P(tu, Su), \frac{1}{2}(d^P(Su, tx_n) + d^P(Sx_n, tu))\}\Big\}.$$

Letting $n \to \infty$,

$$H^P(A, Su) \leqslant \phi\Big\{ \max\{d^P(tu, tu), d^P(tu, A), d^P(tu, Su), \frac{1}{2}(d^P(Su, tu) + d^P(A, tu))\}\Big\}$$

or

$$H^P(A, Su) \leqslant \phi(d^P(tu, Su))$$

or

$$H^P(A, Su) < d^P(tu, Su).$$

Since $tu \in A$, it follows from the definition of Hausdorff distance

$$d^P(tu, Su) \leqslant H^P(A, Su) < d^P(tu, Su), \text{ a contradiction.}$$

Hence, $d^P(tu, Su) = 0 \Rightarrow tu \in Su$, i.e., u is a coincidence point, i.e., $C(t, S) \neq \phi$.
Now by the assumption we have $ttu = tu \in Su$. Taking $x = tu$ and $y = u$ in the condition (2) we have

$$H^P(Stu, Su) \leqslant \phi\Big\{ \max\{d^P(ttu, tu), d^P(ttu, Stu), d^P(tu, Su), \frac{1}{2}(d^P(Su, ttu) + d^P(Stu, tu))\}\Big\}$$

or

$$H^P(Stu, Su) \leqslant \phi(d^P(tu, Stu)) < d^P(tu, Stu).$$

From the definition of Hausdorff distance

$$d^P(tu, Stu) \leqslant H^P(Stu, Su) < d^P(tu, Stu), \text{ a contradiction.}$$

Hence, $tu = ttu \in Stu \Rightarrow tu$ is a common fixed point. The uniqueness follows using condition (2). $\square$

THEOREM 3.6. *Let a hybrid pair (t, S) of a symmetric space (X, d) satisfies:*

(1) *(CLR_t)-I property with respect to S,*

(2) $H(Sx, Sy) \leqslant \phi\Big\{ \max\{d(tx, ty), d(tx, Sx), d(ty, Sy), \frac{1}{2}(d(Sy, tx) + d(Sx, ty))\}\Big\}$, *for all $x, y \in X$. Then t and S have a coincidence point. Furthermore, they have a unique common fixed point in X given that $ttu = tu$ for some $u \in C(t, S) \neq \phi$.*

PROOF. Proof follows on putting $P = 1$ in Theorem 3.5. $\square$

EXAMPLE 3.3. $X = [0, 12]$ and d be the symmetry such that $d(x, y) = (x-y)^2$. Let the mappings t and S be defined as follows:

$$tx = \begin{cases} 4 - x, & 0 \leqslant x \leqslant 2 \\ 12, & 2 < x \leqslant 12, \end{cases} \qquad Sx = \begin{cases} [\frac{3}{2}, 3], & 0 \leqslant x < 2 \\ [2, 4], & 2 \leqslant x \leqslant 12. \end{cases}$$

Consider a sequence $\{x_n\}$ for all $n \geqslant 1$ such that $x_n = 2 - e^{-n}$. Clearly,

$$\lim_{n\to\infty} tx_n = 2 \in [\frac{3}{2}, 3] = \lim_{n\to\infty} Sx_n$$

and $\lim_{n\to\infty} Sx_n = [\frac{3}{2}, 3] \neq [2, 4] = Su$. Hence, the pair (t, S) satisfies CLR_t-I property with respect to S. The point $u = 2$ is a coincidence point and $uu2 = u2$.

For $x, y \in [0, 2]$, we have $H(Sx, Sy) = 0$.

For $x \in [0, 2)$ and $y \in (2, 12]$, we have $H(Sx, Sy) = 1 \leqslant \frac{1}{60}(8 + x)^2 = \frac{d(tx,ty)}{60}$.

For $x = 2$ and $y \in [2, 12]$,we have $H(Sx, Sy) = 0$.

For $x \in (2, 12]$ and $y \in [0, 2)$,we have $H(Sx, Sy) = 1 \leqslant \frac{1}{60}(8 + y)^2 = \frac{d(tx,ty)}{60}$.

For $x \in (2, 12]$ and $y = 2$, we have $H(Sx, Sy) = 1 \leqslant \frac{1}{60}100 = \frac{d(tx,ty)}{60}$.

For $x, y \in [2, 4]$, we have $H(Sx, Sy) = 0$.

Thus, t and S satisfy all the conditions of Theorem 3.6 for $\phi(t) = \frac{t}{60}$ and 2 is their common fixed point of t and S. Further, t and S are both discontinuous mappings and $tx \not\subset Sx$.

REMARK 3.1. All the theorems generalize and improve the related theorems included in Imdad and Chauhan [**2**], Sintunavarat and Kumam [**5**], Yamaod and Sintunavarat [**7**] and others without using completeness, continuity or containment of involved mappings.

4. Applications

4.1. Application to Dynamic Programming Problem. Consider a multistage process, reduced to the system of functional equations

$$q_i(x) = \sup_{y\in D}\{g(x, y) + G_i(x, y, q(\tau(x, y)))\}, x \in W, i \in \{1, 2\} \tag{4.1}$$

where $\tau : W \times D \to W$, $g : W \times D \to R$ and $G_i : W \times D \times R \to R$ are mappings, while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space and U, V are Banach spaces.

Let $B(W)$ be the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = sup_{x\in W}|h(x)|^2$ with respective metric d. Also, $(B(W), \|.\|)$ is a Banach space wherein convergence is uniform. Therefore, if we consider a Cauchy sequence $\{h_n\} \in B(W)$, then the sequence $\{h_n\}$ converges uniformly to a function, say $h^* \in B(W)$.

THEOREM 4.1. *Let $T_i : B(W) \to B(W)$ be such that:*

(1) $\exists$ *a continuous monotonic increasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$, such that:* $|G_1(x, y, h_1(x) - G_2(x, y, h_2(x))| \leqslant [\phi(\Theta(h_1, h_2))]^{\frac{1}{2}}$, *where*

$$\Theta(h_1, h_2) = \max\bigg\{d(T_1h_1, T_1h_2), d(T_1h_1, T_2h_1), d(T_1h_2, T_2h_2), \frac{1}{2}(d(T_1h_2, T_2h_1)+$$

$$d(T_2h_2, T_1h_1))\bigg\}, \quad for \ \ h_1, h_2 \in B(W), x \in W, y \in D;$$

(2) $\exists$ *a sequence $\{h_n\}$ in $B(W)$ and $h^* \in B(W)$ satisfying*

$$\lim_{n\to\infty} T_1h_n = \lim_{n\to\infty} T_2h_n = T_1h^*;$$

(3) *$g : W \times D \to R$ and $G_i : W \times D \times R \to R$ are bounded functions, for $i = 1, 2$;*

(4) *$T_1T_1h = T_1h$, whenever $T_1h = T_2h$, $h \in B(W)$.*

Then the functional equations

$$T_ih_i(x) = \sup_{y\in D}\{g(x, y) + G_i(x, y, h_i(\tau(x, y)))\}, h_i \in B(W), x \in W, i \in \{1, 2\} \tag{4.2}$$

has a unique bounded solution.

PROOF. By hypothesis (2), the pair (T_1, T_2) satisfies the CLR_t-property with respect to T_1. Now, let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$. $\exists$ $y_1, y_2 \in D$ satisfying

$$T_2h_1(x) < g(x, y_1) + G_1\bigg(x, y_1, h_1(\tau(x, y_1))\bigg) + \lambda, \tag{4.3}$$

$$T_2h_2(x) < g(x, y_2) + G_2\bigg(x, y_2, h_2(\tau(x, y_2))\bigg) + \lambda. \tag{4.4}$$

Also by definition

$$T_2h_1(x) \geqslant g(x, y_2) + G_1\bigg(x, y_2, h_1(\tau(x, y_2))\bigg), \tag{4.5}$$

$$T_2h_2(x) \geqslant g(x, y_1) + G_2\bigg(x, y_1, h_2(\tau(x, y_1))\bigg). \tag{4.6}$$

Next, by using inequalities (4.3) and (4.6), we obtain

$$T_2h_1(x) - T_2h_2(x) < G_1\Big(x, y_1, h_1(\tau(x, y_1))\Big) - G_2\Big(x, y_1, h_2(\tau(x, y_1))\Big) + \lambda \tag{4.7}$$

$$\leqslant \Big|G_1\Big(x, y_1, h_1(\tau(x, y_1))\Big) - G_2\Big(x, y_1, h_2(\tau(x, y_1))\Big)\Big| + \lambda$$

$$\leqslant [\phi(\Theta(h_1, h_2))]^{\frac{1}{2}} + \lambda. \tag{4.8}$$

Analogously, using inequalities (4.4) and (4.5), we get

$$T_2h_2(x) - T_2h_1(x) < [\phi(\Theta(h_1, h_2))]^{\frac{1}{2}} + \lambda. \tag{4.9}$$

Combining inequalities (4.8) and (4.9), we get

$$|T_2h_1(x) - T_2h_2(x)| < [\phi(\Theta(h_1, h_2))]^{\frac{1}{2}} + \lambda.$$

Implying there by

$$|T_2h_1(x) - T_2h_2(x)| < [\phi(\Theta(h_1, h_2))]^{\frac{1}{2}},$$

which does not depend on $x \in W$ and $\lambda > 0$ is arbitrary. So, on squaring we get

$$d(T_2h_1, T_2h_2) < \phi(\Theta(h_1, h_2)).$$

Using(4), for each $t = T_1, S = T_2$ all the conditions of Theorem 3.2 are verified for the pair (T_1, T_2). Hence, the operators T_1 and T_2 have a unique common fixed point, implying there by that the system of functional equations (4.2) has a unique bounded solution. □

4.2. Application to Volterra integral inclusions. Inspired by Tŭrkoğlu and Altun [**6**], in this section, we establish the existence of solutions of integral inclusion of the type

$$x(t) \in q(t) + \int_a^{\sigma(t)} k(t, s)F(s, x(s))ds \tag{4.10}$$

for $t \in J$, where $\sigma : J \to J, q : J \to E, k : J \times J \to R$ are continuous, $F : J \times E \to C(E)$, E is a real Banach space with norm $\|.\|_E$, $C(E)$ denotes the class of all non-empty closed subsets of E and $J = [a, b]$ in R is a closed and bounded interval. By a solution for the integral inclusion (4.10), we mean a continuous function $x : J \to E$ such that

$$x(t) = q(t) + \int_a^{\sigma(t)} k(t, s)v(s)ds$$

for some $v \in B(J, E)$ satisfying $v(t) \in F(t, x(t))$, $t \in J$, where $B(J, E)$ is the space of all E-valued Bochner-integrable functions on J. Let $C(J, E)$ denote the space of all continuous E-valued functions on J. Define a norm $\|.\|$ on $C(J, E)$ by $\|x\| = sup_{t\in J}|x(t)|_E^2$. We use the following definitions.

Definition 4.1. A multivalued function $\beta : J \times E \to 2^E$ is called Carathèodory if

(1) $t \to \beta(t,x)$ is measurable for each $x \in E$, and
(2) $x \to \beta(t,x)$ is upper semi-continuous almost everywhere for $t \in J$.

Denote $\|F(t,x)\| = \sup\{\|u\|_E^2 : u \in F(t,x)\}$

DEFINITION 4.2. A Carathèodory multifunction $F(t,X)$ is L^1- Carathèodory if for every real number $r > 0$ there exists a function $h_r \in L^1(J,R)$ such that $\|F(t,x)\| \leqslant h_r t$ for almost every $t \in J$ and $x \in E$ with $\|x\|_E \leqslant r$.

Denote $S_F^1 = \{v \in B(J,E) : v(t) \in F(t,x(t))$ a. e. $t \in J\}$

LEMMA 4.1 ([**3**]). *If diam(E)* $< \infty$ *and* $F : J \times E \to 2^E$ *is* L^1- *Carathèodory, then* $S_F^1 \neq \phi$ *for each* $x \in C(J,E)$.

LEMMA 4.2 ([**3**]). *Let* E *be a Banach space,* F *a Carathèodory multi-mapping with* $S_F^1 \neq \phi$ *and* $L : L^1(J,E) \to C(J,E)$ *be a continuous linear mapping. Then the operator* $LoS_F^1 : C(J,E) \to 2^{C(J,E)}$ *is a closed graph operator on* $C(J,E) \times C(J,E)$.

Suppose that the following set of hypotheses hold:
(1) The function $k(t,s)$ is non-negative on $J \times J$ with $M = \sup_{t,s\in J}[k(t,s)]^2$;
(2) the multivalued function $F(t,x)$ is Carathèodory;
(3) the multivalued function $F(t,x)$ is nondecreasing in x almost everywhere for $t \in J$;
(4) there exists a continuous monotonic increasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(0) = 0$ and $\phi(t) < t$ for each $t > 0$ such that

$$|F(s,x(s)) - F(s,y(s))| \leqslant \frac{1}{M}\phi(\Delta(x,y)),$$

for all $s \in J, x \in E$, where

$$\Delta(x,y) = \max\left\{d(tx,ty), d(tx,Sx), d(ty,Sy), \frac{1}{2}(d(Sy,tx) + d(Sx,ty))\right\};$$

(5) $S_F^1 \neq \phi$ for each $x \in C(J,E)$;
(6) $\exists$ a sequence $\{x_n\} \in C(J,E)$ and $A \in 2^{C(J,E)}$ satisfying $\lim\limits_{n\to\infty} x_n = z \in A = \lim\limits_{n\to\infty} Sx_n$, for $z \in C(J,E)$.

THEOREM 4.2. *Assume that hypotheses (1)-(6) hold. Then the integral inclusion* (4.10) *has a solution in* $[a,b]$ *defined on* J.

PROOF. Let $X = C(J,E)$ and consider the interval $[a,b] \in X$. Define the multivalued mapping $S : [a,b] \to 2^X$ for $u \in [a,b]$ as

$$Sx = \left\{u(t) = q(t) + \int_a^{\sigma(t)} k(t,s)v(s)ds\right\}$$

$v \in S_F^1(x)$ for every $t \in [a,b]$. Clearly S is well-defined, since, from (5), $S_F^1 \neq \phi$. For all $t \in [a,b]$ by (2) and (4) we get

$$|Sx - Sy| = \left|\int_a^{\sigma(t)} \Big(k(t,s)v_1(s) - k(t,s)v_2(s)\Big)ds\right|_E.$$

Squaring both sides

$$|Sx - Sy|^2 \leqslant \int_a^{\sigma(t)} |k(t,s)|^2 |v_1(s) - v_2(s)|_E^2 ds$$

$$\leqslant \sup_{t,s\in J} [k(t,s)]^2 |F(s,x(s)) - F(s,y(s))| = M|F(s,x(s)) - F(s,y(s))|.$$

This implies that

$$H(Sx, Sy) \leqslant \phi(\Delta(x,y)),$$

for each $t \in J$. We deduce that the operator S satisfy condition (4), where t is an identity mapping. Also, using (6), S satisfies all conditions of Theorem 3.4 on $[a,b]$ and we conclude that the given integral inclusion has a unique solution. □

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