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PROPERTIES OF RAZUMIKHIN CLASS OF FUNCTIONS AND PPF DEPENDENT FIXED POINTS OF WEAKLY CONTRACTIVE TYPE MAPS

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Abstract. We discuss the properties of Razumikhin class of functions. We introduce weakly contractive type maps and prove the existence of PPF dependent fixed points of weakly contractive type mappings in the Razumikhin class of functions. Further, we prove the uniqueness of PPF dependent fixed points under certain assumption. We provide examples to illustrate our results.

1. Introduction

The Banach contraction principle is one of the fundemental and useful result in fixed point theory and it plays an important role in solving problems related to non-linear functional analysis. In 1997, Alber and Gurre-Delabriere [**1**] introduced weakly contractive maps which are extensions of contraction maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [**12**] extended this concept to metric spaces.

In 1997, Bernfeld, Lakshmikantham and Reddy [**4**] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point. Furthermore, they introduced notation of Banach type contraction for a non-self mapping and proved the existence of PPF dependent fixed point theorems of Banach type contractive mappings in the Razumikhin class. In 2007, Dirci, McRae and Vasundhara Devi [**7**] developed a technique to establish the existence of PPF dependent fixed points for a non-linear operator in partially ordered metric spaces, and an application to a periodic boundary value problem with delay is given. In 2013, Hussain, Khaleghizadeh,

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Salimi and Akbar [**8**] extended this study to prove the existence of PPF dependent fixed point and coincidence points using admissible mappings. For more works and its applications, we refer [**3, 11, 6, 2, 10, 9, 5**].

In Section 2, we discuss the properties of Razumikhin class of functions. In Section 3, we introduce weakly contractive type maps and prove the existence of PPF dependent fixed points. In fact, PPF dependent fixed points of weakly contractive maps need not be unique (Example 3.2). But under certain additional assumption, uniqueness holds. Examples are provided in support of our results.

Throughout this paper, $\mathbb{R}$ denotes the real line, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{N}$ denotes the set of all natural numbers, $(E, ||.||_E)$ is a Banach space, $I = [a, b] \subseteq \mathbb{R}$ and $E_0 = C(I, E)$ denote the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup\limits_{a \leqslant t \leqslant b} ||\phi(t)||_E$ for $\phi \in E_0$.

2. Razumikhin class of functions and its properties

For a fixed $c \in I$, the Razumikhin class R_c of functions in E_0 is defined by $R_c = \left\{\phi \in E_0 / \, ||\phi||_{E_0} = ||\phi(c)||_E\right\}$. Clearly every constant function from I to E belongs to R_c so that R_c is a non-empty subset of E_0.

Definition 2.1. Let R_c be the Razumikhin class of continuous functions in E_0. Then we say that

i) The class R_c is algebraically closed with respect to the difference if $\phi - \psi \in R_c$ whenever $\phi, \psi \in R_c$.

ii) The class R_c is topologically closed if it is closed with respect to the topology on E_0 by the norm $||.||_{E_0}$.

The Razumikhin class of functions R_c has the following properties.

Theorem 2.1. *Let R_c be the Razumikhin class of functions in E_0. Then*

i) $E_0 = \bigcup\limits_{c \in [a,b]} R_c$.

ii) *For any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.*

iii) *The Razumikhin class R_c is topologically closed with respect to the norm defined on E_0.*

Proof. i) Let $\phi \in E_0$. Then $\phi : I \to E$ is continuous. Since I is compact , we have $\phi(I)$ is compact. By the definition of $||\phi||_{E_0}$, there exists a sequence $\{t_n\}$ in $[a, b]$ such that

$$\lim_{n \to \infty} ||\phi(t_n)||_E = ||\phi||_{E_0} . \tag{2.1}$$

Since $\phi(I)$ is compact , it is sequentially compact and so there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\phi(t_{n_k}) \to l$ as $k \to \infty$, $l \in \phi(I)$ and so $l = \phi(c)$ for some $c \in I$. Hence $\phi(t_{n_k}) \to \phi(c)$ as $k \to \infty$. Therefore $\lim\limits_{k \to \infty} ||\phi(t_{n_k})||_E = ||\phi(c)||_E$. Hence, from (2.1), we have $||\phi||_{E_0} = ||\phi(c)||_E$. It shows that $\phi \in R_c$ for some $c \in [a, b]$ so that $\phi \in \bigcup\limits_{c \in [a,b]} R_c$. Hence $E_0 \subseteq \bigcup\limits_{c \in [a,b]} R_c$. The other inclusion is trivial and hence (i) holds.

ii) Let $\phi \in R_c$. Then $||\phi||_{E_0} = ||\phi(c)||_E$, and

$$||\alpha\phi||_{E_0} = \sup_{a\leqslant t\leqslant b} ||(\alpha\phi)(t)||_E = |\alpha| \sup_{a\leqslant t\leqslant b} ||\phi(t)||_E = |\alpha| \, ||\phi(c)||_E = ||(\alpha\phi)(c)||_E \,.$$

iii) Let $\{\phi_n\}$ be a sequence in R_c that converges to ϕ. Since the norm is continuous, we have $||\phi_n||_{E_0} \to ||\phi||_{E_0}$ as $n \to \infty$. Since $\phi_n \in R_c$, we have $||\phi_n||_{E_0} = ||\phi_n(c)||_E \to ||\phi(c)||_E$ as $n \to \infty$. By the uniqueness of the limits, we have $||\phi||_{E_0} = ||\phi(c)||_E$ so that $\phi \in R_c$. Thus R_c is topologically closed with respect to the norm defined on E_0. □

The following example shows that, there may exist a Razumikhin class of functions in E_0 which is not algebraically closed with respect to the difference for any $c \in [a, b]$.

EXAMPLE 2.1. Let $I = [a, b],\ E = \mathbb{R},\ E_0 = C(I, \mathbb{R})$. Let $c \in [a, b]$. We define $\phi : I \to E$ by $\phi(x) = 1$ for $x \in [a, b]$ and $\psi : I \to E$ by

$$\psi(x) = \begin{cases} \frac{1}{2} + \frac{x-a}{2(c-a)} & \text{if } x \in [a, c] \\ \frac{1}{2} + \frac{x-b}{2(c-b)} & \text{if } x \in [c, b]. \end{cases}$$

We consider the following cases.

Case (i) : $a < c < b$.
Clearly $||\phi||_{E_0} = 1 = ||\phi(c)||_E$ and $||\psi||_{E_0} = 1 = ||\psi(c)||_E$ so that $\phi, \psi \in R_c$. Now

$$(\phi - \psi)(x) = \begin{cases} \frac{1}{2} - \frac{x-a}{2(c-a)} & \text{if } x \in [a, c] \\ \frac{1}{2} - \frac{x-b}{2(c-b)} & \text{if } x \in [c, b], \end{cases}$$

and $||\phi - \psi||_{E_0} = \frac{1}{2} \neq 0 = ||(\phi - \psi)(c)||_E$ so that $\phi - \psi \notin R_c$.

Case (ii) : $c = a$.
We define $\phi : I \to E$ by $\phi(x) = 1$ for $x \in [a, b]$ and $\psi : I \to E$ by

$$\psi(x) = \frac{1}{2} + \frac{x-b}{2(a-b)} \quad \text{for } x \in [a, b].$$

Clearly $||\phi||_{E_0} = 1 = ||\phi(c)||_E$ and $||\psi||_{E_0} = 1 = ||\psi(c)||_E$ so that $\phi, \psi \in R_c$. Now $(\phi - \psi)(x) = \frac{1}{2} - \frac{x-b}{2(a-b)}$ for $x \in [a, b]$ and $||\phi - \psi||_{E_0} = \frac{1}{2} \neq 0 = ||(\phi - \psi)(c)||_E$. So, that $\phi - \psi \notin R_c$.

Case (iii) : $c = b$.
We define $\phi : I \to E$ by $\phi(x) = 1$ for $x \in [a, b]$ and $\psi : I \to E$ by

$$\psi(x) = \frac{1}{2} + \frac{x-a}{2(b-a)} \text{ for } x \in [a, b].$$

We have $||\phi||_{E_0} = 1 = ||\phi(c)||_E$ and $||\psi||_{E_0} = 1 = ||\psi(c)||_E$ so that $\phi, \psi \in R_c$. Now, $(\phi - \psi)(x) = \frac{1}{2} - \frac{x-a}{2(b-a)}$ for $x \in [a, b]$ and $||\phi - \psi||_{E_0} = \frac{1}{2} \neq 0 = ||(\phi - \psi)(c)||_E$ so that $\phi - \psi \notin R_c$. From all the above cases, it follows that R_c is not algebraically closed with respect to the difference in E_0.

DEFINITION 2.2. ([**4**]). Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a PPF dependent fixed point of T if $T(\phi) = \phi(c)$ for some $c \in I$.

The PPF dependent fixed point ϕ of the operator T depends on the available class of continuous functions that are present, the past time c whose unaltered future value under the operator T is $\phi(c)$.

DEFINITION 2.3. ([**4**]). Let $T : E_0 \to E$ be a mapping. Then T is called a Banach type contraction if there exists $k \in [0,1)$ such that $||T\phi - T\psi||_E \leqslant k\,||\phi - \psi||_{E_0}$ for all $\phi, \psi \in E_0$.

Bernfeld et al. ([**4**]) established the existence and uniqueness of a PPF dependent fixed point of a Banach type contraction mapping.

THEOREM 2.2 ([**4**]). *Let $T : E_0 \to E$ be a Banach type contraction. Let R_c be an algebraically closed with respect to the difference and topologically closed. Then T has a unique PPF dependent fixed point in R_c.*

REMARK 2.1. By Theorem 2.1 of (iii), R_c is topologically closed with respect to the norm topology on E_0 and hence in the hypothesis of Theorem 2.2 'R_c is topologically closed' is redundant. Hence the following is the modified version of Theorem 2.2.

THEOREM 2.3 (Modified version of Theorem 2.2). *Let $T : E_0 \to E$ be a Banach type contraction. If R_c is algebraically closed with respect to the difference then T has a unique PPF dependent fixed point in R_c.*

The following lemma is useful to prove our main results of Section 3.

LEMMA 2.1. *Let $(E, ||.||_E)$ be a Banach space, $I = [a, b] \subseteq \mathbb{R}$ and $E_0 = C(I, E)$ be the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup\limits_{a\leqslant t\leqslant b} ||\phi(t)||_E$ for $\phi \in E_0$. Let $\{\phi_n\}$ be a sequence in E_0 such that $||\phi_n - \phi_{n+1}||_{E_0} \to 0$ as $n \to \infty$. If $\{\phi_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that $\big|\big|\phi_{m(k)} - \phi_{n(k)}\big|\big|_{E_0} \geqslant \epsilon$, $\big|\big|\phi_{m(k)} - \phi_{n(k)-1}\big|\big|_{E_0} < \epsilon$ and*

i) $\lim\limits_{k\to\infty} \big|\big|\phi_{m(k)} - \phi_{n(k)}\big|\big|_{E_0} = \epsilon$, ii) $\lim\limits_{k\to\infty} \big|\big|\phi_{m(k)} - \phi_{n(k)-1}\big|\big|_{E_0} = \epsilon$,

iii) $\lim\limits_{k\to\infty} \big|\big|\phi_{m(k)-1} - \phi_{n(k)}\big|\big|_{E_0} = \epsilon$, iv) $\lim\limits_{k\to\infty} \big|\big|\phi_{m(k)-1} - \phi_{n(k)-1}\big|\big|_{E_0} = \epsilon$.

PROOF. Runs as that of Lemma 1.4 of [**2**] □

3. PPF dependent fixed points of weakly contractive type maps

We denote $\mathsf{F} = \{f/f : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $f(t) = 0$ if and only if $t = 0$ for $t \in [0,\infty)\}$.

DEFINITION 3.1. A mapping $T : E_0 \to E$ is said to be weakly contractive type mapping if there exists $f \in \mathsf{F}$ such that $||T\phi - T\psi||_E \leqslant ||\phi - \psi||_{E_0} - f(||\phi - \psi||_{E_0})$ for all $\phi, \psi \in E_0$.

THEOREM 3.1. *Let $(E, ||.||_E)$ be a Banach space, $I = [a, b] \subseteq \mathbb{R}$. Let $E_0 = C(I, E)$ denote the set of all continuous functions on I equipped with the supremum* *norm* $||.||_{E_0}$ *and we define it by* $||\phi||_{E_0} = \sup_{a \leqslant t \leqslant b} ||\phi(t)||_E$ *for* $\phi \in E_0$. *Fix* $c \in I$. *Let* R_c *be the Razumikhin class of functions in* E_0. *If* $T : E_0 \to E$ *is a weakly contractive type mapping, then* T *has a PPF dependent fixed point in* R_c.

Proof. Let $\phi_0 \in R_c \subseteq E_0$. Clearly $T\phi_0 \in E$. Let $x_1 = T\phi_0$. We define $\phi_1 : I \to E$ by $\phi_1(t) = x_1$ for $t \in I$. Then $\phi_1 \in E_0$ with $||\phi_1||_{E_0} = ||x_1||_E = ||\phi_1(c)||_E$. Hence, we choose $\phi_1 \in R_c$ such that $T\phi_0 = x_1 = \phi_1(c)$. Let $x_2 = T\phi_1$. We Define $\phi_2 : I \to E$ by $\phi_2(t) = x_2$ for $t \in I$. Then $\phi_2 \in E_0$ with $||\phi_2||_{E_0} = ||x_2||_E = ||\phi_2(c)||_E$. Hence, we choose $\phi_2 \in R_c$ such that $T\phi_1 = x_2 = \phi_2(c)$ and $||\phi_2 - \phi_1||_{E_0} = \sup_{a \leqslant t \leqslant b} ||\phi_2(t) - \phi_1(t)||_E = \sup_{a \leqslant t \leqslant b} ||x_2 - x_1||_E = ||\phi_2(c) - \phi_1(c)||_E$. Let $x_3 = T\phi_2$. We define $\phi_3 : I \to E$ by $\phi_3(t) = x_3$ for $t \in I$. Then $\phi_3 \in E_0$ with $||\phi_3||_{E_0} = ||x_3||_E = ||\phi_3(c)||_E$. Hence we choose $\phi_3 \in R_c$ such that $T\phi_2 = x_3 = \phi_3(c)$ and $||\phi_3 - \phi_2||_{E_0} = \sup_{a \leqslant t \leqslant b} ||\phi_3(t) - \phi_2(t)||_E = \sup_{a \leqslant t \leqslant b} ||x_3 - x_2||_E = ||\phi_3(c) - \phi_2(c)||_E$. On continuing this process , we define a sequence $\{\phi_n\}$ inductively by $T\phi_n = x_{n+1} = \phi_{n+1}(c)$ and $||\phi_{n+1} - \phi_n||_{E_0} = ||\phi_{n+1}(c) - \phi_n(c)||_E$ for $n \in \mathbb{N} \cup \{0\}$.

If $\phi_{n+1} = \phi_n$ for some $n \in \mathbb{N} \cup \{0\}$, then $T\phi_n = \phi_{n+1}(c) = \phi_n(c)$, so that T has a PPF dependent fixed point in R_c. Suppose that $\phi_{n+1} \neq \phi_n$ for all $n \in \mathbb{N} \cup \{0\}$. We consider,

$$\begin{aligned} ||\phi_{n+1} - \phi_n||_{E_0} &= ||T\phi_n - T\phi_{n-1}||_E \\ &\leqslant ||\phi_n - \phi_{n-1}||_{E_0} - f(||\phi_n - \phi_{n-1}||_{E_0}) \\ &\leqslant ||\phi_n - \phi_{n-1}||_{E_0}. \end{aligned}$$

Therefore, the sequence $\{||\phi_{n+1} - \phi_n||_{E_0}\}$ is a decreasing sequence in $\mathbb{R}^+$. Let $||\phi_{n+1} - \phi_n||_{E_0} \to r$ as $n \to \infty$. We consider,

$$||\phi_{n+1} - \phi_n||_{E_0} = ||T\phi_n - T\phi_{n-1}||_E \leqslant ||\phi_n - \phi_{n-1}||_{E_0} - f(||\phi_n - \phi_{n-1}||_{E_0}).$$

Now on taking limits as $n \to \infty$, we get

$$\lim_{n\to\infty} ||\phi_{n+1} - \phi_n||_{E_0} \leqslant \lim_{n\to\infty} ||\phi_n - \phi_{n-1}||_{E_0} - f(\lim_{n\to\infty} ||\phi_n - \phi_{n-1}||_{E_0}),$$

and hence this shows that $r \leqslant r - f(r)$. Therefore $f(r) = 0$ which implies that $r = 0$. This shows that $||\phi_{n+1} - \phi_n||_{E_0} \to 0$ as $n \to \infty$.

We now show that $\{\phi_n\}$ is a Cauchy sequence. If it is not a Cauchy, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$\left|\left|\phi_{m(k)} - \phi_{n(k)}\right|\right|_{E_0} \geqslant \epsilon, \left|\left|\phi_{m(k)} - \phi_{n(k)-1}\right|\right|_{E_0} < \epsilon.$$

Now,

$$\begin{aligned} \epsilon \leqslant \left|\left|\phi_{m(k)} - \phi_{n(k)}\right|\right|_{E_0} &= \left|\left|\phi_{m(k)}(c) - \phi_{n(k)}(c)\right|\right|_E \\ &= \left|\left|T\phi_{m(k)-1} - T\phi_{n(k)-1}\right|\right|_E \\ &\leqslant \left|\left|\phi_{m(k)-1} - \phi_{n(k)-1}\right|\right|_{E_0} - f(\left|\left|\phi_{m(k)-1} - \phi_{n(k)-1}\right|\right|_{E_0}) \end{aligned}$$

and hence

$$\epsilon \leqslant \left|\left|\phi_{m(k)-1} - \phi_{n(k)-1}\right|\right|_{E_0} - f(\left|\left|\phi_{m(k)-1} - \phi_{n(k)-1}\right|\right|_{E_0}). \tag{3.1}$$

Since $||\phi_{n+1} - \phi_n||_{E_0} \to 0$ as $n \to \infty$, by Lemma 2.1, we have

$$\lim_{k\to\infty} \left|\left|\phi_{m(k)-1} - \phi_{n(k)-1}\right|\right|_{E_0} = \epsilon.$$

By applying limit as $k \to \infty$ to the inequality (3.1) on both sides, we get $\epsilon \leqslant \epsilon - f(\epsilon)$ which imlies that $f(\epsilon) = 0$ and hence $\epsilon = 0$, a contradiction. Therefore $\{\phi_n\}$ is a Cauchy sequence in $R_c \subseteq E_0$. Since E_0 is a Banach space, we have $\{\phi_n\}$ converges and $\lim_{n\to\infty} \phi_n = \phi^*$ (say), $\phi^* \in E_0$. Since R_c is topologically closed, we have $\phi^* \in R_c$. Now we show that ϕ^* is a PPF dependent fixed point of T. We consider

$$\begin{aligned} ||T\phi^* - \phi^*(c)||_E &\leqslant ||T\phi^* - T\phi_n||_E + ||T\phi_n - \phi^*(c)||_E \\ &\leqslant ||\phi^* - \phi_n||_{E_0} - f(||\phi^* - \phi_n||_{E_0}) + ||\phi_{n+1}(c) - \phi^*(c)||_{E_0}. \end{aligned}$$

By applying limits as $n \to \infty$ on both sides we get $||T\phi^* - \phi^*(c)||_E \leqslant 0$, which implies that $T\phi^* = \phi^*(c)$. Therefore ϕ^* is a PPF dependent fixed point of T in R_c. □

Example 3.1. Let $I = [0, 1]$, $E = \mathbb{R}$. Fix $c = \frac{1}{2} \in [0, 1]$. Let $E_0 = C(I, \mathbb{R})$. Let $f : [0, \infty) \to [0, \infty)$ be a function defined by $f(x) = \frac{x}{7}$ for $x \in [0, \infty)$. Then $f \in \mathsf{F}$. We define $T : E_0 \to E$ by $T(\phi) = \frac{4}{7}\phi(\frac{1}{2}) + \frac{3}{28}, \phi \in E_0$. Clearly T is a weakly contractive type mapping. Hence T satisfies all the hypotheses of Theorem 3.1 and T has a PPF dependent fixed point. We now compute this PPF dependent fixed point. We define $\phi : I \to E$ by

$$\phi(x) = \begin{cases} x^2 & \text{if } x \in [0, \frac{1}{2}] \\ \frac{1}{4} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, $||\phi||_{E_0} = \frac{1}{4} = \left|\left|\phi(\frac{1}{2})\right|\right|_E$. Therefore $\phi \in R_c$ and $T(\phi) = \phi(\frac{1}{2})$, so that ϕ is a PPF dependent fixed point of T in R_c.

The weakly contractive type mapping T may have more than one PPF dependent fixed point in R_c. The following example shows that if the weakly contractive type mapping T have more than one PPF dependent fixed point in R_c, then R_c is not algebraically closed with respect to the difference.

Example 3.2. Let $I = [0, 1]$, $E = \mathbb{R}$. Fix $c = \frac{1}{2} \in [0, 1]$. Let $E_0 = C(I, \mathbb{R})$. Let $f : [0, \infty) \to [0, \infty)$ be a function defined by $f(x) = \frac{3x}{4}$ for $x \in [0, \infty)$. Then $f \in \mathsf{F}$. We define $T : E_0 \to E$ by $T(\phi) = \frac{1}{4}\phi(\frac{1}{2}) + \frac{3}{16}$. Clearly T is a weakly contractive type mapping. We define $\phi : I \to E$ by

$$\phi(x) = \begin{cases} x^2 & \text{if } x \in [0, \frac{1}{2}] \\ \frac{1}{4} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, $||\phi||_{E_0} = \frac{1}{4} = \left|\left|\phi(\frac{1}{2})\right|\right|_E$ and $T\phi = \phi(\frac{1}{2})$. Therefore ϕ is a PPF dependent fixed point of T in R_c. We define $\psi : I \to E$ by

$$\psi(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{1}{4} & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, $||\psi||_{E_0} = \frac{1}{4} = \left|\left|\psi(\frac{1}{2})\right|\right|_E$ and $T\psi = \psi(\frac{1}{2})$. Therefore ψ is a PPF dependent fixed point of T in R_c. Hence ϕ and ψ are two PPF dependent fixed points of T in

R_c. Here we observe that $\phi - \psi \notin R_c$. For,

$$(\phi - \psi)(x) = \begin{cases} x^2 - \frac{x}{2} & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

and $||\phi - \psi||_{E_0} = \frac{1}{16} \neq 0 = \left|\left|\phi(\frac{1}{2}) - \psi(\frac{1}{2})\right|\right|_E$, so that $\phi - \psi \notin R_c$. Therefore R_c is not algebraically closed with respect to the difference with $c = \frac{1}{2}$.

To prove the uniqueness of PPF dependent fixed point of T, we use the following 'condition (H)'.

(**H**) : If ϕ and ψ are any two PPF dependent fixed points in R_c, then $\phi - \psi \in R_c$.

THEOREM 3.2. *In addition to the hypotheses to Theorem 3.1, if R_c satisfies condition (**H**), then every weakly contractive type mapping $T : E_0 \to E$ has a unique PPF dependent fixed point in R_c.*

PROOF. Let us suppose ϕ and ψ be two PPF dependent fixed points of a weakly contractive type mapping T. Hence $T\phi = \phi(c)$ and $T\psi = \psi(c)$. By condition (H), we have $\phi - \psi \in R_c$. Therefore $||\phi - \psi||_{E_0} = ||\phi(c) - \psi(c)||_E$. We consider

$$\begin{aligned} ||\phi - \psi||_{E_0} = ||\phi(c) - \psi(c)||_E &= ||T\phi - T\psi||_E \\ &\leqslant ||\phi - \psi||_{E_0} - f(||\phi - \psi||_{E_0}). \end{aligned}$$

It follows that $f(||\phi - \psi||_{E_0}) \leqslant 0$ so that $f(||\phi - \psi||_{E_0}) = 0$. Therefore $||\phi - \psi||_{E_0} = 0$ and hence $\phi = \psi$. □

In the following, we show that every function f in E_0 that attains maximum value at some point $c \in [a, b]$ is a PPF dependent fixed point.

COROLLARY 3.1. *If any function $f \in E_0$ attains maximum value at some $c \in [a, b]$, then it is the unique PPF dependent fixed point in some algebraically closed linear subspace F_0 of E_0 which is contained in R_c.*

PROOF. Let $f \in E_0$ be such that $||f||_{E_0} = ||f(c)||_E$ for some $c \in [a, b]$. Then $f \in R_c$. We define $F_0 = \{\alpha f / \alpha \in \mathbb{R}\}$. Then by Theorem 2.1(ii), we have $F_0 \subseteq R_c$. Clearly F_0 is an algebraically closed linear subspace of E_0 contained in R_c.

We define $T : E_0 \to E$ by $T(\phi) = f(c)$. For any $\phi, \psi \in E_0$ and $g \in \mathsf{F}$, clearly we have

$$||T\phi - T\psi||_E = 0 \leqslant ||\phi - \psi||_{E_0} - g(||\phi - \psi||_{E_0}),$$

so that T is a weakly contractive type mapping. Hence by Theorem 3.1, it follows that, T has a PPF dependent fixed point and it is f, since $T(f) = f(c)$. We observe that it is unique in F_0. □

REMARK 3.1. Thoerem 2.3 follows from Theorem 3.2 as a corollary.

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