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EQUITABLE TOTAL DOMINATING GRAPHS

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Abstract. The equitable total dominating graph $D_{qt}(G)$ of a graph $G = (V, E)$ is the graph with the vertex set $V \cup S$ where S is the collection of all minimal equitable total dominating sets of G with two vertices $u, v \in V \cup S$ adjacent in $D_{qt}(G)$ if $u \in V$ and v is a minimal equitable total dominating set in S containing u. In this paper, we initiate a study of this new graph and obtain basic properties of $D_{qt}(G)$ like, connectedness, covering invariants, connectivity, traversability and planarity.

1. Introduction

All graphs considered here are simple, finite, connected and nontrivial. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the *vertex set* and $E(G)$ be the *edge set* of G. The vertex $v \in V$ is called a *pendant vertex*, if $deg_G(v) = 1$ and an *isolated vertex* if $deg_G(v) = 0$, where $deg_G(x)$ is the *degree of a vertex* $x \in V(G)$. A vertex which is adjacent to a pendant vertex is called a *support vertex*. We denote $\delta(G)(\Delta(G))$ as the *minimum(maximum) degree* and $n = |V(G)|$, $m = |E(G)|$ the *order* and *size* of G respectively. A *spanning subgraph* is a subgraph containing all the vertices of G. A shortest $u - v$ path is often called a *geodesic*. The *diameter* $diam(G)$ of a connected graph G is the length of any longest geodesic. The *neighborhood* of a vertex u in V is the set $N(u)$ consisting of all vertices v which are adjacent with u. The *closed neighborhood* is $N[u] = N(u) \cup \{u\}$. A subset $S \subseteq V(G)$ is said to be vertex covering set if every edge of G is incident to at least one vertex in S. The minimum cardinality among all vertex covering sets is called vertex covering number. It is denoted by $\alpha_0(G)$. A subset $F \subseteq E(G)$ is said to be edge covering set if every vertex of G is incident to at least one edge in F. The minimum cardinality among all edge covering sets is called edge covering number. It is denoted by $\alpha_1(G)$.

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The cardinality of maximum independent set of vertices(respectively edges) of a graph G is called vertex(respectively edge) independence number. It is denoted by $\beta_0(G)$ (respectively $\beta_1(G)$). The vertex connectivity is the minimum number of vertices are required to disconnect a graph. It is denoted by $\kappa(G)$. Similarly, the edge connectivity is the minimum number of edges are required to disconnect a graph. It is denoted by $\lambda(G)$.

A subset D of V is called a *dominating set* of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of G is minimal if for every vertex $v \in D$, $D - \{v\}$ is not a dominating set of G. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set.

Cockayne et. al [**3**] introduced the concept of total domination in graphs. A dominating set D of G is called a *total dominating set* if $\langle D\rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of G is called the *total domination number* of G and is denoted by $\gamma_t(G)$.

A subset D of V is called an *equitable dominating set* if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leqslant 1$. The minimum cardinality of such a dominating set is called the *equitable domination* number and is denoted by $\gamma^e(G)$. This concept was introduced by Swaminathan et. al [**12**].

A subset D of V is called an *equitable total dominating set* of G, if D is an equitable dominating set and $\langle D\rangle$ has no isolated vertices. The minimum cardinality taken over all equitable total dominating sets is the *equitable total domination number*[**2**] and is denoted by $\gamma_t^e(G)$.

The minimal dominating graph of G is an intersection graph on the minimal dominating sets of vertices of G. This concept was introduced by Kulli and Janakiram [**8**].

In [**9**], the concept of common minimal dominating graph of G was defined as the graph having same vertex set as G with two vertices adjacent if there is a minimal dominating set containing them. The concept of vertex minimal dominating graph $M_VD(G)$ of G was introduced in [**10**], as the graph having $V(M_VD(G)) = V(G) \cup S(G)$, where $S(G)$ is the set of all minimal dominating sets of G with two vertices u, v adjacent if they are adjacent in G or $v = D$ is a minimal dominating set containing u.

The edge dominating graph $E_D(G)$ of G as the graph with $V(E_D(G)) = E(G) \cup S(G)$, where $S(G)$ is the set of all minimal edge dominating sets of G with two vertices $u, v \in V(E_D(G))$ adjacent, if $u \in E$ and $v = S$ is a minimal edge dominating set containing u [**1**].

In this paper, we introduce the concept of equitable total dominating graph, which is defined as follows:

Definition 1.1. The *equitable total dominating graph* $D_{qt}(G)$ of a graph $G = (V, E)$ is the graph with the vertex set $V \cup S$ where S is the collection of all minimal equitable total dominating sets of G with two vertices $u, v \in V \cup S$ adjacent in $D_{qt}(G)$ if $u \in V$ and v is a minimal equitable total dominating set in S containing u..

In Figure 1, a graph G and its equitable total dominating graph $D_{qt}(G)$ are shown: Here the minimal equitable total dominating sets of a graph G are: $D_1 = \{1,2\}$, $D_2 = \{1,3\}$, $D_3 = \{2,3\}$.

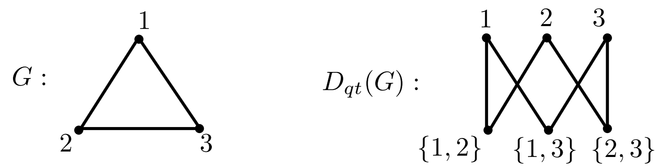

Figure 1: A graph and its equitable total dominating graph.

2. Observations

(1) For any graph G, $D_{qt}(G)$ is bipartite, hence bicolorable.
(2) $V(D_{qt}(G)) = V \cup S$, where S is the set of all minimal equitable total dominating sets in G. No two vertices of G and $S(G)$ are adjacent vertices in $D_{qt}(G)$.
(3) For star graph, $D_{qt}(G)$ does not exist.
(4) $V(D_{qt}(G)) \leqslant \frac{n(n+1)}{2}$, where p is the number of vertices in G.

3. Connectedness of $D_{qt}(G)$

THEOREM 3.1. *For any graph G, the equitable total dominating graph $D_{qt}(G)$ of G is connected if and only if*

(1) $v_i \cap \{s_1, s_2, s_3, \cdots, s_k\} \neq \phi$, *where* $v_i \in V$; $1 \leqslant i \leqslant p$;
(2) $\bigcup_{j=1}^{k} s_j = V$.

PROOF. Let G be a graph with vertex set $V = \{v_1, v_2, v_3, \cdots, v_p\}$ and let S be set of all minimal equitable total dominating sets of G i.e $S = \{s_1, s_2, s_3, \cdots, s_k\}$ for some positive integer k. Suppose G does not satisfies the conditions (1) and (2), then there exist a vertex say $v_i \in V(G)$ which belongs to the minimal equitable total dominating set $s_j \in S(G)$ such that $v_i \cap s_j = \phi$. Hence there is no path between the vertex v_i or s_j to any other vertex of $D_{qt}(G)$. Hence $D_{qt}(G)$ is disconnected.

Conversely, suppose G satisfies the conditions (1) and (2) then there is a path between any pair of vertices in $D_{qt}(G)$. Hence $D_{qt}(G)$ is connected. □

THEOREM 3.2. *For any connected graph G, $D_{qt}(G) = S_1(G)$ if and only if every pair of adjacent vertices of G forms an equitable total dominating set in G, where $S_1(G)$ is the subdivision graph of G.*

PROOF. Let $D_{qt}(G) = S_1(G)$. Suppose there exists a pair of adjacent vertices in G which do not forms an equitable total dominating set in G, then in $D_{qt}(G)$ these two vertices are independent and have no common point to join them, a contradiction. Hence every pair of adjacent vertices of G forms an equitable total dominating set in G.

Converse is obvious. □

PROPOSITION 3.1. *For any disconnected graph G with n-components ($G \neq \bigcup_{i=1}^{n} K_2$), $D_{qt}(G)$ is (n+1) partite graph.*

PROOF. Suppose G is connected then $D_{qt}(G)$ is bipartite by Observation (1). That is if G has only one component then $D_{qt}(G)$ is bipartite. Suppose G has two components then we get 3 pair of independent vertex sets i.e., $D_{qt}(G)$ is tripartite. Hence for n-components of G, $D_{qt}(G)$ is (n+1) partite graph. $\square$

4. Covering invariants for $D_{qt}(G)$

THEOREM 4.1. *For any graph G, $\beta_0(D_{qt}(G)) = max\{|V|, |S|\}$, where V is the vertex set of a graph G and S is the set of all minimal equitable total dominating sets of G.*

PROOF. Let G be a graph with vertex set $V = \{v_1, v_2, v_3, \cdots, v_p\}$ and let S be set of all minimal equitable total dominating sets of G i.e., $S = \{s_1, s_2, s_3, \cdots, s_k\}$ for some positive integer k. Notice that $D_{qt}(G)$ is a bipartite graph with the partition $V \cup S$. Hence no two vertices in the set V or in the set S are adjacent. Therefore, the set of maximum cardinality will form a vertex independence number for $D_{qt}(G)$. Thus, $\beta_0(D_{qt}(G)) = max\{|V|, |S|\}$. $\square$

COROLLARY 4.1. *For any graph G, $\alpha_0(D_{qt}(G)) = \begin{cases} |S|, & if\ |V| > |S| \\ |V|, & if\ |S| > |V| \end{cases}$, where V is the vertex set of a graph G and S is the set of all minimal equitable total dominating sets of G.*

PROOF. By Theorem 4.1, $\beta_0(D_{qt}(G)) = max\{|V|, |S|\}$. Also we know that $V(D_{qt}(G)) = V \cup S$. Hence the theorem follows from the fact for any graph G, $\alpha_0(G) + \beta_0(G) = p$ [**4**]. $\square$

THEOREM 4.2. *For any graph G, $\beta_1(D_{qt}(G)) = \begin{cases} |S|, & if\ |V| > |S| \\ |V|, & if\ |S| > |V| \end{cases}$, where V is the vertex set of a graph G and S is the set of all minimal equitable total dominating sets of G.*

PROOF. Let G be any graph. Then by Observation (1), $D_{qt}(G)$ is bipartite. Since for any bipartite graph G, $\alpha_0(G) = \beta_1(G)$. Therefore the result follows from Corollary 4.1. $\square$

COROLLARY 4.2. *For any graph G, $\alpha_1(D_{qt}(G)) = |V| + |S| - \beta_1(D_{qt}(G))$ where V is the vertex set of a graph G and S is the set of all minimal equitable total dominating sets of G.*

PROOF. Let G be any graph. By the definition of $D_{qt}(G)$, $V(D_{qt}(G)) = V(G) \cup S(G)$, i.e., $|V(D_{qt}(G))| = |V| + |S(G)|$. Further from Theorem 4.2 and the fact that $\alpha_1(G) + \beta_1(G) = |V(G)|$, we have the following useful observations

- Suppose $|V(G)| < |S(G)|$, then $\alpha_1(D_{qt}(G)) = |S(G)|$.
- Suppose $|V(G)| > |S(G)|$, then $\alpha_1(D_{qt}(G)) = |V(G)|$.

□

COROLLARY 4.3. *For any graph G, $\alpha_1(D_{qt}(G)) \leqslant \frac{p(p+1)}{2} - \beta_1(D_{qt}(G))$ where V is the vertex set of a graph G and S is the set of all minimal equitable total dominating sets of G.*

PROOF. Proof follows from Observation (4) and Corollary 4.2. □

5. Connectivity of $D_{qt}(G)$

THEOREM 5.1. *For any graph G,*

$$\kappa(D_{qt}(G)) = min\{min_{1\leqslant i\leqslant p}(deg(v_i)), min_{1\leqslant j\leqslant p}|s_j|\}$$

where s_j are the minimal equitable total dominating sets in G.

PROOF. Let G be a (p, q) graph. Clearly vertex set and equitable total dominating sets of G are independent. We consider the following cases.

$Case1$: Let $u \in v_i$ for some i, having minimum degree among all v_i's in $D_{qt}(G)$. If the degree of u is less than any other vertex in $D_{qt}(G)$, then by deleting those vertices of $D_{qt}(G)$ which are adjacent with u results in a disconnected graph.

$Case2$: Let $w \in s_j$ for some j having minimum degree among all vertices of s_j's. If degree w is less than any other vertices in $D_{qt}(G)$, then by deleting those vertices which are adjacent with w, results in a disconnected graph. Thus

$$\kappa(D_{qt}(G)) = min\{min_{1\leqslant i\leqslant p}(deg(v_i)), min_{1\leqslant j\leqslant p}|s_j|\}.$$

□

THEOREM 5.2. *For any graph G,*

$$\lambda(D_{qt}(G)) = min\{min_{1\leqslant i\leqslant p}(deg(v_i)), min_{1\leqslant j\leqslant p}|s_j|\}$$

where s_j are the minimal equitable total dominating sets in G.

PROOF. Let G be a (p, q) graph. Clearly vertex set and equitable total dominating sets of G are independent. We consider the following cases.

$Case1$: Let $u \in e_i$ for some i, having minimum degree among all e_i's in $D_{qt}(G)$. If the degree of u is less than any other edge in $D_{qt}(G)$, then by deleting those edges of $D_{qt}(G)$ which are adjacent with u results in a disconnected graph.

$Case2$: Let $w \in s_j$ for some j having minimum degree among all edges of s_j's. If degree w is less than any other edge in $D_{qt}(G)$, then by deleting those edges which are adjacent with w, results in a disconnected graph. Thus

$$\lambda(D_{qt}(G)) = min\{min_{1\leqslant i\leqslant p}(deg(v_i)), min_{1\leqslant j\leqslant p}|s_j|\}.$$

□

6. Traversability of $D_{qt}(G)$

THEOREM 6.1. *For any graph G, the equitable total dominating graph $D_{qt}(G)$ of G is eulerian if and only if the following conditions are satisfied,*

(1) $v_i \cap \{s_1, s_2, s_3, \cdots, s_k\} \neq \phi$, *where* $v_i \in V$; $1 \leqslant i \leqslant n$;

(2) $\bigcup_{j=1}^{k} s_j = V$,

(3) $|s_i| = 2x$; $1 \leqslant i \leqslant k$, $x \in \mathbb{Z}^+$.

PROOF. Let G be a graph having vertex set $V = \{v_1, v_2, v_3, \cdots, v_p\}$ and let S be set of all minimal equitable total dominating sets of G i.e $S = \{s_1, s_2, s_3, \cdots, s_k\}$ for some positive integer k. Suppose G satisfies the conditions (1) and (2), then by Theorem 4.1 $D_{qt}(G)$ is connected. Now, suppose G satisfies the condition (3), then the degree of a vertex $s_j \in V(D_{qt}(G))$ and $v_i \in V(D_{qt}(G))$ will be even. Hence there exist a eulerian path in $D_{qt}(G)$. Therefore $D_{qt}(G)$ is eulerian.

Conversely, suppose G does not satisfies any of the above conditions (1), (2) and (3). Then $D_{qt}(G)$ either disconnected or containing a vertex of odd degree. Hence G must satisfies the conditions (1)-(3). □

THEOREM 6.2. *For any graph G, the equitable total dominating graph $D_{qt}(G)$ of G is hamiltonian if the following conditions are satisfied,*

(1) $v_i \cap \{s_1, s_2, s_3, \cdots, s_k\} \neq \phi$, *where* $v_i \in V$; $1 \leqslant i \leqslant p$,

(2) $\bigcup_{j=1}^{k} s_j = V$

(3) $|s_i| = 2$; $1 \leqslant i \leqslant k$.

PROOF. Suppose G satisfies the conditions (1) and (2) then by Theorem 4.1 $D_{qt}(G)$ is connected. Now we have to show that $D_{qt}(G)$ contains a Hamiltonian cycle. To show this let us assume that G satisfies the condition (3), then the degree of every vertex in $D_{qt}(G)$ will be two, and hence there exists a spanning cycle in $D_{qt}(G)$. Hence $D_{qt}(G)$ contains a Hamiltonian cycle. Therefore, $D_{qt}(G)$ is Hamiltonian. □

7. Planarity of $D_{qt}(G)$

THEOREM 7.1. *For any graph G, the equitable total dominating graph $D_{qt}(G)$ of G is planar if and only if G does not satisfies the following conditions,*

(1) $v_i \cap \{s_i, s_j, s_k\} \neq \phi$
(2) $v_j \cap \{s_i, s_j, s_k\} \neq \phi$
(3) $v_k \cap \{s_i, s_j, s_k\} \neq \phi$

for any $v_i, v_j, v_k \in V(G)$ *and* $s_i, s_j, s_k \in S$.

PROOF. Let G be a graph having vertex set $V = \{v_1, v_2, v_3, \cdots, v_p\}$ and let S be set of all minimal equitable total dominating sets of G i.e., $S = \{s_1, s_2, s_3, \cdots, s_k\}$ for some positive integer k. Now we have to prove that the equitable total dominating graph is planar if and only if it satisfies the hypothesis of the theorem. By well known Kurtowski's theorem, a graph G is nonplanar if and only if K_5 or $K_{3,3}$ is not a subgraph of G. By Observation (1), the equitable total dominating graph is bipartite. Therefore, K_5 will not be a subgraph of $D_{qt}(G)$ because it contains odd cycles. Now we have left with a choice of recognizing the complete bipartite graph $K_{3,3}$ as a subgraph in $D_{qt}(G)$. Let us assume that the hypothesis of the theorem is true, then the degree of every vertex i.e v_i, v_j, v_k, s_i, s_jands_k will be 3 and by Observation (1), $D_{qt}(G)$ is bipartite. Hence $D_{qt}(G)$ contains a subgraph $K_{3,3}$ as a subgraph. Therefore, $D_{qt}(G)$ is planar if hypothesis of the theorem is not satisfied by G.

Conversely, if G does not satisfies the hypothesis of the theorem, then $D_{qt}(G)$ does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$. Hence $D_{qt}(G)$ is planar. □

THEOREM 7.2. *For any graph G, the equitable total dominating graph $D_{qt}(G)$ of G is outerplanar if and only if G does not satisfies the following conditions,*

(1) $v_i \cap \{s_i, s_j\} \neq \phi$
(2) $v_j \cap \{s_i, s_j\} \neq \phi$
(3) $v_k \cap \{s_i, s_j\} \neq \phi$

for any $v_i, v_j, v_k \in V(G)$ and $s_i, s_j \in S$.

PROOF. Let G be a graph having vertex set $V = \{v_1, v_2, v_3, \cdots, v_p\}$ and let S be set of all minimal equitable total dominating sets of G i.e $S = \{s_1, s_2, s_3, \cdots, s_k\}$ for some positive integer k. Now we have to prove that the equitable total dominating graph is outertplanar if and only if it satisfies the hypothesis of the theorem. We know that a graph G is outerplanar if and only if G does not contain K_4 or $K_{2,3}$ as a subgraph. By Observation (1), the equitable total dominating graph is bipartite. Therefore, K_4 will not be a subgraph of $D_{qt}(G)$ because it contains odd cycles. Now we have left with a choice of recognizing the complete bipartite graph $K_{2,3}$ as a subgraph in $D_{qt}(G)$. Let us assume that the hypothesis of the theorem is true, then the degree of every vertex i.e v_i, v_j, v_k is 2 and the degree of s_i, s_j will be 3 and by Observation (1), $D_{qt}(G)$ is bipartite. Hence $D_{qt}(G)$ contains $K_{2,3}$ as a subgraph. Therefore, $D_{qt}(G)$ is outerplanar if hypothesis of the theorem is not satisfied by G.

Conversely, if G does not satisfies the hypothesis of the theorem, then $D_{qt}(G)$ does not contain a subgraph homeomorphic to K_4 or $K_{2,3}$. Hence $D_{qt}(G)$ is outerplanar. □

THEOREM 7.3. *For any graph G, the equitable total dominating graph $D_{qt}(G)$ of G is maximal planar if and only if G does not satisfies the following conditions,*

(1) $v_i \cap \{s_i, s_j, s_k\} \neq \phi$
(2) $v_j \cap \{s_i, s_j, s_k\} \neq \phi$
(3) $v_k \cap \{s_i, s_j\} \neq \phi$

for any $v_i, v_j, v_k \in V(G)$ and $s_i, s_j, s_k \in S$.

PROOF. Let G be a graph having vertex set $V = \{v_1, v_2, v_3, \cdots, v_p\}$ and let S be set of all minimal equitable total dominating sets of G i.e $S = \{s_1, s_2, s_3, \cdots, s_k\}$ for some positive integer k. Now we have to prove that the equitable total dominating graph is maximal planar if and only if it satisfies the hypothesis of the theorem. We know that a graph G is maximal planar if and only if G does not contain $K_5 - x$ or $K_{3,3} - x$ (where x is any edge) as a subgraph. By Observation (1), the equitable total dominating graph is bipartite. Therefore, $K_5 - x$ will not be a subgraph of $D_{qt}(G)$ because it contains odd cycles. Now we have left with a choice of recognizing the graph $K_{3,3} - x$ as a subgraph in $D_{qt}(G)$. Let us assume that the hypothesis of the theorem is true, then the degree of every vertex i.e v_i, v_k, s_i, s_j is 3 and the degree of the vertices v_j, s_k are 2 and by Observation 1, $D_{qt}(G)$ is bipartite. Hence $D_{qt}(G)$ contains $K_{3,3} - x$ as a subgraph. Hence $D_{qt}(G)$ is maximal planar if hypothesis of the theorem is not satisfied by G.

Conversely, if G does not satisfies the hypothesis of the theorem, then $D_{qt}(G)$ does not contain a subgraph homeomorphic to $K_5 - x$ or $K_{3,3} - x$. Hence $D_{qt}(G)$ is maximal planar. □

References

[1] B. Basavanagoud and S. M. Hosamani. Edge dominating graph of a gaph. *Tamakang J. Math.*, **43**(4)(2013), 603-608.

[2] B. Basavanagoud, V. R. Kulli and Vijay V. Teli. Equitable total domination in graphs. *J. Comp. Math. Sci.*, **5**(2)(2014), 235-241.

[3] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi. Total domination in graphs. *Networks*, **10**(3)(1980), 211-219.

[4] F. Harary. *Graph Theory.* Addison-Wesley, Reading, Mass, 1969.

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, Inc., New York, 1998.

[6] V. R. Kulli. *Theory of Domination in Graphs.* Vishwa International Publications, Gulbarga, India, 2010.

[7] V. R. Kulli and Radha R. Iyer. The Total Minimal Dominating Graph. In: V. R. Julli (Ed.). *Advances in Domination Theory I* (pp. 121–125). Vishwa International Publications, Gulbarga, India, 2012.

[8] V. R. Kulli and B.Janakiram. The minimal dominating graph. *Graph Theory Notes of New York*, Vol. **28**, Academy of Sciences, New York, 1995, (pp. 12-15).

[9] V. R. Kulli and B. Janakiram. The common minimal dominating graph. *Indian J. Pure. Appl. Math*, **27**(2)(1996), 193-196.

[10] V. R. Kulli, B.Janakiram and K.M.Niranjan. The vertex minimal dominating graph. *Acta Ciencia Indica*, **28**(3)(2002), 435-440.

[11] V. R. Kulli, B. Janakiram and K. M. Niranjan. The dominating graph. *Graph Theory Notes of New York*, Vol. **46**, New York Academy of Sciences, New York, 2004, (pp. 5-8).

[12] V. Swaminathan and K. M. Dharmalingam. Degree equitable domination on graphs. *Kragujevac J. Math.*, **35**(1)(2011), 191-197.

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